$$(8.10)\quad D(\alpha-1,x) \sim x^{\alpha-1}e^{-x}\sum_{s=0}^{\infty}(-1)^s\frac{(1-\alpha)_s}{s!x^s}\int_0^\infty dt\;\frac{t^s e^{-t}}{1-e^{-(x+t)}}\;.$$

9. SOME FUNCTIONAL RELATIONS.

It is easy to show that

$$(9.1)\quad \frac{d}{dx}\left[x^{-\rho}\Psi(\alpha,\beta,\gamma;x)\right] = -x^{-(\rho+1)}\{(\rho-\alpha+\gamma\beta)\Psi(\alpha,\beta,\gamma;x) + \\ +\gamma\left[\Psi(\alpha+1,\beta,\gamma;x)-\Psi(\alpha+1,\beta,\gamma-1;x)-\beta\Psi(\alpha,\beta,\gamma-1;x)\right]\}.$$

In fact, we have

$$(9.2)\quad \frac{d}{dx}\left[x^{-\rho}\Psi(\alpha,\beta,\gamma,x)\right] = -x^{-(\rho+1)}\{\rho\Psi(\alpha,\beta,\gamma;x)+x^{\alpha}\left[1-\left(1-\frac{e^{-x}}{x^{\beta}}\right)^{\gamma}\right]\}.$$

The result (9.1) is thus achieved with the help of the recurrence relation (4.5).

Putting $\rho = \alpha-\gamma\beta$, Eq. (9.2) takes the form

$$(9.3)\quad \frac{d}{dx}\left[x^{-(\alpha-\gamma\beta)}\Psi(\alpha,\beta,\gamma;x)\right] = -\gamma x^{-(\alpha-\gamma\beta+1)}\cdot \\ \cdot\left[\Psi(\alpha+1,\beta,\gamma;x)-\Psi(\alpha+1,\beta,\gamma-1;x)-\beta\Psi(\alpha,\beta,\gamma-1;x)\right],$$

which for $\gamma = 1$ becomes the well-known functional relation for the incomplete Γ-function :

$$\frac{d}{dx}\left[x^{-(\alpha-\beta)}\Gamma(\alpha-\beta,x)\right] = -x^{-(\alpha-\beta+1)}\Gamma(\alpha-\beta+1,x),$$

where $\qquad \Gamma(\alpha-\beta, x) \equiv \Psi(\alpha, \beta, 1; x),$

and $\qquad \Psi(\alpha, \beta, 0; x) = 0.$

Following the same procedure used in deriving Eq. (9.1) one can also demonstrate the more complicated relation

$$\frac{d^2}{dx^2}\left[x^{-(\alpha-\gamma\beta)}\Psi(\alpha,\beta,\gamma;x)\right] = \gamma x^{-(\alpha-\beta\gamma+1)}\Big[\gamma\Psi(\alpha+2,\beta,\gamma;x) - (2\gamma-1)\Psi(\alpha+2,\beta,\gamma-1,$$

$$(9.4) \qquad -2\beta(\gamma-1)\Psi(\alpha+1,\beta,\gamma-1;x) + (\gamma-1)(\beta+1)\Psi(\alpha+2,\beta,\gamma-2;x) +$$

$$+\beta(\beta-1)\Psi(\alpha,\beta,\gamma-1;x) + 2\beta(\gamma-1)\Psi(\alpha+1,\beta,\gamma-2;x) + \beta^2(\gamma-1)\Psi(\alpha,\beta,\gamma-2;x)\Big].$$

One could at this point look for a general expression for the n-th derivative of the function $x^{-(\alpha-\beta)}\Psi(\alpha,\beta,\gamma;x)$ with respect to x. This task is quite cumbersome; here we limit ourselves to provide such a generalization for the case $\beta = 0$ (the parameters α and β being left free).

To this end, setting in (9.3) $\beta = 0$ one has

$$(9.5) \qquad \frac{d}{dx}\left[x^{-\alpha}K(\alpha,\gamma;x)\right] = -\gamma x^{-(\alpha+1)}\left[K(\alpha+1,\gamma;x) - K(\alpha+1,\gamma-1,x)\right],$$

where the symbol $K(\alpha,\gamma;x)$ stands for the function $\Psi(\alpha,0,\gamma;x)$.

Now with the help of Eq. (9.5) we obtain

$$(9.6) \qquad \frac{d^2}{dx^2}\left[x^{-\alpha}K(\alpha,\gamma;x)\right] = x^{-(\alpha+2)}\Big[\gamma^2K(\alpha+2,\gamma;x) + \gamma(1-2\gamma)K(\alpha+2,\gamma-1;x) - \gamma(\gamma-1)K(\alpha+2,\gamma-2;x)\Big].$$

In the same manner we can write generally

$$\frac{d^n}{dx^n}\left[x^{-\alpha}K(\alpha,\gamma;x)\right] = x^{-(\alpha+n)}\left[a_n^{(n)}K(\alpha+n,\gamma;x) + a_{n-1}^{(n)}K(\alpha+n,\gamma-1,x) + \cdots + a_0^{(n)}K(\alpha+n,\gamma-n;x)\right], \tag{9.7}$$

where the coefficients $a_{n-i}^{(n)}$ $(i=0,1,2,\dots,n)$ have to be determined.

In doing so, let us differentiate the expression (9.7) with respect to x. Using the result (9.5) one gets

$$\frac{d^{n+1}}{dx^{n+1}}\left[x^{-\alpha}K(\alpha,\gamma;x)\right] = x^{-(\alpha+n+1)}\Big\{-\gamma a_n^{(n)}K(\alpha+n+1,\gamma;x) +$$

$$+\left[\gamma a_n^{(n)} - (\gamma-1)a_{n-1}^{(n)}\right]K(\alpha+n+1,\gamma-1;x) + \tag{9.8}$$

$$+\left[(\gamma-1)a_{n-1}^{(n)} - (\gamma-2)a_{n-2}^{(n)}\right]K(\alpha+n+1,\gamma-2;x) + \cdots$$

$$\cdots + (\gamma-n)a_0^{(n)}.$$

On the other hand, comparing (9.8) with the expression which one obtains from (9.7) replacing n by n+1, we are led to the following relations

$$a_{n+1}^{(n+1)} = -\gamma a_n^{(n)} \tag{9.9}$$

$$a_0^{(n+1)} = (\gamma - n)\, a_0^{(n)}, \tag{9.10}$$

and

$$a_{n-i}^{(n+1)} = (\gamma - i)\, a_{n-i}^{(n)} - (\gamma - i - 1)\, a_{n-i-1}^{(n)}, \tag{9.11}$$

where $i = 0,1,2,\dots, n-1$.

The coefficients $a_n^{(n)}$ and $a_0^{(n)}$ are easily found. Indeed, iterating (9.9)

$$a^{(n+1)}_{n+1} = (-1)^{n+1}\gamma^{n+1}, \tag{9.12}$$

where the relation $a^{(1)}_{1} = -\gamma$ has been used. Eqs.(9.12) and (9.9) give

$$a^{(n)}_{n} = (-1)^{n}\gamma^{n}. \tag{9.13}$$

Now from (9.10) we deduce that

$$\begin{aligned} a^{(n+1)}_{0} &= (\gamma-n)a^{(n)}_{0} = (\gamma-n)(\gamma-n+1)\, a^{(n-1)}_{0} = \ldots = \\ &= (\gamma-n)(\gamma-n+1)\ldots(\gamma-1)\gamma, \end{aligned} \tag{9.14}$$

where we have substituted $a^{(1)}_{0} = \gamma$. Eqs. (9.14) and (9.10) yield

$$a^{(n)}_{0} = (\gamma-n+1)(\gamma-n+2)\ldots(\gamma-1)\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-n+1)}. \tag{9.15}$$

Let us now calculate the coefficients $a^{(n)}_{n-1}$.

To this end, consider the relation (9.11) for i=0, i.e.

$$a^{(n+1)}_{n} = \gamma a^{(n)}_{n} - (\gamma-1)\, a^{(n)}_{n-1}. \tag{9.16}$$

By iteration and with the help of (9.13), from (9.16) one has

$$\begin{aligned} a^{(n+1)}_{n} &= \gamma a^{(n)}_{n} - (\gamma-1)\left[\gamma a^{(n-1)}_{n-1} - (\gamma-1)\, a^{(n-1)}_{n-2}\right] = \ldots \\ \ldots &= (-1)^{n}\left[\gamma^{n+1}+\gamma^{n}(\gamma-1)+\ldots+\gamma^{2}(\gamma-1)^{n-1}+\gamma(\gamma-1)^{n}\right]. \end{aligned} \tag{9.17}$$

Since the expression between the sauare brakets is a geometrical progression of the ratio $(\gamma-1)/\gamma$, Eq.(9.17) yields

$$(9.18)\qquad a_n^{(n+1)} = (-1)^n\gamma\left[\gamma^{n+1} - (\gamma-1)^{n+1}\right].$$

Finally from (9.18) and (9.16) we obtain

$$(9.19)\qquad a_{n-1}^{(n)} = (-1)^n\gamma\left[(\gamma-1)^n - \gamma^n\right].$$

Our purpose now is to calculate the coefficient $a_{n-i}^{(n)}$ for any i=0,1,2,..., n-1. To this end, we start from (9.11) and iterate $a_{n-i-1}^{(n)}$.

We have

$$a_{n-i}^{(n+1)} = (\gamma-i)\, a_{n-i}^{(n)} - (\gamma-i-1)\, a_{n-i-1}^{(n)} =$$

$$(9.20)\qquad = \sum_{k=0}^{n-i-1} a_{n-i-k}^{(n-k)}\,(\gamma-i)(\gamma-i-1)^k(-1)^k + (-1)^{n-i}(\gamma-i-1)^{n-i}\, a_0^{(i+1)}.$$

Finally, Eq. (9.20) gives

$$(9.21)\qquad a_{n-i}^{(n)} = (\gamma-i+1)\sum_{k_1=0}^{n-1-k_1} a_{n-i-k_1}^{(n-1-k_1)}\,(\gamma-i)^{k_1}(-1)^{k_1} + (-1)^{n-i}(\gamma-i)^{n-i}\, a_0^{(i)},$$

where

$$(9.22)\qquad a_0^{(i)} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)}, \quad i = 0,1,2,\dots,n-1.$$

To derive explicitly the coefficients $a^{(n)}_{n-i}$ in terms of n, i and γ, we use repeated iterations of (9.21). To facilitate our task, it is advisable to take into account that: the index i appearing in (9.21) can be interpreted as the difference between the upper and lower indices of the coefficients $a^{(n)}_{n-i}$, and $(n-i)$ can be regarded as the lower index of $a^{(n)}_{n-i}$. Hence, in virtue of these considerations from (9.21) we deduce that:

$$(9.23)\qquad {}^{(n-1-k_1)}_{n-i-k_1} = (\gamma-i+2)\sum_{k_2=0}^{n-i-k_1-1} a^{(n-2-k_1-k_2)}_{n-i-k_1-k_2}(\gamma-i+1)^{k_2}(-1)^{k_2} + (-1)^{n-i-k_1}(\gamma-i+1)^{n-i-k_1}a_0^{(i-1)},$$

for $i \geqslant 2$.

Inserting (9.23) into (9.21), we obtain:

$$(9.24)\qquad a^{(n)}_{n-i} = (\gamma-i+1)(\gamma-i+2)\sum_{k_1=0}^{n-i-1}(\gamma-i)^{k_1}(-1)^{k_1}\sum_{k_2=0}^{n-i-k_1-1}(\gamma-i+1)^{k_2}\cdot(-1)^{k_2}a^{(n-2-k_1-k_2)}_{n-i-k_1-k_2} +$$

$$+(\gamma-i+1)\sum_{k_1=0}^{n-i-1}(\gamma-i)^{k_1}(-1)^{n-i}(\gamma-i+1)^{n-i-k_1}a_0^{(i-1)} +$$

$$+ (-1)^{n-i}(\gamma-i)^{n-i}a_0^{(i)}.$$

After (i-1) iterations ($i \geq 2$), Eq. (9.21) yields

$$a_{n-i}^{(n)} = (\gamma-i+1)(\gamma-i+2)\cdots(\gamma-1)\gamma \cdot \sum_{k_1=0}^{n-i-1} (\gamma-i)^{k_1}(-1)^{k_1} \cdot$$

$$\cdot \sum_{k_2=0}^{n-i-k_1-1} (\gamma-i+1)^{k_2}(-1)^{k_2} \cdots \sum_{k_i=0}^{n-i-\sum_{m=1}^{i-1} k_m} (\gamma-1)^{k_i}(-1)^{k_i} a_{(n-i-\sum_{m=1}^{i} k_m)}^{(n-i-\sum_{m=1}^{i} k_m)}$$

(9.25)
$$+ (\gamma-i+1)\cdots(\gamma-1)a_0^{(1)}(-1)^{n-i}\sum_{k_1=0}^{n-i-1}(\gamma-i)^{k_1}(-1)^{k_1} \cdot \cdots$$

$$\cdots \cdot \sum_{k_{i-1}=0}^{n-i-\sum_{m=1}^{i-2} k_m} (\gamma-2)^{k_{i-1}}(\gamma-1)^{n-i-\sum_{m=1}^{i-1} k_m} + \cdots$$

$$\cdots + (-1)^{n-i}(\gamma-i+1)\sum_{k_1=0}^{n-i-1}(\gamma-i)^{k_1}(\gamma-i+1)^{n-i-k_1} a_0^{(i-1)} +$$

$$+ (-1)^{n-1}(\gamma-i)^{n-i} a_0^{(i)} .$$

Taking account of

(9.26)
$$a_{n-i-\sum_{m=1}^{i} k_m}^{(n-i-\sum_{m=1}^{i} k_m)} = (-1)^{n-i-\sum_{m=i}^{i} k_m} \cdot \gamma^{n-i-\sum_{m=1}^{i} k_m} ,$$

(see (9.13), and of

$$(9.27) \qquad (\gamma-i+1)\dots(\gamma-j)\, a_0^{(j)} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)} \,,$$

where

$$(9.28) \qquad a_0^{(j)} = (\gamma-j+1)(\gamma-j+2)\ \dots\ (\gamma-1)\gamma \,,$$

(j = 1,2,..., i-1), Eq.(9.25) can also be written as

$$a_{n-i}^{(n)} = (-1)^{n-i}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)}\Bigg\{\gamma^{n-i}\prod_{j=1}^{i}\sum_{k_j=0}^{n-1-\sum_{m=1}^{j-1}k_m}(\gamma-i+j-1)^{k_j}\gamma^{-}$$

$$+\sum_{j=1}^{i-1}(\gamma-j)^{n-i}\prod_{\ell=1}^{i-j}\sum_{k_\ell=0}^{n-1-\sum_{m=1}^{\ell-1}k_m}(\gamma-i+\ell-1)^{k_\ell}(\gamma-j)^{-k_\ell} +$$

$$(9.29) \qquad + (-1)^{n-i}(\gamma-i)^{n-i}\Bigg\} =$$

$$= (-1)^{n-i}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)}\Bigg\{\sum_{j=0}^{i-1}(\gamma-j)^{n-i}\prod_{\ell=1}^{i-j}\sum_{k_\ell=0}^{n-1-\sum_{m=1}^{\ell-1}k_m}(\gamma-i+\ell-1)^{k_\ell}(\gamma-j)^{-}$$

$$+ (\gamma-i)^{n-i}\Bigg\} .$$

Finally, using the notation

$$(9.30)\quad \alpha_{n\ell,ij} = (n-1)(1-\delta_{ij}) - \theta(\ell-2)\sum_{m=1}^{\ell-1} k_m ,$$

where δ_{ij} is Kronecker's symbol and

$$(9.31)\quad \theta(\ell-2) = \begin{cases} 1 & \text{for } \ell \geq 2, \\ 0 & \text{for } \ell < 2, \end{cases}$$

the coefficient $a_{n-i}^{(n)}$ as given by (9.29) can also be expressed in the more compact form

$$(9.32)\quad a_{n-i}^{(n)} = (-1)^{n-i}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)}\sum_{j=0}^{i}(\gamma-j)^{n-i}\prod_{\ell=1-\delta_{ij}}^{i-j}\sum_{k_\ell=0}^{\alpha_{n\ell,ij}}(\gamma-i+\ell-1)^{k_\ell}(\gamma-j),$$

where $i = 2,3,\ldots, n-1$.

In virtue of (9.13), (9.15), (9.19) and (9.32), we have determined explicitly the expression (9.7) for the n-th derivative of the function $x^{-\alpha}K(\alpha,\gamma;x)$.

Remark 9.1

For $\gamma = 1$, Eq.(9.7) reduces to the functional relation for the incomplete Γ -function :

$$(9.33)\quad \frac{d^n}{dx^n}\left[x^{-\alpha}\Gamma(\alpha,x)\right] = (-1)^n x^{-(\alpha+n)}\,\Gamma(\alpha+n,x).$$

In fact, when $\gamma = 1$ from (9.13) and (9.15) we have resepctively $a_{n-i}^{(n)} = (-1)^n$ and $a_0^{(n)} = 0$. Furthermore, Eq.

(9.21) yields $a^{(n)}_{n-i} = 0$ for $i = 1, 2, \ldots, n-1$. The result (9.33) thus follows immediately from (9.7).

We point out that for $\gamma = -1$, Eqs.(9.13), (9.15) and (9.19) become respectively

$$a^{(n)}_{n} = 1, \tag{9.34}$$

$$a^{(n)}_{0} = (-1)^n n!\ , \tag{9.35}$$

and

$$a^{(n)}_{n-1} = 1 - 2^n. \tag{9.36}$$

On the other hand, from (9.32) we have that

$$a^{(n)}_{n-i} = i! \sum_{j=0}^{i} (-1)^i (1+j)^{n-i} \prod_{\ell=1-\delta_{ij}}^{\alpha_{n\ell,ij}} (2+i-\ell)^{k_\ell} (1+j)^{-k_\ell}, \tag{9.37}$$

for $i = 2, 3, \ldots, n-1$.

Since (see (6.3))

$$K(\alpha_1 - 1; x) \equiv -D(\alpha - 1, x), \tag{9.38}$$

with the help of (9.34), (9.35), (9.36) and (9.37), Eq.(9.7) gives a relation for the n-th derivative of the function $x^{-\alpha} D(\alpha - 1, x)$, where $D(\alpha - 1, x)$ is the Debye function defined by (6.3). As far as we know, this formula is new.

We close this Section by noticing that, analogous to the manner in which we derived (9.7), one can obtain a formula for the n-derivative of the function $e^{-x} K(\alpha,\gamma;x)$.

10. ANOTHER RECURRENCE FORMULA.

The use of (5.1) and the relation (9.1) allows us to write down another recurrence formula besides (4.1).

In fact, by integrating term by term (9.1) with $-\rho=\mu$ and applying (5.1), we find the following relation:

$$(\mu+\alpha-\gamma\beta)\psi(\alpha+\mu,\beta,\gamma;x)-\gamma\psi(\alpha+\mu+1,\beta,\gamma;x)+\gamma\psi(\alpha+\mu+1,\beta,\gamma-1;x)+$$

$$+\gamma\beta\psi(\alpha+\mu,\beta,\gamma-1;x)+x^{\mu}\Big[(-\alpha+\gamma\beta)\,\psi\,(\alpha,\beta,\gamma;x)+$$

$$+\gamma\psi(\alpha+1,\beta,\gamma;x)-\gamma\psi(\alpha+1,\beta,\gamma-1;x)-\gamma\beta\psi(\alpha,\beta,\gamma-1;x)\Big]=0. \tag{10.1}$$

Remark 10.1

When $\gamma=1$, Eq. (10.1) gives the well-known recursive relation for the incomplete Γ-function:

$$(\mu+a)\,\Gamma\,(a+\mu,x)-\Gamma(a+\mu+1,x)+x^{\mu}\Big[\Gamma(a+1,x)-$$

$$-a\Gamma(a,x)\Big]=0, \tag{10.2}$$

where $a=\alpha-\beta$.