1 - JETS.

Let M and N be two C^∞ manifolds.

1 DEFINITION.

The JET SPACE, of order i, OF MAPS $M \to N$, is the set

$$J^i(M,N) \equiv \bigsqcup_{p \in M} \mathfrak{F}_{p/\rho^i_p} ,$$

where

a) $\mathfrak{F}_p$ is the set of C^∞ maps $M \to N$ defined in a neighbourhood of p;

b) ρ^i_p is the equivalence relation in $\mathfrak{F}_p$ given by

$$f \, \rho^i_p \, g \iff T^i_p f = T^i_p g \quad \underline{\cdot}$$

2 DEFINITION.

Let $$f : M \to N$$
be a C^∞ map, perhaps defined locally.

The JET, of order i, of f is the map

$$j^i f : M \to J^i(M,N)$$

given by $$p \to [f]^i_p \quad \underline{\cdot}$$

3 PROPOSITION.

There is a unique C^∞ structure on $J^i(M,N)$, such that

$$\forall f : M \to N \qquad \text{the map} \quad j^i f \quad \text{is} \quad C^\infty .$$

PROOF.

It can be easily seen by means of an atlas of M and N $\underline{\cdot}$

4 PROPOSITIONS.

Let $0 \leq i \leq j$. The natural projection

$$\sigma^{ij} : J^j(M,N) \to J^i(M,N)$$

given by $$[f]^j_p \to [f]^i_p$$

(which is well defined) induces a bundle structure

$$(J^j(M,N),\ \sigma^{ij}\ ,\ J^i(M,N))\quad \underline{.}$$

5 PROPOSITION.

The map

$$J^\circ(M,N) \to M \times N$$

given by

$$[f]^\circ_p \to (p,f(p))$$

(which is well defined) is a diffeomorphism $\underline{.}$

Henceforth we will make the identification

$$J^\circ(M,N) \cong M \times N \ .$$

6 PROPOSITION.

The map

$$J'(M,N) \to T^*M \otimes T\,N$$

$$[f]^i_p \to T_p f \in T^*_p M \otimes T_{f(p)} N$$

(which is well defined) is a diffeomorphism.

PROOF.

We have

$$J'(M,N) = \bigsqcup_{(p,q)\in M\times N} \{\Phi_{(p,q)}\}$$

where

$$\Phi_{(p,q)} : T_p M \to T_q N$$

is any linear map $\underline{.}$

Henceforth we will make the identification

$$J'(M,N) \cong T^*M \otimes T\,N \ .$$

7 THEOREM.

$(J^2(M,N),\sigma'',J'(M,N))$ is an affine bundle, whose vector bundle is

$$(J'(M,N)\times_{M\times N}(T^*M \vee_M T^*M \underset{M\times N}{\otimes} T\,N),\ \bar{\sigma}^{12},J'(M,N)) \ .$$

(where $\vee$ denotes the symmetrized tensor product).

PROOF.

We have

$$J^2(M,N) = \bigsqcup_{\Phi_{(p,q)}\in J'(M\ N)} \{\bar{\Phi}_{\Phi_{(p,q)}}\}$$

where $$\bar{\Phi}_{\Phi_{(p,q)}} : T^2_p M \to T^2_q N$$

is any map such that

a) $\bar{\Phi}_{\Phi_{(p,q)}}$ is a linear bundle homomorphism, hence the following diagram is commutative

$$\begin{array}{ccc} T^2_p M & \xrightarrow{\bar{\Phi}_{\Phi_{(p,q)}}} & T^2_q N \\ {\scriptstyle \Pi_{TM}} \downarrow & & \downarrow {\scriptstyle \Pi_{TN}} \\ T_p M & \xrightarrow{\Phi_{(p,q)}} & T_q N \end{array}$$

b) $\bar{\Phi}_{\Phi(p,q)} \circ s$ is linear

c) $T \Pi_N \circ \bar{\Phi}_{\Phi_{(p,q)}} = \Phi_{(p,q)}$

d) $\underline{\perp\!\perp} \circ \bar{\Phi}_{\Phi_{(p,q)}} \circ \nu = \Phi_{(p,q)} \circ T \Pi_M$.

In fact a) ...,d) characterize the jets of maps $M \to N$.
Moreover, if we fix $\Phi_{(p,q)} \in J'(M,N)$, then the conditions a) and b) determine a vector space structure on the set $\{\bar{\Phi}_{\Phi_{(p,q)}}\}$ and the linear functional conditions c) and d) determine an affine subspace.

The associated vector space is obtained taking $\Phi_{(p,q)} = 0$ in the conditions c) and d). Such maps can be identified with a couple constituted by a bilinear symmetric map $TM x_M TM \to TN$ and a linear map $TM \to TN$ over a same map $M \to N$.

This theorem can be generalized to higer orders.

8 PROPOSITION.

a) We get $$J'(R,N) \cong R \times TN$$
This isomorphism is the unique map $J'(R,N) \to R \times T N$ that makes commutative the following diagram, for each curve $c : R \to N$,

$$\begin{array}{ccccc} J'(R,N) & & \longrightarrow & & R \times T\,N \\ & {\scriptstyle j^1c}\nwarrow & & \nearrow{\scriptstyle (id_R,dc)} & \\ & & R & & \end{array}$$

b) We get $\qquad J'(M,R) \cong R \times T^*M$.

This isomorphism is the unique map $J'(M,R) \to R \times T^*M$ that makes commutative the following diagram, for each function $f : M \to R$,

$$\begin{array}{ccccc} J'(M,R) & & \longrightarrow & & R \times T^*M \\ & {\scriptstyle j^1f}\nwarrow & & \nearrow{\scriptstyle (f,df)} & \\ & & M & & \end{array}$$

c) There is a unique map (which is an isomorphism)

$$J^2(R,N) \to R \times s\,T^2N$$

such that the following diagram is commutative, for each curve $c : R \to N$,

$$\begin{array}{ccccc} J^2(R,N) & & \longrightarrow & & R \times s\,T^2\,N \\ & {\scriptstyle j^2c}\nwarrow & & \nearrow{\scriptstyle (id_R,d^2c)} & \\ & & R & & \end{array}$$

$\underline{\cdot}$

2 - JETS OF SECTIONS.

Let $\eta \equiv (E,p,M)$ be a bundle.

1 DEFINITION.

The JET SPACE, of order i, OF SECTIONS $M \to E$, is the set

where $$J^iE \equiv \bigsqcup_{p\in M} \mathcal{S}_{p/\rho^i_p} ,$$

a) $\mathcal{S}_p$ is the se of C^∞ sections $M \to E$ defined in a neighbourhood of p;

b) ρ^i_p is the restriction of the equivalence relation defined in (1,1) $\underline{\cdot}$

Let us remark that we get