Introduction

The tangent and cotangent spaces of a bundle are well known. But their affine structure is not sufficiently analised. We study deeply this structure, showing its fundamental role in many classical operations, suggesting new points of view, which we want to use in further works on Analytical Mechanics and Field Theory.

Let $\eta \equiv (E,p,M)$ be a bundle. We show that the tangent (cotangent) space $TE(T^*E)$ has an affine structure on the horizontal space $hTE \equiv E\times_M TM$, (vertical space, $\nu T^*E \equiv T^*E/hT^*E$) besides the vector structure on E(§1,2). Specializing the previous results to $E \equiv TM$ and $E \equiv T^*M$, we get a systematical table of canonical structures, in particular we see the exchange and symplectic isomophisms as pull-back maps (§3).
We give an intrinsical definition and we find an explicit intrinsical expression of the Lie derivative of a tensor, by means of the tangent functor (§4).

We define a pseudo-connection on η identifying each affine fiber of TE with its vector space, or, equivalently, choicing a "zero" on each affine fiber. Then we get immediately an affine structure on the space of all connections.We define the linear connections, requiring the previous identifications to be bilinear on the horizontal tangent space.
We get a functional construction of the tensor product of two linear connections and of the dual of a linear connection. In the case $E \equiv TM$ we can explain the classical notions by the previous results (§5).

In the following all manifolds and maps are C^∞. We leave to the reader the coordinate expression of formulas and the proof of some proposition.

1 - THE TANGENT SPACE OF A BUNDLE.

Let $\eta \equiv (E,p,M)$ be a C^∞ bundle.

1 DEFINITION.

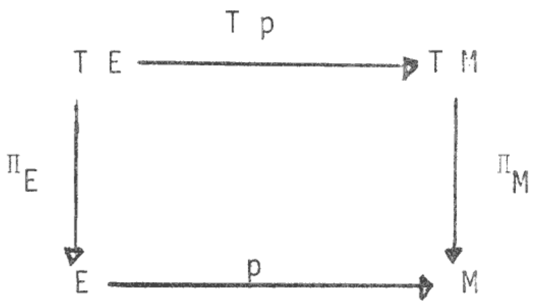
The TANGENT BUNDLE OF E is the vector bundle

$$\tau E \equiv (TE, \pi_E, E) .$$

The TANGENT BUNDLE OF η is the vector bundle

$$\tau \eta \equiv (TE, Tp, TM) .$$

The following diagram is commutative.



2 DEFINITION.

The HORIZONTAL BUNDLE OF TE is the pull-back vector bundle

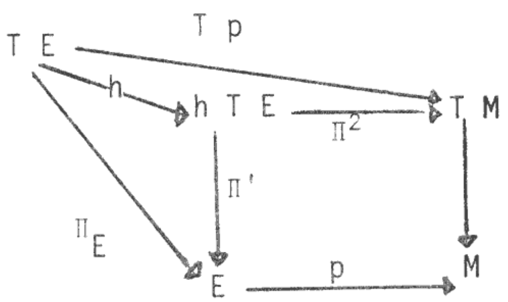
$$h \tau E \equiv h T E, \pi', E),$$

where

$$h T E \equiv E \times_M TM .$$

The map $h \equiv (\pi_E, T p) : T E \to h T E$

in the unique map which makes commutative the following diagram



3 DEFINITION.

The VERTICAL BUNDLE OF TE is the subbundle of τE, kernel of h on E

$$\nu\,\tau\,E \equiv (\nu\,T\,E,\ \Pi_E, E)\ .$$

4 The following diagram is commutative

$$\begin{array}{ccc} \nu\,T\,E & \hookrightarrow & T\,E \\ \downarrow & & \downarrow \\ E & \hookrightarrow & h\,T\,E \end{array}$$

and the following sequence is exact

$$0 \to \nu\,T\,E \to T\,E \xrightarrow{h} h\,T\,E \to 0,$$

hence we have a canonical isomorphism

$$h\,T\,E \hookrightarrow T\,E/\nu\,T\,E.$$

5 PROPOSITION.

We get $$\nu\,T\,E = \bigsqcup_{e\in E} \{d\,c_e(0)\}$$

where $$\{c_e\} \equiv \{c : \mathbb{R} \to E \mid c(o) = e,\ p \circ c = p(c(o)\}\ .$$

Such curves $c : \mathbb{R} \to E$ are called VERTICAL.

6 DEFINITION.

The TANGENT BUNDLE OF E, ON hTE, is the pull-back bundle

$$\tau_h E \equiv (TE,\ h,\ hTE)\ .$$

The VERTICAL BUNDLE OF TE, ON hTE, is the pull-back bundle

$$\bar{\tau}_h E \equiv (\bar{\nu}\ TE, \bar{h}, hT\ E),$$

where $\bar{\nu}\ T\ E \equiv T\ M \times_M \nu T\ E$ and $\bar{h} \equiv id_{TM} \times \Pi_E$.

Hence the following diagram is commutative

$$\begin{array}{ccc} \bar{\nu}\,T\,E & \longrightarrow & \nu\,T\,E \\ \downarrow & & \downarrow \\ h\,T\,E & \longrightarrow & E \end{array}$$

7 PROPOSITION.

The bundle $\bar{\tau}_h E \equiv (T\ E,\ h,\ h\ T\ E)$

is an affine bundle, whose vector bundle is

$$\bar{\tau}_{h}E \equiv (\bar{\tau}\, T\, E, \bar{h}, h\, T\, E)$$

PROOF.

Let $(e,u) \in E \times_M T\, M.$

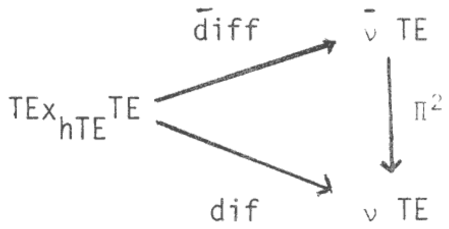
We get $h^{-1}(e,u) = \{\alpha \in T_e E \mid T\, p(\alpha) = u\}$

Since $T_e p : T_e E \to T_{p(e)} M$ is a linear map, then $h^{-1}(e,u)$ is an affine space, whose vector space is $\mathrm{Ker}\ T_e p = \bar{\nu}\, T_e\, E$ $\underline{.}$

Hence $T\, E$ is an affine bundle on $h\, T\, E$ and a vector bundle on E. Let us remark that we can consider the two difference maps, with respect to the two previous structures

$$\overline{\mathrm{diff}}: TE \times_{hTE} TE \to \bar{\nu}TE \qquad \text{and} \qquad \mathrm{diff}: TE \times_{hTE} TE \to \nu TE$$

and the following diagram is commutative

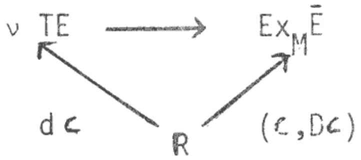


8 PROPOSITION.

Let $\eta \equiv (E,p,M)$ be an affine bundle, whose vector bundle is $\bar{\eta} \equiv (\bar{E},\bar{p},M)$. There is a unique diffeomorphism

$$\nu\, TE \hookrightarrow E \times_M \bar{E}$$

such that, for each vertical map $c : \mathbb{R} \to E$, the following diagram is commutative



Such a diffeomorphism is an isomorphism over E $\underline{.}$

We will make often the identification

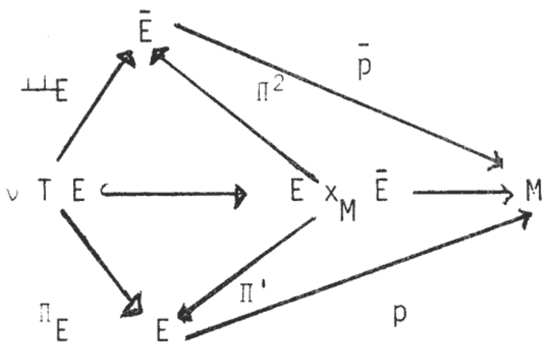
$$\nu T E \cong E \times_M \bar{E} .$$

9 We define the map

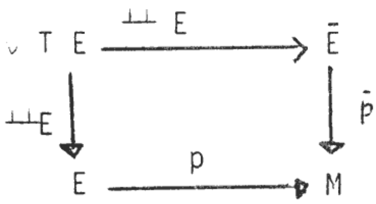
$$\perp\!\!\perp_E : \nu T E \to \bar{E}$$

by the composition $\nu T E \to E \times_M \bar{E} \xrightarrow{\pi^2} \bar{E}$.

Then we get the commutative diagram



and the homomorphism



is an isomorphism of fibers.

10 PROPOSITION.
Let $\eta \equiv (E,p,M)$ be a vector bundle. Then $T\eta \equiv (TE, Tp, TM)$ has a natural structure of vector bundle.

PROOF.

Let $\alpha, \beta \in T E$ be such that $T p(\alpha) = T p(\beta)$.

There exist $c_\alpha : \mathbb{R} \to E$ and $c_\beta : \mathbb{R} \to E$ such that

$$p \circ c_\alpha = p \circ c_\beta \quad \text{and} \quad d\, c_\alpha(o) = \alpha \, , \; d\, c_\beta(o) = \beta .$$

We can define $c \equiv c_\alpha + c_\beta : \mathbb{R} \to E$, for which we get

$p \circ c_\alpha = p \circ c = p \circ c_\beta$ and $T\,p(\alpha) = T\,p \circ d\,c = T\,p(\beta)$.

Since $d\,\gamma(o)$ depends only on α and β, we can put

$$\alpha + \beta \equiv d\,c(o) \qquad \underline{.}$$

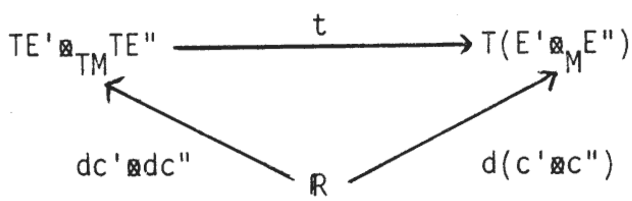
11 PROPOSITION.

Let $\eta' \equiv (E',p',M)$ and $\eta'' \equiv (E'',p'',M)$ be vector bundles.

There is a unique map

$$t : T\,E' \otimes_{TM} TE'' \longrightarrow T(E' \otimes_M E'')$$

such that the following diagram is commutative



for each $c':\mathbb{R} \to E'$ and $c'':\mathbb{R} \to E''$ such that

$$p' \circ c' = p'' \circ c'' .$$

This map is a surjective linear homomorphism over TM $\underline{.}$

2. - THE COTANGENT SPACE OF A BUNDLE.

Let $\eta \equiv (E,p,M)$ be a C^∞ bundle.

1 DEFINITION.

The COTANGENT BUNDLE OF E is the vector bundle

$$\tau^* E \equiv (T^* E, \rho_E, E) \quad \underline{.}$$

2 DEFINITION.

The HORIZONTAL BUNDLE OF $T^* E$ is the pull-back vector bundle

$$h\,\tau^*\,E \equiv (hT^* E, \Pi^1, E),$$

where $$h\,T^*\,E \equiv E \times_M T^*\,M \quad \underline{.}$$

Hence the following diagram is commutative

$$\begin{array}{ccc} h\,T^*E & \xrightarrow{\;\pi^2\;} & T^*M \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \rho_M} \\ E & \xrightarrow{\;p\;} & M \end{array}$$

3. PROPOSITION.

The transpose map of $h : TE \to hTE$ over E is an injective map

$$h\,T^*E \to T^*E \quad .$$

The following diagram is commutative

$$\begin{array}{ccc} h\,T^*E & \hookrightarrow & T^*E \\ & {\scriptstyle \pi'}\searrow \quad \swarrow{\scriptstyle \rho_E} & \\ & E & \end{array}$$

PROOF.

In fact $h\,T^*E$ is the dual of $h\,TE$ and h is surjective .

4 PROPOSITION.

The inclusion $h\,T^*E \to T^*E$ identifies $h\,T^*E$ with the orthogonal of νTE.

PROOF.

In fact νTE is the kernel of h .

5 DEFINITION.

The VERTICAL BUNDLE OF T^*E is the quotient vector bundle

$$\nu\,\tau^*E \equiv (\nu T^*E, \tilde{\rho}_E, E)$$

where
$$\nu\,T^*E \equiv T^*E/hT^*E \quad .$$

The following sequence is exact and the diagram commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h\,T^*E & \hookrightarrow & T^*E & \xrightarrow{\;\nu\;} & \nu T^*E & \longrightarrow & 0 \\ & & & {\scriptstyle \pi'}\searrow & \downarrow{\scriptstyle \rho_E} & \swarrow{\scriptstyle \tilde{\rho}_E} & & & \\ & & & & E & & & & \end{array}$$

6 DEFINITION.

The COTANGENT BUNDLE OF E, ON νT^*E, is

$$\tau^*_\nu E \equiv (T^*E, \nu, \nu T^*E).$$

The HORIZONTAL BUNDLE OF T^*E, ON νT^*E, is the pull-back vector bundle

$$\bar{\tau}^*_\nu E \equiv (\bar{h}T^*E, \bar{\nu}, \nu T^*E)$$

where $\bar{h}T^*E \equiv \nu T^*E \times_E h\, T^*E$ and $\bar{\nu} \equiv \Pi'$.

Hence the following diagram is commutative

$$\begin{array}{ccc} \bar{h}\, T^*\, E & \xrightarrow{\Pi^2} & h\, T^*E \\ \Pi' \downarrow & & \downarrow \Pi' \\ T^*E & \xrightarrow{\tilde{\rho}_E} & E \end{array}$$

7 PROPOSITION.

The bundle $\tau^*_\nu E \equiv (T^*E, \nu, \nu T^*E)$

is an affine bundle, whose vector bundle is

$$\bar{\tau}^*_\nu E \equiv (\bar{h}T^*E, \bar{\nu}, \nu T^*E) .$$

PROOF.

Let $[\beta] \in \nu T^*_e E$

We get $\nu^{-1}\, [\beta] = \{\alpha \in T^*_e E \mid \nu(\alpha) = [\beta]\}.$

Since $\nu_e : T^*_e E \to \nu T^*_e E$ is a linear map, then $\nu^{-1}\, [\beta]$ is an affine space,

whose vector space is $\mathrm{Ker}\, \nu_e = h\, T^*_e E$.

Hence T^*E is an affine bundle on νT^*E and a vector bundle on E.

8 PROPOSITION.

Let $\eta \equiv (E,p,M)$ be an affine bundle, whose vector bundle is $\bar{\eta} \equiv (\bar{E},\bar{p},M)$.

Let $\nu^* : T^*E \to E\times_M \bar{E}^*$

be the transpose map of the inclusion

$$E\times_M \bar{E} \cong \nu TE \longrightarrow TE .$$

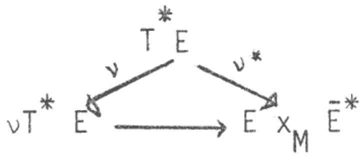
The following sequence is exact

$$0 \to hT^*E \hookrightarrow T^*E \to E\times_M \bar{E}^* \to 0$$

Then there is a unique homomorphism over E

$$\nu T^*E \to E\times_M \bar{E}^*$$

such that the following diagram is commutative



Such a map is an isomorphism $\underline{.}$

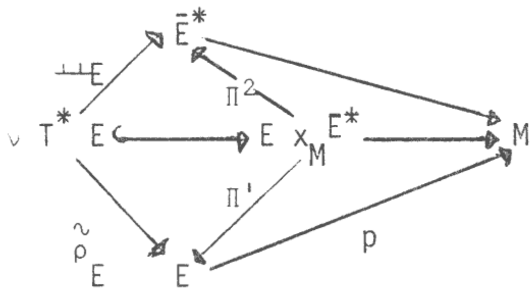
We will often make the identification

$$\nu T^*E \cong E \times_M \bar{E}^*$$

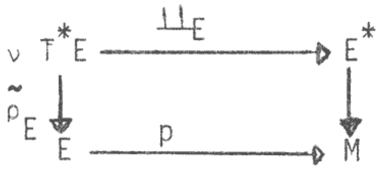
9 We define the map $\quad \perp\!\!\perp_E : \nu T^*E \to \bar{E}^*$

by the composition $\quad \nu T^*E \to E\times_M \bar{E}^* \xrightarrow{\pi^2} \bar{E}^*$

Then we get the commutative diagram



and the homomorphism



is an isomorphism on fibers.

3 - THE SECOND TANGENT AND COTANGENT SPACES OF A MANIFOLD.

1 As a particular case of the previous results, let us consider

$$\eta \equiv (TM, \pi_M, M) \quad \text{or} \quad \eta \equiv (T^*M, \rho_M, M) \quad .$$

Then we get the following spaces

$h\,T\,T\,M = TM \times_M T\,M$ $\qquad \nu\,T\,T\,M = T\,M \times_M T\,M$

$h\,T\,T^*M = T^*M \times_M T\,M$ $\qquad \nu\,T\,T^*M = T^*M \times_M T^*M$

$h\,T^*T\,M = T\,M \times_M T^*M$ $\qquad \nu\,T^*T\,M = T\,M \times_M T^*M$

$h\,T^*T^*M = T^*M \times_M T^*M$ $\qquad \nu\,T^*T^*M = T^*M \times_M T\,M$

and the following maps

$(\pi_{TM}, T\pi_M) \equiv h : TT\,M \to h\,T\,T\,M$ $\qquad \nu\,T\,T\,M \to T\,T\,M$

$(\pi_{T^*M}, T\rho_M) \equiv h : TT^*M \to h\,T\,T^*M$ $\qquad \nu\,T\,T^*M \to T\,T^*M$

$h\,T^*T\,M \to T^*TM$ $\qquad \nu : T^*T\,M \to \nu T^*T\,M$

$h\,T^*T^*M \to T^*T^*M$ $\qquad \nu : T^*T^*M \to T^*T^*M$

$\amalg_{TM} : \nu T\,T\,M \to T\,M$ $\qquad \amalg_{T^*M} : \nu TT^*M \to T^*M$

$\amalg_{T^*M} : \nu\,T^*T\,M \to T^*M$ $\qquad \amalg_{T^*M} : \nu T^*T^*M \to T\,M$

2 Taking into account that

$$h\,T\,T\,M = \nu T\,T\,M \quad \text{and} \quad \nu\,T^*T\,M = h\,T^*T\,M\,,$$

we define the following maps

$$\nu : T\,T\,M \to T\,T\,M$$

given by $\quad T\,T\,M \overset{h}{\to} h\,T\,T\,M = \nu T\,T\,M \to T\,T\,M$

and $\quad h : T^*T\,M \to T^*T\,M$

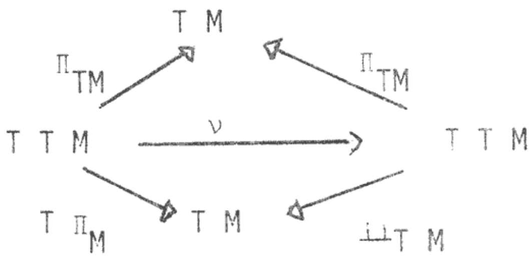
given by $\quad T^*T\,M \overset{\nu}{\to} \nu\,T^*T\,M = h\,T^*T\,M \to T^*T\,M\,.$

3 PROPOSITION.

a) The vertical endomorphism is the unique map

$$\nu : T\,T\,M \to T\,T\,M$$

which makes commutative the following diagram:



b) The horizontal endomorphism

$$h : T^{*}T\,M \;\rightarrow\; T^{*}T\,M$$

is the transpose of the vertical endomorphism $\nu : T\,T\,M \rightarrow T\,T\,M$, as

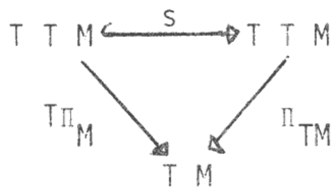
$(T^{*}T\,M,\ \rho_{TM},\ T\,M)$ is the dual of $(T\,T\,M\ ,\ \pi_{TM}\ \ T\,M)$ $\dot{-}$

4 PROPOSITION.

a) $(T\,T\,M,\ h,\ T\,M\ x_M\ T\,M)$ is the pull-back bundle of
$(T\,T\,M,\ h,\ T\,M\ x_M\ T\,M)$ with respect to the exchange endomorphism $ex : T\,M\ x_M\ T\,M \rightarrow TM\ x_M\ T\,M$.

The induced map $s \equiv (ex)^{*} : T\,T\,M \rightarrow T\,T\,M$

is an involutive automorphism such that the following diagram is commutative



b) $(\ T\,T^{*}M, h, T^{*}M\ x_M T\,M)$ is the pull-back bundle of
$(\ T^{*}T\,M, \nu, T\,M\ x_M T^{*}M)$ with respect to the exange map

$$ex : T^{*}M\ x_M T\,M \rightarrow TM\ x_M\ T^{*}M\ .$$

The induced map $s \equiv (ex)^{*} : T\,T^{*}M \rightarrow T^{*}T\,M$

is an isomorphism such that the following diagram is commutative

$$\begin{array}{ccc} T\,T^{*}M & \xrightarrow{\;s\;} & T^{*}T\,M \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle \nu} & \\ & T\,M \times_M T^{*}M & \end{array}$$

b)' Reversing all the terms of b), we get the inverse isomorphism

$$s^{-1} : T^{*}T\,M \hookrightarrow T\,T^{*}M.$$

c) $(\,T\,T^{*}M, h, T^{*}M \times_M T\,M)$ is the pull-back bundle of
$(T^{*}T^{*}M, \nu, T^{*}M \times_M T\,M)$ with respect to the map

$$i \equiv (id_{T^{*}M} \times (-\,id_{TM})) : T^{*}M \times_M TM \to T^{*}M \times_M T\,M.$$

The induced map $\omega \equiv i^{*} : T\,T^{*}M \hookrightarrow T^{*}T^{*}M$

is an isomorphism (the SYMPLECTIC ISOMORPHISM) such that the following diagram is commutative

$$\begin{array}{ccc} T\,T^{*}M & \xrightarrow{\;\omega\;} & T^{*}T^{*}M \\ {\scriptstyle (\pi_{T^{*}M}, T\rho_M)}\downarrow & & \downarrow{\scriptstyle \nu} \\ T^{*}M \times_M T\,M & \xrightarrow{\;i\;} & T^{*}M \times_M T\,M \end{array}$$

c)' In an analogous way we get the isomorphism

$$\omega \circ s^{-1} : T^{*}T\,M \hookrightarrow T^{*}T^{*}M \quad \underline{.}$$

5 DEFINITION.

The SYMMETRIC SUBMANIFOLD of $T\,T\,M$ is

$$s\,T\,T\,M \equiv \{\alpha \in T\,T\,M \mid s(\alpha) = \alpha\} \quad \underline{.}$$

4 - <u>Lie derivative of tensors.</u>

1 Let $\mathcal{M}$ be the category, whose objects are manifolds and whose morphisms are diffeomorphisms.

Let $$T_{(r,s)} : \mathcal{M} \to \mathcal{M}$$

be the covariant functor defined as follows:

a) $$T_{(r,s)} M \equiv \otimes_r T M \otimes_s T^* M$$

b) if $f : M \to N$ is a diffeomorphism,

then $$T_{(r,s)} f \equiv \otimes_r T f \otimes_s T^* f^{-1} : T_{(r,s)} M \to T_{(r,s)} N.$$

2 Let M and N be manifolds.

Let $$\phi : R \times M \to N$$

be a map (defined at least locally). Then

$$\partial\phi : M \to T N$$

is the map given by

$$\partial\phi(x) = T\phi_x(0,1) .$$

3 Let M be a manifold.

Let $u : M \to T M$ be a vector field and

let $C : R \times M \to M$ be the (locally defined)

group of local diffeomorphisms generated by u.

Namely we have $u = \partial\, c$.

Let $v : M \to T_{(r,s)} M$ be a tensor field .

Let $C\, v : R \times M \to T_{(r,s)} M$ be the (locally defined)

map, given, $\forall\, \lambda \in R$, by the tensor field

$$(C\, v)_\lambda \equiv T_{(r,s)}\, C_\lambda^{-1} \circ v \circ C_\lambda : M \to T_{(r,s)} M .$$

Let us remark that $\partial(Cv)$ takes its values in the subspace

$\nu\, T\, T_{(r,s)} M \hookrightarrow T\, T_{(r,s)} M$, since $(C\, v)_\lambda$ is a section of $T_{(r,s)} M$.

Then we can give the following definition.

4 DEFINITION.

The LIE DEFIVATIVE of $v : M \to T_{(r,s)} M$ with respect to $u : M \to TM$ is

the tensor field

$$L_u v \equiv \coprod_{T_{(r,s)}M} \circ\ \partial(Cv) \ : \ M \to T_{(r,s)}M \qquad \dot{-}$$

5 LEMMA

Let $u : M \to T\,M$ be a section. There is a unique map

$$\alpha(Tu) \ : \ T^*M \ \to T^*T\,M$$

such that the following diagram is commutative and exact in T^*M

$$\begin{array}{ccccccc}
 & & T^*M & \xleftarrow{\ \Pi^2 \circ \nu\ } & & & \\
 & \nearrow^{id_{T^*M}} & & & & & \\
T^*M & & \xrightarrow{\quad \alpha(Tu) \quad} & T^*TM_{|u(M)} & \xrightarrow{\quad T^*u \quad} & T^*M & \longrightarrow M \times 0 \quad \dot{-} \\
\downarrow{\rho_M} & & & \downarrow{\rho_{TM}} & & & \\
M & & \xrightarrow{\quad u \quad} & TM & & &
\end{array}$$

6 PROPOSITION.

We have

$$L_u v = \coprod_{T(r,s)M} \circ \ (Tv\circ u \ - \ t \ \circ(\underset{r}{\otimes} \ s \ \circ \ Tu \ \underset{s}{\otimes} \ s \ \circ(\alpha(Tu))\circ \ v) \ . \ (*)$$

PROOF.

It suffices to give the proof for $v : M \to T_{(1,0)}M$ and $v : M \to T_{(0,1)}M$.

In the first case, we get

$$Cv = (T_2C^{-1}) \circ (\Pi', v) \circ (\Pi^1, C) \ : \ R \times M \ \to TM,$$

hence

$$\begin{aligned}
\partial(Cv) &= (TT_2C^{-1}) \circ (\Pi', Tv) \circ (\Pi', T_1c)_{(0,1)} = \\
&= ((\partial T_2C^{-1}) \circ \Pi_{TM} + T_2T_2C_o^{-1}) \circ (Tv) \circ \partial C = \\
&= (s \circ T\partial C^{-1}) \circ \Pi_{TM} + id_{TTM}) \circ (Tv) \circ u = \\
&= - s \circ T u \circ v + T v \circ u \ .
\end{aligned}$$

In the second case, we get

$$Cv = (T_2^*C) \circ (\Pi^1, v) \circ (\Pi^1, c) \ : \ R \times M \ \to T^*M$$

hence $\partial(Cv) = (TT_2^* C) \circ (\pi^1, Tv) \circ (\pi^1, T_1 C)_{(0,1)} =$

$$= ((\partial T_2^* C) \circ \pi_{TM} + T_2 T_2^* C_o) \circ (TV) \circ \partial C =$$

$$= (- s \circ \alpha(T_2 \partial C) \circ \pi_{TM} + id_{TT^*M}) \circ (Tv) \circ u =$$

$$= - s \circ \alpha(Tu) \circ v + Tv \circ u \quad .$$

Let us remark that both tensors in (*) are on the same affine fiber on $h\, TT_{(r,s)} M$.

5 Connection on a bundle.

Let $\eta \equiv (E,p,M)$ be a bundle.

1 DEFINITION.

A PSEUDO-CONNECTION on η is an affine bundle morphism on $h\, T\, E$

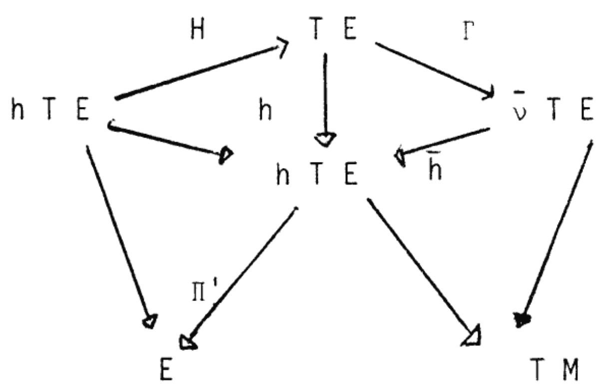
$$\Gamma : TE \to \bar{\nu}\, T\, E$$

whose fiber derivatives are 1.

A PSEUDO-HORIZONTAL SECTION is a section

$$H : h\, TE \to TE \quad .$$

Hence the following diagram is commutative



Let us remark that $\Gamma : T\, E \to \bar{\nu}\, T\, E$ is characterized by the map $\Gamma' : TE \to \nu\, T\, E$ given by $TE \xrightarrow{\Gamma} \bar{\nu}\, TE \xrightarrow{\Pi^2} \nu\, T\, E$.

2 PROPOSITION.

The maps α and β between the set of pseudo connections and the set of pseudo-horizontal sections, given by

$$\alpha : \Gamma \to H ,$$

where H is the unique horizontal section such that $\Gamma \circ H = 0$, and

$$\beta : H \to \Gamma \equiv id_{TE} - H \circ h ,$$

are inverse bijection $\dot{-}$

Henceforth we will consider Γ and H as mutually related . Hence giving a pseudo-connection is the choice of a point for each affine fiber of TE, getting in this way an identification of the affine fibers with their vector spaces.

3 PROPOSITION.

Let $c : R \to E$ be a map. The following condition are equivalect :

a) $H \circ h \circ d c \equiv H \circ (c, d(p \circ c)) = d c$

b) $\bar{\ } \circ d c = 0$.

4 DEFINITION.

A curve $c : R \to E$ is HORIZONTAL if the previous conditions are satisfied.

5. PROPOSITION.

The set $\mathcal{J}^{\cdot}$ of all pseudo-connections is the affine space of the sections of the affine bundle $\tau_h E$, whose vector space is the space of the sections of the vector bundle $\bar{\tau}_h E$ $\dot{-}$

6. PROPOSITION.

The following conditions are equivalent

a) $\Gamma : TE \to \bar{\nu} T E$ is a linear morphism on E

b) $H : hTE \to TE$ is a linear morphism on E.

Moreover, if such conditions are verified, then we get

$$T E = h T E \oplus_E \nu TE .$$

PROOF.

a) $\Longleftrightarrow$ b trivial.

For the splitting it suffices to take into account the two exact sequences on E

$$0 \to \nu TE \hookrightarrow TE \overset{h}{\twoheadrightarrow} hTE \to 0$$

$$0 \to hTE \overset{H}{\hookrightarrow} TE \overset{\Gamma'}{\twoheadrightarrow} \nu TE \to 0 \quad .$$

7 DEFINITION.

A CONNECTION (HORIZONTAL SECTION) is a pseudo connection(pseudo-horizontal section) satisfyng the condition (a),(b) .

Hence giving a connections allows us to make a comparison between "close" fibers of E.

8 PROPOSITION.

Let η be a vector bundle. Let Γ be a connection.

The following conditions are equivalent

a) $\Gamma : TE \to \bar{\nu}\, TE$ is a vector bundle morphism on TM

b) $H : hTE \to TE$ is a vector bundle morphism on TM .

9 DEFINITION.

A connection (horizontal section) is LINEAR if the previous conditions hold . Hence giving a linear connection allows us to make a comparison between "close" fibers of E by means of isomorphisms.

10 The set $\widetilde{\mathcal{J}}_\ell$ of all linear connections is an affine subspace of $\widetilde{\mathcal{J}}$, whose vector space is the space of bilinear sections of $\bar{\tau}_h E$(this vector space is naturally isomorphic to the space of sections $M \to T^*M \otimes E^* \otimes E$).

11 PROPOSITION.

Let Γ' and Γ'' be two linear connections on η' and η'', respectively.

The map $\quad H \equiv t \circ (H' \otimes H'') : hT(E' \otimes_M E'') \to t(E' \otimes_M E'')$
is a linear connections on $\eta' \otimes \eta''$ $\underline{.}$

Hence the following diagram is commutative:

$$\begin{array}{ccc}
TE' \otimes_{TM} TE'' & \xrightarrow{\quad t \quad} & T(E' \otimes_M E'') \\
{\scriptstyle H' \otimes H''} \nwarrow & & \nearrow {\scriptstyle H} \\
& (E' \otimes_M E'') \times_M TM &
\end{array}$$

12 DEFINITION.

The TENSOR PRODUCT of Γ' and Γ'' is the connection associated with the horizontal section H previously defined $\underline{.}$

13 PROPOSITION.

Let Γ be a linear connection on η .
There is a unique linear connection Γ^* on η^* such that the following diagram is commutative

$$\begin{array}{ccc}
TE \times_{TM} TE^* & \xrightarrow{\quad \dot{b} \quad} & R \\
\uparrow {\scriptstyle (H,H^*)} & & \uparrow \\
(E \times_M E^*) \times_M TM & \longrightarrow & 0
\end{array}$$

where $b : E \times_M E^* \to R$ is the inner product and $\dot{b} = \pi^2 \circ Tb$ $\underline{.}$

14 DEFINITION.

The DUAL connection of Γ is the connection associated with the horizontal section H^* previously defined $\underline{.}$

15 DEFINITION.

Let Γ be a linear connection on $\eta \equiv \tau M$.

The TORSION of Γ is the bilinear map

$$\Theta \equiv \amalg_{TM} \circ (H - s \circ H \circ ex) : TM \times_M TM \to TM.$$

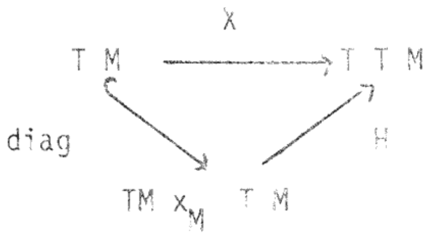
The connection Γ is SYMMETRICAL if $\Theta = 0$ $\underline{.}$

16 DEFINITION.

A QUADRATIC SPRAY is a second order differential equation

$$X : T M \to T T M$$

which is factorizable by a symmetrical linear horizontal section as follows



$\doteq$

17 PROPOSITION.

The previous diagram determines a bijection between quadratic sprays and symmetrical linear connections $\doteq$

The quadratic sprays are homogeneous with degree two .

18 DEFINITION.

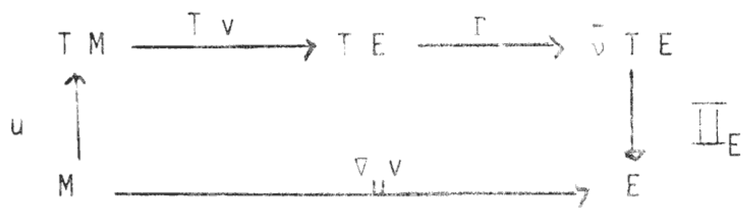
Let Γ be a linear connection on $\eta \equiv (E,p,M)$.

Let $v : M \to E$ be a section and let $u : M \to T M$ be a vector field.

The COVARIANT DERIVATIVE of v with respect u is the section

$$\nabla_u v \equiv \overline{\amalg}_E \circ \Gamma \circ T v \circ u : M \to E \quad \doteq$$

Hence the following diagram is commutative



Let us remark that we have

$$\nabla_u v = \overline{\amalg}_E \circ \Gamma \circ \partial(v \circ c),$$

where $c : R \times M \to M$ is the group of local diffeomorphisms generated by u.

19 PROPOSITION.

Let ∇ be a linear connection on $\eta \equiv (E,p,M)$.

We have

$$\nabla_{fu} = f\, \nabla_u v$$

$$\nabla_{u+u'} v = \nabla_u v + \nabla_{u'} v$$

$$\nabla_u (v+v') = \nabla_u v + \nabla_u v'$$

$$\nabla_u (fv) = f\, \nabla_u\, v + (u, f)\, v$$

If $U \subset M$ is open , then

$$\nabla_{u/_V}\, v_{/_V} = (\nabla_u v)_{/U} \, .$$

If ∇^* is the dual connection of Γ , we have

$$\nabla_u \langle \omega v \rangle = \langle \nabla_u \omega, v\rangle + \langle \omega, \nabla_u\, v\, \rangle \, .$$

If is the tensor product of the linear connection Γ' and Γ'' ,
we have $\nabla_u (v' \otimes v'') = \nabla_u\, v' \otimes v'' + v' \otimes \nabla_u\, v''$ $\doteq$

20 PROPOSITION.

Let Γ be a linear connection on $\eta \equiv \tau M$.

We have $\nabla_u v = \nabla_v u + L_u v + \Theta \circ (u,v)$.

PROOF.

$$\nabla_u v - \nabla_v u = \coprod_{TM} \circ \Gamma \circ (Tv \circ u - s \circ Tv \circ u) + \Theta \circ (u,v) = L_u v + \Theta \circ (u,v)$$

PROPOSITION.

Let $g : TM \times_M TM \to R$ be a non degenerate symmetrical linear map. Let us denote by the same notation the associated maps

$g : TM \to R$, $\quad g : M \to T_{(0,2)} M$ and $\quad g : TM \to T^* M$.

Each one of the following conditions characterize the same symmetrical linear connection Γ on τM.

a) The following diagram is commutative

$$\begin{array}{ccc} TTM & \xrightarrow{\dot g} & \\ \downarrow & & R \\ TTM & \xrightarrow{\dot g} & \end{array}$$

b) The following diagram is commutative

$$TTM \times_{TM} TTM \quad \dot g \qquad\qquad R$$

$$(H,H)$$

$$(TM \times_M TM) \times_M TM \qquad\qquad 0$$

c) We have $\quad {}_u g = 0 \quad , \qquad \forall\, u : M \to TM$.

d) The following diagram is commutative

$$TM \times_M TM \quad\quad H \quad\quad TTM$$

$$id_{TM} \times g^{-1} \qquad\qquad\qquad T g$$

$$H^*$$

$$TM \times_M T^*M \quad\quad\quad\quad TT^*M$$

R E F E R E N C E S

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