hence $\partial(Cv) = (TT_2^{*}C) \circ (\pi^1, Tv) \circ (\pi^1, T_1 C)_{(0,1)} =$

$$=((\partial T_2^{*}C) \circ \Pi_{TM} + T_2 T_2^{*} C_o) \circ (TV) \circ \partial C =$$

$$=(- s \circ \alpha(T_2 \partial C) \circ \Pi_{TM} + id_{TT^{*}M}) \circ (Tv) \circ u =$$

$$= - s \circ \alpha(Tu) \circ v + Tv \circ u$$

Let us remark that both tensors in (*) are on the same affine fiber on $h\ TT_{(r,s)}M$.

5 Connection on a bundle.

Let $\eta \equiv (E,p,M)$ be a bundle.

1 DEFINITION.

A PSEUDO-CONNECTION on $\eta$ is an affine bundle morphism on $h\ T\ E$

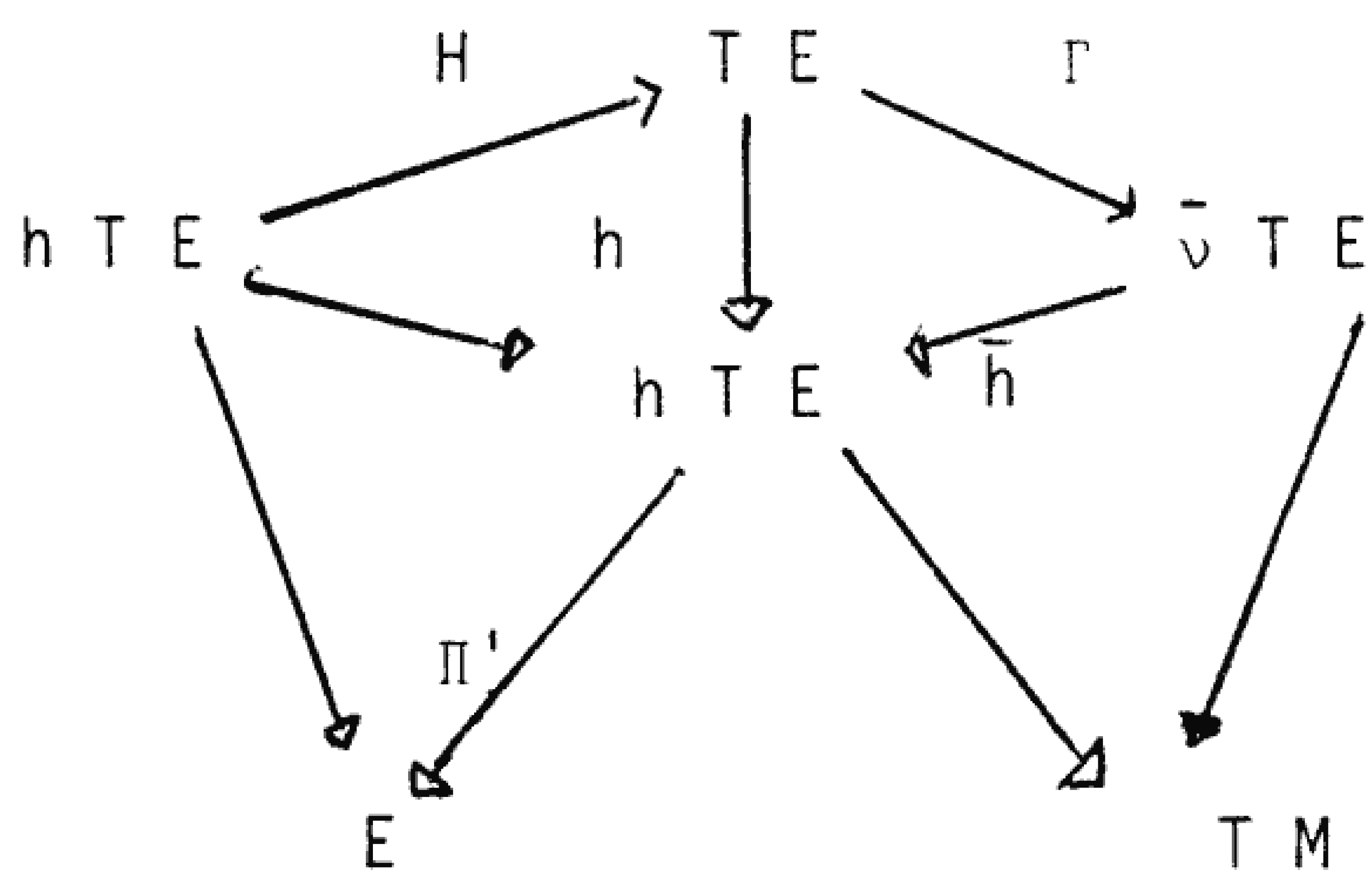
$$\Gamma : TE \to \bar{\nu}\ T\ E$$

whose fiber derivatives are 1.

A PSEUDO-HORIZONTAL SECTION is a section

$$H : h\ TE \to TE$$

Hence the following diagram is commutative




Let us remark that $\Gamma : T\ E \to \bar{\nu}\ T\ E$ is characterized by the map $\Gamma' : TE \to \nu\ T\ E$ given by $TE \xrightarrow{\Gamma} \bar{\nu}\ TE \xrightarrow{\Pi^2} \nu\ T\ E$ .

2 PROPOSITION.

The maps $\alpha$ and $\beta$ between the set of pseudo connections and the set of pseudo-horizontal sections, given by

$$\alpha : \Gamma \rightarrow H ,$$

where $H$ is the unique horizontal section such that $\Gamma \circ H = 0$, and

$$\beta : H \rightarrow \Gamma \equiv id_{TE} - H \circ h ,$$

are inverse bijection $\dot{-}$

Henceforth we will consider $\Gamma$ and $H$ as mutually related . Hence giving a pseudo-connection is the choice of a point for each affine fiber of $TE$, getting in this way an identification of the affine fibers with their vector spaces.

3 PROPOSITION.

Let $c : R \rightarrow E$ be a map. The following condition are equivalect :

a) $H \circ h \circ d c \equiv H \circ (c, d(p \circ c)) = d c$

b) $\Gamma \circ d c = 0$ .

4 DEFINITION.

A curve $c : R \rightarrow E$ is HORIZONTAL if the previous conditions are satisfied.

5. PROPOSITION.

The set $\mathcal{J}$ of all pseudo-connections is the affine space of the sections of the affine bundle $\tau_h E$, whose vector space is the space of the sections of the vector bundle $\bar{\tau}_h E$ $\dot{-}$

6. PROPOSITION.

The following conditions are equivalent

a) $\Gamma : TE \rightarrow \bar{\nu} T E$ is a linear morphism on $E$

b) $H : hTE \rightarrow TE$ is a linear morphism on $E$.

Moreover, if such conditions are verified, then we get

$$T E = h T E \oplus_E \nu TE .$$

PROOF.

a) $\Longleftrightarrow$ b trivial.

For the splitting it suffices to take into account the two exact sequences on E

$$0 \to \nu TE \hookrightarrow TE \xrightarrow{h} hTE \to 0$$

$$0 \to hTE \xrightarrow{H} TE \xrightarrow{\Gamma'} \nu TE \to 0 \quad .$$

7 DEFINITION.

A CONNECTION (HORIZONTAL SECTION) is a pseudo connection(pseudo-horizontal section) satisfyng the condition (a),(b) .

Hence giving a connections allows us to make a comparison between "close" fibers of E.

8 PROPOSITION.

Let $\eta$ be a vector bundle. Let $\Gamma$ be a connection.

The following conditions are equivalent

a) $\Gamma : TE \to \bar{\nu}\, TE$ is a vector bundle morphism on TM

b) $H : hTE \to TE$ is a vector bundle morphism on TM .

9 DEFINITION.

A connection (horizontal section) is LINEAR if the previous conditions hold .

Hence giving a linear connection allows us to make a comparison between "close" fibers of E by means of isomorphisms.

10 The set $\tilde{\mathcal{J}}_\ell$ of all linear connections is an affine subspace of $\tilde{\mathcal{J}}$,whose vector space is the space of bilinear sections of $\bar{\tau}_h E$(this vector space is naturally isomorphic to the space of sections $M \to T^*M \otimes E^* \otimes E$).

11 PROPOSITION.

Let $\Gamma'$ and $\Gamma''$ be two linear connections on $\eta'$ and $\eta''$, respectively.

The map $H \equiv t \circ (H' \otimes H'') : hT(E' \otimes_M E'') \to t(E' \otimes_M E'')$ is a linear connections on $\eta' \otimes \eta''$ $\underline{.}$

Hence the following diagram is commutative:

$$\begin{array}{ccc} TE' \otimes_{TM} TE'' & \xrightarrow{\quad t \quad} & T(E' \otimes_M E'') \\ & {\scriptstyle H'\otimes H''} \nwarrow \quad \nearrow {\scriptstyle H} & \\ & (E' \otimes_M E'') \times_M TM & \end{array}$$

12 DEFINITION.

The TENSOR PRODUCT of $\Gamma'$ and $\Gamma''$ is the connection associated with the horizontal section $H$ previously defined $\underline{.}$

13 PROPOSITION.

Let $\Gamma$ be a linear connection on $\eta$ .
There is a unique linear connection $\Gamma^*$ on $\eta^*$ such that the following diagram is commutative

$$\begin{array}{ccc} TE \times_{TM} TE^* & \xrightarrow{\quad \dot{b} \quad} & R \\ \uparrow {\scriptstyle (H,H^*)} & & \uparrow \\ (E \times_M E^*) \times_M TM & \longrightarrow & 0 \end{array}$$

where $b : E \times_M E^* \to R$ is the inner product and $\dot{b} = \Pi^2 \circ Tb$ $\underline{.}$

14 DEFINITION.

The DUAL connection of $\Gamma$ is the connection associated with the horizontal section $H^*$ previously defined $\underline{.}$

15 DEFINITION.

Let $\Gamma$ be a linear connection on $\eta \equiv \tau M$.

The TORSION of $\Gamma$ is the bilinear map

$$\Theta \equiv \underline{||}_{TM} \circ (H - s \circ H \circ ex) : TM \times_M TM \to TM.$$

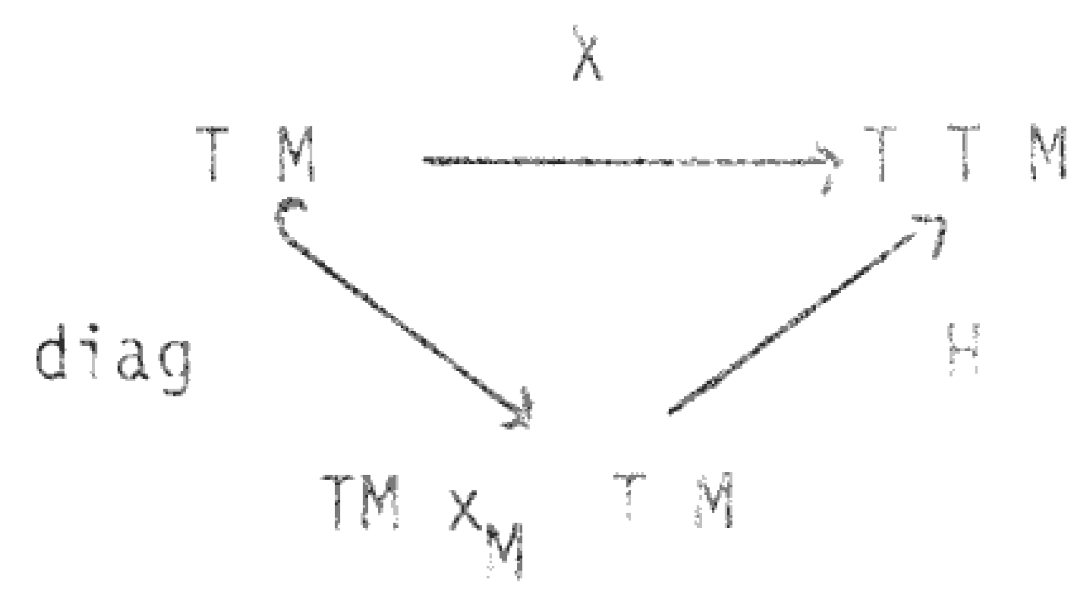
The connection $\Gamma$ is SYMMETRICAL if $\Theta = 0$ $\underline{.}$

16 DEFINITION.

A QUADRATIC SPRAY is a second order differential equation

$$X : T M \to T T M$$

which is factorizable by a symmetrical linear horizontal section as follows




∸

17 PROPOSITION.

The previous diagram determines a bijection between quadratic sprays and symmetrical linear connections ∸

The quadratic sprays are homogeneous with degree two .

18 DEFINITION.

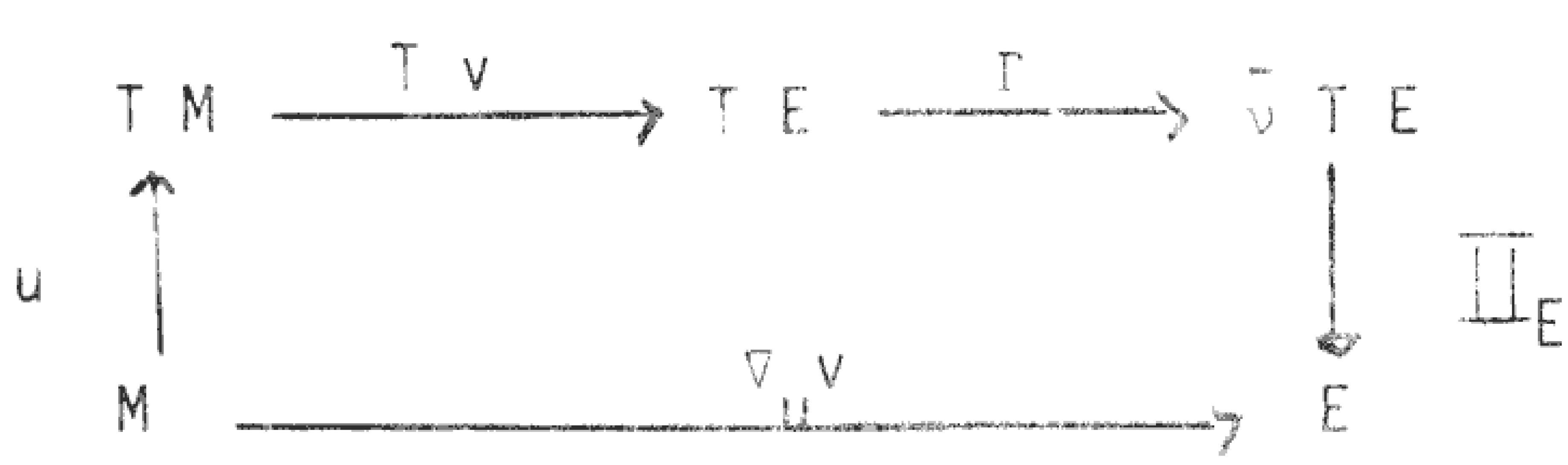
Let $\Gamma$ be a linear connection on $\eta \equiv (E,p,M)$.

Let $v : M \to E$ be a section and let $u : M \to T M$ be a vector field.

The COVARIANT DERIVATIVE of $v$ with respect $u$ is the section

$$\nabla_u v \equiv \overline{\amalg}_E \circ \Gamma \circ T v \circ u : M \to E$$ ∸

Hence the following diagram is commutative




Let us remark that we have

$$\nabla_u v = \overline{\amalg}_E \circ \Gamma \circ \partial(v \circ c),$$

where $C : R \times M \to M$ is the group of local diffeomorphisms generated by u.

19 PROPOSITION.

Let $\nabla$ be a linear connection on $\eta \equiv (E,p,M)$.

We have

$$\nabla_{fu} = f \nabla_u v$$

$$\nabla_{u+u'} v = \nabla_u v + \nabla_{u'} v$$

$$\nabla_u (v+v') = \nabla_u v + \nabla_u v'$$

$$\nabla_u (fv) = f \nabla_u v + (u,f) v$$

If $U \subset M$ is open, then

$$\nabla_{u/v} v_{/v} = (\nabla_u v)_{/U} .$$

If $\nabla^*$ is the dual connection of $\nabla$, we have

$$\nabla_u \langle \omega, v\rangle = \langle \nabla_u \omega, v\rangle + \langle \omega, \nabla_u v\rangle .$$

If $\nabla$ is the tensor product of the linear connection $\nabla'$ and $\nabla''$,

we have $\nabla_u (v' \otimes v'') = \nabla_u v' \otimes v'' + v' \otimes \nabla_u v''$ .

20 PROPOSITION.

Let $\nabla$ be a linear connection on $\eta \equiv \tau M$.

We have $\nabla_u v = \nabla_v u + L_u v + \Theta\, o(u,v)$ .

PROOF.

$$\nabla_u v - \nabla_v u = \coprod_{TM} o \nabla o (Tv \; o \; u - s \; o \; Tv \; o \; u) + \Theta \; o \; (u,v) = L_u v + \Theta \; o(u,v) .$$

PROPOSITION.

Let $g : TM \times_M TM \to R$ be a non degenerate symmetrical linear map. Let us denote by the same notation the associated maps

$g : TM \to R$, $g : M \to T_{(0,2)} M$ and $g : TM \to T^* M$ .

Each one of the following conditions characterize the same symmetrical linear connection $\nabla$ on $\tau M$.

a) The following diagram is commutative

$$\begin{array}{ccc} T\,T\,M & \xrightarrow{\dot{g}} & \\ r' \downarrow & & R \\ T\,T\,M & \xrightarrow{\dot{g}} & \end{array}$$

b) The following diagram is commutative

$$\begin{array}{ccc} T\,T\,M \times_{T\,M} TTM & \dot{g} & R \\ & (H,H) & \\ (T\,M \times_M TM) \times_M T\,M & & 0 \end{array}$$

c) We have $\quad {}_u\, g = 0 \quad , \qquad \forall\, u : M \to T\,M$ .

d) The following diagram is commutative

$$\begin{array}{ccc} T\,M \times_M T\,M & H & T\,T\,M \\ id_{TM} \times g^{-1} & & T\,g \\ T\,M \times_M T^* M & H^* & T\,T^* M \end{array}$$