POINCARE RECURRENCE THEOREM FOR FINITELY ADDITIVE MEASURES

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SUMMARY.- *In this paper we study the validity of Poincaré recurrence theorem for finitely additive measures.*

§ 1.- DEFINITIONS AND PROBLEM

Let X be an arbitrary non empty point set, and $T : X \to X$ a trasformation on X. If $(X,\mathcal{Q},\mu)$ is a charge space, i.e. $\mathcal{Q}$ is a field of subsets of X and μ is a nonnegative charge (usually called finitely additive measure) the transformation T is called a measurable transformation if

$$(1.1) \qquad \forall A \in \mathcal{Q} : T^{-1}(A) \in \mathcal{Q}$$

A measurable trasformation T is said to be measure preserving if

$$(1.2) \qquad \forall A \in \mathcal{Q} : \mu(T^{-1}(A)) = \mu(A).$$

If T is a measure preserving transformation and $E \in \mathcal{Q}$ then a point $x \in E$ is called recurrent if

$$\exists\, n \in \mathbb{N}^{(2)} \text{ such that } T^n x \in E$$

and x is called strongly recurrent if

$$T^n x \in E \qquad \text{for infinitely many values of } n.$$

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(2) $\mathbb{N}$ is the set $\{1,2,3,\dots\}$ of positive integers, $\mathbb{N}_0 = \{0,1,2,\dots\}$ and $\mathbb{Z} = \{\dots -2,-1,0,1,2,\dots\}$

Now, the classical Poincaré's recurrence theorem, very usefull in the Ergodic theory, asserts:

If $\mathcal{A}$ is a σ-field of subsets of a set X,

μ is a coutably additive measure (c.a.m.), T is measure preserving, $\mu(X) < +\infty$ and $E \in \mathcal{A}$, then almost every point of E is strongly recurrent.

This theorem is due essentially to H. Poincaré ([1]) p. 67-72) but the first rigorous proof was given in [2] by C. CARATHEODORY.

In this paper we study the validity of Poincaré recurrence theorem for charges.

§ 2.- Results

Theorem 1.- *If T is measure preserving μ a charge, $\mu(X) < +\infty$, $\mathcal{A}$ is a σ-field and $E \in \mathcal{A}$, then almost every point of E is recurrent.*

PROOF.

We consider the set

(2.1) $\quad F = \{x \in E : T^n x \notin E \quad \forall n \in \mathbb{N}\}$

because of the identity

$$F = E - \bigcup_{n=1}^{\infty} \{x \in E : T^n x \in E\} = E - \bigcup_{n=1}^{\infty} T^{-n}(E) ,$$

F is measurable.

Furthermore we have

$$F \cap T^{-n}(F) = \emptyset$$

and all the sets

$$F , T^{-1}(F) , T^{-2}(F) \dots$$

are mutually disjoint since

$$T^{-n}(F) \cap T^{-(n+p)}(F) = T^{-n}(F \cap T^{-p}(F)) = T^{-n}(\emptyset) = \emptyset .$$

Because T is measure preserving we have

$$\mu(T^{-n}(F)) = \mu(F) \qquad \text{for} \qquad n = 1,2,3,\ldots.$$

and soif $\mu(F) > 0$ we would have

$$+\infty = \sum_{n=1}^{\infty} \mu(F) = \sum_{n=1}^{\infty} \mu(T^{-n}(F)) \leq \mu(\bigcup_{n=1}^{\infty} T^{-n}(F)) \leq \mu(X) < +\infty.$$

This is a contradiction. ∎

It is well known that for $\mu(X) = +\infty$ theorem 1 is not necessarily true even if μ is a c.a.m. .

If μ is a c.a.m., we have also the strong recurrence, i.e. almost every point of E is strongly recurrent, but this is not true if μ is only finitely additive.

In fact if for instance $X = \mathbb{R}$ it is well known ([3]) pag. 243) that there is a charge ν on $\mathcal{P}(\mathbb{R})$ satisfying the following conditions:

(i) $0 \leq \nu(A) \leq 1$ for all $A \subset \mathbb{R}$

(ii) $\nu(A) = 1$ if $[\alpha,\infty[\subset A$ for some $\alpha \in \mathbb{R}$

(iii) $\nu(A) = 0$ if A is bounded above

(iv) $\nu(A+a) = \nu(A)$ for all $A \subset \mathbb{R}$ and $a \in \mathbb{R}$

Now if we consider

$$T(x) = x-1 \quad \text{and} \quad A =]0,1] \cup]2,3] \cup]4,5] \cup \ldots$$

because $A \cup A+1 =]0,+\infty]$ it follows that $\nu(A \cup A+1) = 1$

and since $A \cap A+1 = \emptyset$ and $\nu(A) = \nu(A+1)$

(by (iv) we have

$$\nu(A) = \frac{1}{2}.$$

For every $x \in A$ the set $\{n : T^n(x) \in A\}$ is finite and x is not strongly recurrent.

So the strong version of recurrence theorem is not true, but we can give a result very near.

We need the following definition: a point $x \in E$ is called <u>n-times recurrent</u> (for $n = 1,2,3,\ldots$.) if there are n different values of $k \in \mathbb{N}$ such that

$$T^k x \in E$$

<u>Theorem 2</u>. *If T is measure preserving, μ is a charge, $\mu(X) < +\infty$ $\mathfrak{A}$ is a σ-field and $E \in \mathfrak{A}$, then for each $n \in \mathbb{N}$, almost every point of E is n-times recurrent.*

<u>PROOF</u>.

Let $S(1,E) = \{x \in E : T^k(x) \in E$ for at least one $k \in \mathbb{N}\}$

Since (see (2.1)

$$F = E - S(1,E)$$

we have $\quad S(1,E) \in \mathfrak{A}$ and

(2.2) $$\mu(E) = \mu(S(1,E))$$

We define in general

$$S(n,E) = \{x \in E : T^k(x) \in E \text{ for at least } n \text{ different values of } k \in \mathbb{N}\}$$

We can easily recognise that

$$S(1,S(1,E)) = S(2,E)$$

so we have for the same reason of (2.2):

$$\mu(S(1,E)) = \mu(S(2,E))$$

and also

$$\mu(S(2,E)) = \mu(E)$$

In general $$S(1,S(n-1,E)) = S(n,E)$$

and so $$\mu(S(n,E)) = \mu(E) \qquad \text{for every } n \in \mathbb{N}.$$

This means that the set

$$F_n = E - S(n,E) = \{x \in E : x \text{ is not n-times ricurrent}\}$$

is measurable and $\mu(F_n) = 0$

Remark 1.-

In theorem 1 we have used the hypothesis that $\mathcal{A}$ is a σ-field, in proving that F is measurable. Is this hypothesis essential? We do not know the answer but we give an exemple where if $\mathcal{A}$ is only a field F is not measurable. Let $X = \mathbb{N} \times Z$, $E = \{(m,0); m \in \mathbb{N}_\circ\} \cup \{(m,-m); m \in \mathbb{N}_\circ\} \cup \{0,-m); m \in \mathbb{N}_\circ\}$, $T : X \to X'$ be defined by $T(n,m) = (n,m+1)$.

Let $\mathcal{A}$ be the smallest field that contains A and such that T verifies (1.1). Such an $\mathcal{A}$ is the collection of all finite unions of sets of the form

$$T^{-n_1}(E) \cap T^{-n_2}(E) \cap \ldots \cap T^{-n_k}(E) \cap T^{-m_1}(E') \cap \ldots \cap T^{-m_h}(E') \qquad (E' = X - E)$$

for some integers $n_1, n_2, \ldots, n_k, m_1, \ldots, m_h$ in $\mathbb{N}_\circ$.

Now the set $F = \{x \in E : T^n x \notin E \text{ for all } n \in \mathbb{N}\} = \{(m,0); m \in \mathbb{N}_\circ\}$ is not an element of $\mathcal{A}$.

In fact if F was an element of $\mathcal{A}$ there exist $n_1, n_2, \ldots, n_k, m_1, \ldots, m_h \in \mathbb{N}_\circ$ such that

$$F \supset C = T^{-n_1}(E) \cap T^{-n_2}(E) \cap \ldots \cap T^{-n_k}(E) \cap T^{-m_1}(E') \cap \ldots \cap T^{-m_h}(E')$$

and the latter element is nonempty .

Because if $n_i \neq 0$ then $T^{-n_i}(E) \cap F = \emptyset$, it must be $n_1 = n_2 = \ldots = n_k = 0$ and $T^{-n_1}(E) \cap \ldots \cap T^{-n_k}(E) = E$.

But $E \cap T^{-m_1}(E') \cap \ldots \cap {}^{-m_h}(E')$ contains $\{(m,-m) : m \geq p\}$ for some $p \in \mathbb{N}$. Thus F cannot contain $C \neq \emptyset$.

Now by a technique developed in Theorem 2 of [4] we can in fact get a nonnegative charge μ on $\mathcal{A}$ such that $\mu(X) = 1$ and $\mu(E) = \frac{1}{2}$, because fon any integer m there is an $x \in E$ such that $T^n x \in E$ for all $n \leq m$. This can be even seen directly by the Hahn-Banach Theorem.

Remark 2.

Observing that the proof of Theorem 2 holds for conservative transformations (there does not exist a set $F \in \mathcal{A}$ with $\mu(F) > 0$ such that the sets

$F, T^{-1}(F), T^{-2}(F), \ldots$ are pairwise disjoint) one can see that in a charge space a trasformation is conservative iff for every set A of positive charge and for every n, almost every point of A is n-times ricurrent.

The other aspects of Ergodic Theory for charges are being worked out by the Authors.

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