$$(7.34)\quad \sum_{m=o}^{\infty} \frac{\Gamma(m+1-\gamma)}{(m+1)!(m+1)^N} \left| \frac{\Gamma(N-\alpha+(m+1)\beta)}{\Gamma(1-\alpha+(m+1)\beta)} \right| x^{-m\beta} e^{-mx},$$

which appears on the right of (7.18), converges uniformly for any x greater than a certain $\bar{x}$ verifying the inequality $e^{-x} < x$. Furthermore, one has

$$(7.35)\quad \left| x^{N-1} \{\psi(\alpha,\beta,\gamma;x)x^{\beta-\alpha+1}e^x - \sum_{s=o}^{N-2} A_s(x)\right| \frac{1}{x^s} \rightarrow \text{const},$$

as $x \rightarrow +\infty$, $A_s(x)$ being defined by (7.17) and

$$(7.36)\quad \text{const} \le |\gamma|(N-1)(1+\varepsilon)^N \left| \frac{\Gamma(N-\alpha+\beta)}{\Gamma(1-\alpha+\beta)} \right| ,$$

where $N \ge 2$ and ε is any arbitrary positive number".

Proof. The first part of the lemma follows directly from Lemma (7.28).

As a consequence, the results (7.35) and (7.36) arise immediately from (7.18).

In virtue of the series of lemmas from (7.3) to (7.7), the basic Theorem 7.2 is thus completely proved.

8. SOME SPECIAL CASES.

a) "Asymptotic expansion of the incomplete Γ-function".

The expression (7.6) can be written as

$$A_s(x)=(-1)^{s+1}\{(-\gamma)\quad \frac{\Gamma(s-\alpha+\beta+1)}{\Gamma(-\alpha+\beta+1)} \quad +$$

(8.1)

$$+ (1-\gamma)(-\gamma) \frac{\Gamma(s-\alpha+\beta+1)\ e^{x}}{2!\,2^{s+1}\,\Gamma(-\alpha+2\beta+1)\ x^{\beta}} + \ldots\} ,$$

where the relation

$$(8.2) \qquad \frac{\Gamma(m+1-\gamma)}{\Gamma(-\gamma)} = (m-\gamma)(m-1-\gamma)\ \ldots\ (2-\gamma)(1-\gamma)(-\gamma)$$

has been employed.

Putting $\gamma=1$ into (8.1) and using the symbol

$$(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)} ,$$

we obtain

$$(8.3) \qquad A_s(x) = (-1)^s\ (1-\alpha+\beta)_s .$$

Then Eq. (7.5) becomes

$$(8.4) \quad \psi(\alpha,\beta,1;x) \equiv \Gamma(\alpha-\beta,x) \sim x^{\alpha-\beta-1} e^{-x} \sum_{s=0}^{\infty} (-1)^s \frac{(1-\alpha+\beta)_s}{x^s} ,$$

which gives the well-known asymptotic expansion for the incomplete Γ-function for fixed $(\alpha-\beta)$ and large x [22].

b) "Asymptotic expansion of the incomplete Debye function".

Let us remember that for $\beta=0$ and $\gamma=-1$, the function $-\psi(\alpha,\beta,\gamma;x)$ reduces to the incomplete Debye function $D(\alpha-1,x)$ as given by (6.8).

In this case, from (7.6) one gets

$$(8.5)\qquad A_s(x) = (-1)^{s+1} \frac{\Gamma(s-\alpha+1)}{\Gamma(1-\alpha)} \sum_{m=o}^{\infty} \frac{e^{-mx}}{(m+1)^{s+1}} .$$

Recalling now the function [23]

$$(8.6)\qquad \Phi(z,s,v) = \sum_{m=o}^{\infty} \frac{z^m}{(m+v)^s} ,$$

defined for $|z|<1$, $v\neq 0,-1,-2, \ldots$, the series on the right of (8.5) can be expressed by $\Phi(e^{-x}, s+1, 1)$.

Therefore (8.5) becomes

$$(8.7)\qquad A_s(x) = (-1)^{s+1}(1-\alpha)_s \Phi(e^{-x}, s+1, 1).$$

Taking account of (8.7), (7.5) specializes to

$$(8.8)\qquad D(\alpha-1,x) \sim x^{\alpha-1}\, e^{-x} \sum_{s=0}^{\infty} (-1)^s (1-\alpha)_s \frac{\Phi(e^{-x}, s+1,1)}{x^s}$$

for fixed α and large values of $x>0$.

Finally, let us point out that since [23] :

$$(8.9)\qquad \Phi(z,s,v) = \frac{1}{\Gamma(s)} \int_o^{\infty} dt \; \frac{t^{s-1}\, e^{-vt}}{1-z\, e^{-t}} ,$$

for $\mathrm{Re}\, v > 0$, the relation (8.8) can also be written as

$$(8.10)\quad D(\alpha-1,x) \sim x^{\alpha-1}e^{-x}\sum_{s=0}^{\infty}(-1)^s\,\frac{(1-\alpha)_s}{s!\,x^s}\int_0^{\infty}dt\;\frac{t^s\,e^{-t}}{1-e^{-(x+t)}}\;.$$

9. SOME FUNCTIONAL RELATIONS.

It is easy to show that

$$(9.1)\quad \frac{d}{dx}\left[x^{-\rho}\Psi(\alpha,\beta,\gamma;x)\right] = -x^{-(\rho+1)}\Big\{(\rho-\alpha+\gamma\beta)\Psi(\alpha,\beta,\gamma;x) + \gamma\left[\Psi(\alpha+1,\beta,\gamma;x)-\Psi(\alpha+1,\beta,\gamma-1;x)-\beta\Psi(\alpha,\beta,\gamma-1;x)\right]\Big\}.$$

In fact, we have

$$(9.2)\quad \frac{d}{dx}\left[x^{-\rho}\Psi(\alpha,\beta,\gamma,x)\right] = -x^{-(\rho+1)}\Big\{\rho\Psi(\alpha,\beta,\gamma;x)+x^{\alpha}\Big[1-\Big(1-\frac{e^{-x}}{x^{\beta}}\Big)^{\gamma}\Big]\Big\}.$$

The result (9.1) is thus achieved with the help of the recurrence relation (4.5).

Putting $\rho = \alpha-\gamma\beta$, Eq. (9.2) takes the form

$$(9.3)\quad \frac{d}{dx}\left[x^{-(\alpha-\gamma\beta)}\Psi(\alpha,\beta,\gamma;x)\right] = -\gamma x^{-(\alpha-\gamma\beta+1)}\cdot\left[\Psi(\alpha+1,\beta,\gamma;x)-\Psi(\alpha+1,\beta,\gamma-1;x)-\beta\Psi(\alpha,\beta,\gamma-1;x)\right],$$

which for $\gamma = 1$ becomes the well-known functional relation for the incomplete Γ-function:

$$\frac{d}{dx}\left[x^{-(\alpha-\beta)}\Gamma(\alpha-\beta,x)\right] = -x^{-(\alpha-\beta+1)}\Gamma(\alpha-\beta+1,x),$$