## 5. SOME INTEGRALS INVOLVING $\psi(\alpha,\beta,\gamma, ;x)$.

In this section, we derive some integrals involving the function (1.6).

The following theorems hold:

THEOREM 5.1. *Let $\alpha_2$, $\beta$ and $\gamma$ be (real) arbitrary parameters. Then the following relation holds:*

$$(5.1)\qquad \int_x^{\infty} dt\ t^{\alpha_1-1}\ \psi(\alpha_2,\beta,\gamma;t) =$$

$$= -\frac{1}{\alpha_1}\left[ x^{\alpha_1}\psi(\alpha_2,\beta,\gamma;x) - \psi(\alpha_1+\alpha_2,\beta,\gamma;x)\right] ,$$

*for* $\alpha_1 \neq 0$ *and* $x > 0$ *such that* $e^{-x} < x^{\beta}$.

*Proof.* The proof of (5.1) is easily obtained by integration by parts, and using the fact that

$$\lim_{t\to+\infty} t^{\alpha_1}\psi(\alpha_2,\beta,\gamma;t) = 0,$$

for $\alpha_1 > 0$.

*Remark* 5.2. From (5.1) one obtains for $\gamma = 1$:

$$(5.2)\quad \int_x^{\infty} dt\ t^{\alpha_1-1}\ \Gamma(\alpha_2-\beta,t) = -\frac{1}{\alpha_1}\left[x^{\alpha_1}\Gamma(\alpha_2-\beta,x) - \Gamma(\alpha_1+\alpha_2-\beta,x)\right],$$

which produces the well-known relation [16] for the incomplete $\Gamma$-function:

$$(5.3)\quad \int_0^{\infty} dt\ t^{\alpha_1-1}\Gamma(\alpha_2-\beta,t) = \frac{1}{\alpha_1}\ \Gamma(\alpha_1+\alpha_2-\beta) ,$$

for $\alpha_1 > 0$ and $\alpha_1+\alpha_2>\beta$ .

Using THEOREM 5.1, on the basis of PROPOSITIONS 1.1 and 1.2 we are led to the following

COROLLARY 5.3. *Let* $\alpha$ *and* $\beta$ *be such that* $-e < \beta < 0$ *and* $\alpha > -|\beta|$. *Then*

$$\psi(\alpha,\beta,\gamma;0) = \int_0^\infty dt\ \psi(\alpha-1,\beta,\gamma;t) \quad , \tag{5.4}$$

*for any value of* $\gamma$.

COROLLARY 5.4. *The relation (5.4) holds also when* $\beta = 0$, *provided that* $\gamma > 0,\ \alpha > 0$ *or* $\gamma < 0,\ \alpha > |\gamma|$.

THEOREM 5.5. *Assuming all the hypotheses of Theorem 5.1, then the following transform holds:*

$$\begin{aligned}\int_x^\infty dt\ e^{-nt}\ \psi(\alpha,\beta,\gamma;t) &= \frac{1}{n}\ e^{-nx}\ \psi(\alpha,\beta,\gamma;x) \\ &- \frac{1}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(\alpha+n\beta,\beta,\gamma+k;x) + \\ &+ \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{k+j+1} \binom{n}{k}\binom{k}{j} j^{(j-n)\beta-\alpha}\ \Gamma\left(-(j-n)\beta+\alpha, jx\right),\end{aligned} \tag{5.5}$$

*where n is a positive integer.*

*Proof.* Consider the function $e^{-nt}\ \psi(\alpha,\beta,\gamma;t)$ , $n$ being a positive integer, and integrate by parts from $x > 0$ to infinity. One has

$$\begin{aligned}\int_x^\infty dt\ e^{-nt}\ \psi(\alpha,\beta,\gamma;t) &= \frac{1}{n}\ e^{-nx}\ \psi(\alpha,\beta,\gamma;x) \\ &- \frac{1}{n} \int_x^\infty dt\ t^{\alpha-1}\ e^{-nt} \left[1-\left(1-\frac{e^{-t}}{t^\beta}\right)^\gamma\right] .\end{aligned} \tag{5.6}$$

Now, by using the relation

$$(5.7)\qquad \frac{e^{-nt}}{t^{n\beta}} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(1-\frac{e^{-t}}{t^{\beta}}\right)^k ,$$

Eq. (5.6) becomes

$$(5.8)\qquad \int_x^{\infty} dt\, e^{-nt}\, \psi(\alpha,\beta,\gamma;t) = \frac{1}{n}\, e^{-nx}\, \psi(\alpha,\beta,\gamma;x)$$

$$- \frac{1}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[\psi(\alpha+n\beta,\beta,\gamma+k;x)-\psi(\alpha+n\beta,\beta,k;x)\right]$$

Since k is a nonnegative integer, we may express $\psi(\alpha+n\beta,\beta,k;x)$ as a finite sum of incomplete $\Gamma$-functions, namely

$$(5.9)\qquad \psi(\alpha+n\beta,\beta,k;x) = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} j^{(j-n)\beta-\alpha}\, \Gamma((n-j)\beta+\alpha, jx) .$$

Inserting (5.9) into (5.8), one achieves the result (5.5).

THEOREM 5.6

"Suppose that the conditions $-e < \beta < 0$ and $\alpha > -|\beta|$ are valid. Then one has

$$(5.10)\qquad \int_0^{\infty} dt\{-e^{-t}\psi(\alpha,\beta,\gamma;t) - \psi(\alpha-1,\beta,\gamma;t) + \psi(\alpha+\beta-1,\beta,\gamma;t)$$

$$-\psi(\alpha+\beta-1,\beta,\gamma+1;t)\} = \Gamma(\alpha),$$

for any value of the parameter $\gamma$".

Proof. By putting n=1 in (5.5), we obtain

$$\int_x^\infty dt\ e^{-t}\ \psi(\alpha,\beta,\gamma;t) = e^{-x}\ \psi(\alpha,\beta,\gamma;x)-\psi(\alpha+\beta,\beta,\gamma;x) +$$

(5.11)
$$+\psi(\alpha+\beta,\beta,\gamma+1;x)-\Gamma(\alpha,x).$$

In virtue of Proposition 1.1 the relation (5.11) is valid also when x=0. Using then (5.4) the assertion is proved.

## 6. SOME FUNCTIONS AND RELATIONS CONNECTED WITH THE $\psi$-FUNCTION.

a) "Case" $\gamma = 0$.

Obviously one has $\psi(\alpha,\beta,0;x) = 0$.

b) "Case" $\gamma = 1$.

For $\gamma = 1$ the function (1.6) specializes to the incomplete $\Gamma$-function. In fact, we have

(6.1)
$$\psi(\alpha,\beta,1;x) = \int_x^\infty dt\ t^{\alpha-\beta-1}e^{-t} = \Gamma(\alpha-\beta,x).$$

c) "Case" $\gamma = n$ (positive integer).

As we have already noted (see Sec.3), the function (1.6) can be expressed as a finite sum of incomplete $\Gamma$-functions.

d) "Case" $\gamma = -1$, $\alpha = n+1$, $\beta = 0$.

For $\gamma = -1$ the function (1.6) becomes

(6.2)
$$\psi(\alpha,\beta,-1;x) = -\int_x^\infty dt\ \frac{t^{\alpha-1}}{e^t\ t^\beta-1} .$$