TM and if M is compact and orientable, the Gauss-Bonnet theorem says that

$$\int_M f^* \Omega^{(n)} = \chi(M)$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M and $f: M \to \tilde{G}_n(M)$ is an orientation of M, $\tilde{G}_n(M)$ being the Grassmann bundle of the oriented n-planes tangent to M.

2. A vanishing theorem

We can now prove the following theorem ([3])

(2.1). Theorem

Let M be a riemannian smooth orientable manifold of dimension n which admits k complementary (smooth) distributions of oriented n_i-planes $(i = 1,\dots,k)$. Then the real Pontrjagin classes $P_r(M)$ are null for $2r > \max(n_1,\dots,n_k)$

Proof.

Let $\tilde{E}$ be the principal fibre bundle of the orthonormal frames (associated to the tangent). Its structural group is $\tilde{G} = SO(n)$ (the rotation group). We recall that the Lie algebra $\tilde{\mathcal{G}}$ of $SO(n)$ can be identified with space of the skew-symmetric matrices of order n.

Let us consider the subbundle E of $\tilde{E}$ formed of the frames "adapted" to the distributions, viz. the orthogonal frames $\{e_i\}(i=1,\dots,n)$ so that the vectors

$$\{e_{\alpha_j}\} \qquad \alpha_j = n_1+\dots+n_{j-1}+1,\dots,n_1+\dots+n_j \qquad (n_\circ = 0)$$

form a basis for T^j. The bundle E can be regarded as having structural group

$$G = SO(n_1) \times SO(n_2) \times \ldots \times SO(n_k).$$

A connection ω on E is represented by a 1-form which takes values in the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{g}$ is the direct product of the Lie algebras of $SO(n_r)$. Hence one obtains

$$\omega_{ii} = 0 \qquad \omega_{ij} + \omega_{ji} = 0 \qquad i,j = 1,\ldots,n$$

$$\omega_{\alpha_i\beta_j} = 0 \quad i \neq j \quad i,j = 1,\ldots,k; \ \alpha_i,\beta_j = n_1+\ldots n_{j-1}+1,\ldots,n_1+\ldots+n_j$$

Analogous relations hold for the components of the curvature form Ω. Therefore, if $2r > \max(n_1,\ldots,n_k)$, then each term in (1.2) will have a factor $\Omega_{\alpha_i\beta_j}$ with $i\neq j$; the assertion $p_r(TM) = P_r(M) = 0$ then follows from (1.1).┘

Remarks

(2.2) If $k = n$ and therefore $n_i = 1$ $\forall i$ (i.e. the manifold is parallelizable) then $P_r(M) = 0$ $\forall r$. It follows that the adapted connection vanishes. The manifold is then flat.

(2.3) For $k = 2$ (and obviously for $k = 1$) the theorem is not meaningful. As for $n_1 + n_2 = n$, $\max(n_1,n_2) \geq [n/2]$ and consequently $P_r(M) = 0$ $\forall\ 2r > \max(n_1,n_2)$.

(2.4) It is worth noticing that the existence of a distribution of q-planes implies the existence of a distribution of (n-q)-planes. An argument analogous to the one followed above, using the Gauss curvature form, yields the following

(2.5) Theorem

Let M be a smooth compact orientable manifold of dimension even n and suppose that it has a distribution of oriented q-planes with q odd ($1 \leq q < n$). Then the Euler-Poincaré characteristic of M is null.

3. Proof of theorems A and B.

(3.1) To every distribution there corresponds a smooth subbundle E_i of the tangent bundle with the fibre of dimension n_i. Denote the quotient bundle by $Q_i = TM/E_i$. From the Whitney duality formula $p(TM)=p(Q_i)p(E_i)$ it follows that

$$(3.2) \qquad P_r(TM) = P_r(Q_i)+P_{r-1}(Q_i)P_1(E_i)+\dots+p_1(Q_i)P_{r-1}(E_i)+p_r(E_i)$$

where the product between classes is the "cup product" in the ring $H^*(M; \mathbb{R})$.

If $2r > n_i$, then $r > n_i/2$ and hence $P_r(E_i) = 0$. If moreover

$$(3.3) \qquad P_h(Q_i)P_s(E_i) = 0 \qquad \forall h,s \geq 1,\ h+s = r$$

then

$$P_r(TM) = P_r(Q_i) \qquad 2r > n_i.$$

Notice that, since the Pontrjagin ring may have divisors of zero, condition (3.3) does not imply that either $P_h(Q_i) = 0$ or $p_s(E_i) = 0$. On account of our assumptions, from theorem (1.1) one can conclude that

$$P_r(Q_i) = 0 \qquad 2r > \max(n_1,\dots,n_k).$$

This proves theorem A.

(3.4) It is well known that if $E_i \subset TM$ is isomorphic to an integrable