III CHAPTER

OBSERVED KINEMATICS

Here we analyse the one- body kinematics in terms of the positions determined by a frame , introducing the observed motion and its velocity and acceleration. By comparison between the absolute and the observed motion we get the "absolute" velocity addition and Coriolis theorem. Finally we make the comparison between the observed motions relative to two frames, getting the velocity addition and Coriolis theorem.

1 OBSERVED KINEMATICS.

Let $\mathcal{P}$ a fixed frame and let M be a fixed motion. We analyse M as viewed by $\mathcal{P}$.

1 We first introduce useful notations.

Let $f : \mathbb{T} \to \mathbb{P}$ be a C^∞ map.

a) We put
$$\check{f} \equiv (id_{\mathbb{T}}, f) : \mathbb{T} \to \mathbb{T} \times \mathbb{P}$$
$$\check{d}f \equiv (id_{\mathbb{T}}, df) : \mathbb{T} \to \mathbb{T} \times T\mathbb{P},$$
$$\check{d}^2 f \equiv (id_{\mathbb{T}}, d^2 f) : \mathbb{T} \to \mathbb{T} \times T^2\mathbb{P} .$$

b) df and d^2f being functions on $\mathbb{T}$, we can choose a natural re presentative of the equivalence classes of $T\mathbb{P}$ and $T^2\mathbb{P}$. So we put

$$df \equiv [f, D_{\mathcal{P}} f]$$

and we get
$$d^2 f \equiv [f, D_{\mathcal{P}} f, D_{\mathcal{P}} f, D^2_{\mathcal{P}} f]$$

where
$$D_{\mathcal{P}} f : \mathbb{T} \to \bar{\mathbb{S}} \quad \text{and} \quad D^2_{\mathcal{P}} f : \mathbb{T} \to \bar{\mathbb{S}}$$

resemble derivatives of affine spaces, but are note properly such.

c) We put
$$\check{\nabla}_p df \equiv \sqcup_p \circ \check{\Gamma}_p \circ \check{d}^2 f : \mathbb{T} \to T\mathbb{P}$$
$$\dot{\nabla}_p df \equiv \sqcup_p \circ \dot{\Gamma}_p \circ \check{d}^2 f : \mathbb{T} \to T\mathbb{P}.$$

<u>Observed motion and absolute velocity addition and Coriolis theorem.</u>

2 The basic definition of observed kinematics is the following.

DEFINITION.

a) The MOTION OF M OBSERVED BY $\mathcal{P}$ is the map

$$M_{\mathcal{P}} \equiv p \circ M : \mathbb{T} \to \mathbb{P} .$$

b) The VELOCITY OF M OBSERVED BY $\mathcal{P}$ is the map

$$(d\, M)_{\mathcal{P}} \equiv T\, p \circ d\, M : \mathbb{T} \to T\, \mathbb{P}.$$

The VELOCITY OF THE OBSERVED MOTION $M_{\mathcal{P}}$ is the map

$$d\, M_p : T \to T\, \mathbb{P}\ .$$

c) The ACCELERATION OF M OBSERVED BY $\mathcal{P}$ is the map

$$(\nabla\, d\, M)_{\mathcal{P}} \equiv T\, p \circ \nabla\, d\, M : \mathbb{T} \to T\, \mathbb{P}\ .$$

The ACCELERATION OF THE OBSERVED MOTION $M_{\mathcal{P}}$ is the map

$$\check{\nabla}_{\mathcal{P}}\, \check{d}\, M_p \equiv \bot\!\!\!\bot_p \circ \check{\Gamma}_{\mathcal{P}} \circ \check{d}^2 M_p : T \to T\, \mathbb{P} \qquad \dot{-}$$

3 We can make the comparison between the observed entities and the entities of the observed motion.

THEOREM. " ABSOLUTE VELOCITY ADDITION AND CORIOLIS THEOREM"

a) $$M = P \circ \check{M}_p$$

i.e., putting $$\mathbb{E} \stackrel{\sim}{=} \mathbb{T} \times \mathbb{P},$$
$$M \stackrel{\sim}{=} \check{M}_p \quad .$$

b) $[M, DM - \bar{P} \circ M] = (\, d\, M)_{\mathcal{P}} = d\, M_p \equiv [M, D_{\mathcal{P}} M_{\mathcal{P}}]$

i.e. $$D\, M - \bar{P} \circ M = D_{\mathcal{P}} M_{\mathcal{P}} \quad .$$

c) $[M, D^2 M] = (\nabla\, d\, M)_{\mathcal{P}} = \check{\nabla}_{\mathcal{P}} \check{d}\, M_p =$

$$= [M, D^2_{\mathcal{P}} M_{\mathcal{P}} + \epsilon_{\mathcal{P}} \circ \check{M}_{\mathcal{P}} (D_{\mathcal{P}} M_{\mathcal{P}}) + 2\Omega_{\mathcal{P}} \circ \check{M}_{\mathcal{P}} \times D_{\mathcal{P}} M_{\mathcal{P}} + \bar{\bar{P}} \circ M_{\mathcal{P}})$$

i.e. $$D^2 M = D^2_{\mathcal{P}} M_{\mathcal{P}} + (\epsilon_{\mathcal{P}} \circ \check{M}_{\mathcal{P}})(D_{\mathcal{P}} M_{\mathcal{P}}) + 2(\Omega_{\mathcal{P}} \circ \check{M}_{\mathcal{P}}) \times D_{\mathcal{P}} M_{\mathcal{P}} + \bar{\bar{P}} \circ \bar{M}_p \ .$$

PROOF.

a) $M = P \circ (t,p) \circ M = P \circ (id_{\mathbb{T}}, M_{\mathcal{P}}) \equiv P \circ \check{M}_p\ .$

b) $(d\,M)_{\mathcal{P}} = T\,p \circ d\,M = d(p \circ M) = d\,M_{\mathcal{P}}$.

c) $(\nabla\,d\,M)_{\mathcal{P}} = T\,p \circ \coprod \circ \Gamma \circ d^2M = \coprod_{\mathcal{P}} \circ T^2p \circ \acute{\Gamma} \circ d^2M =$

$$= \coprod_{\mathcal{P}} \circ \acute{\Gamma}_{\mathcal{P}} \circ \overleftarrow{d}^2M \quad \dot{-}$$

4 COROLLARY.

We have

$$\underset{\frown}{x}^k \circ M_{\mathcal{P}} = M^k \equiv x^k \circ M$$

$$\underset{\frown}{\dot{x}}^k \circ d\,M_{\mathcal{P}} = D\,M^k$$

$$\underset{\frown}{\dot{x}}^k \circ \acute{\nabla}_{\mathcal{P}} d\,M_{\mathcal{P}} = D^2M^k + (\Gamma^k_{ij} \circ \check{M}_P)D\,M^i\,D\,M^j \quad \dot{-}$$

5 COROLLARY.

a) if $\mathcal{P}$ is affine, we have

$$D^2M = D^2_{\mathcal{P}}M_{\mathcal{P}} + \epsilon_{\mathcal{P}}(D_{\mathcal{P}}M_{\mathcal{P}}) + 2\Omega_{\mathcal{P}} \times D_{\mathcal{P}}M_{\mathcal{P}} + \bar{\bar{P}} \circ \check{M}_{\mathcal{P}} .$$

b) If $\mathcal{P}$ is rigid, we have

$$D^2M = D^2_{\mathcal{P}}M_{\mathcal{P}} + 2\Omega_{\mathcal{P}} \times D_{\mathcal{P}}M_{\mathcal{P}} + \bar{\bar{P}} \circ \check{M}_{\mathcal{P}}$$

c) If $\mathcal{P}$ is translating, we have

$$D^2M = D^2_{\mathcal{P}}M_{\mathcal{P}} + \bar{\bar{P}} \quad .$$

d) If $\mathcal{P}$ is inertial, we have

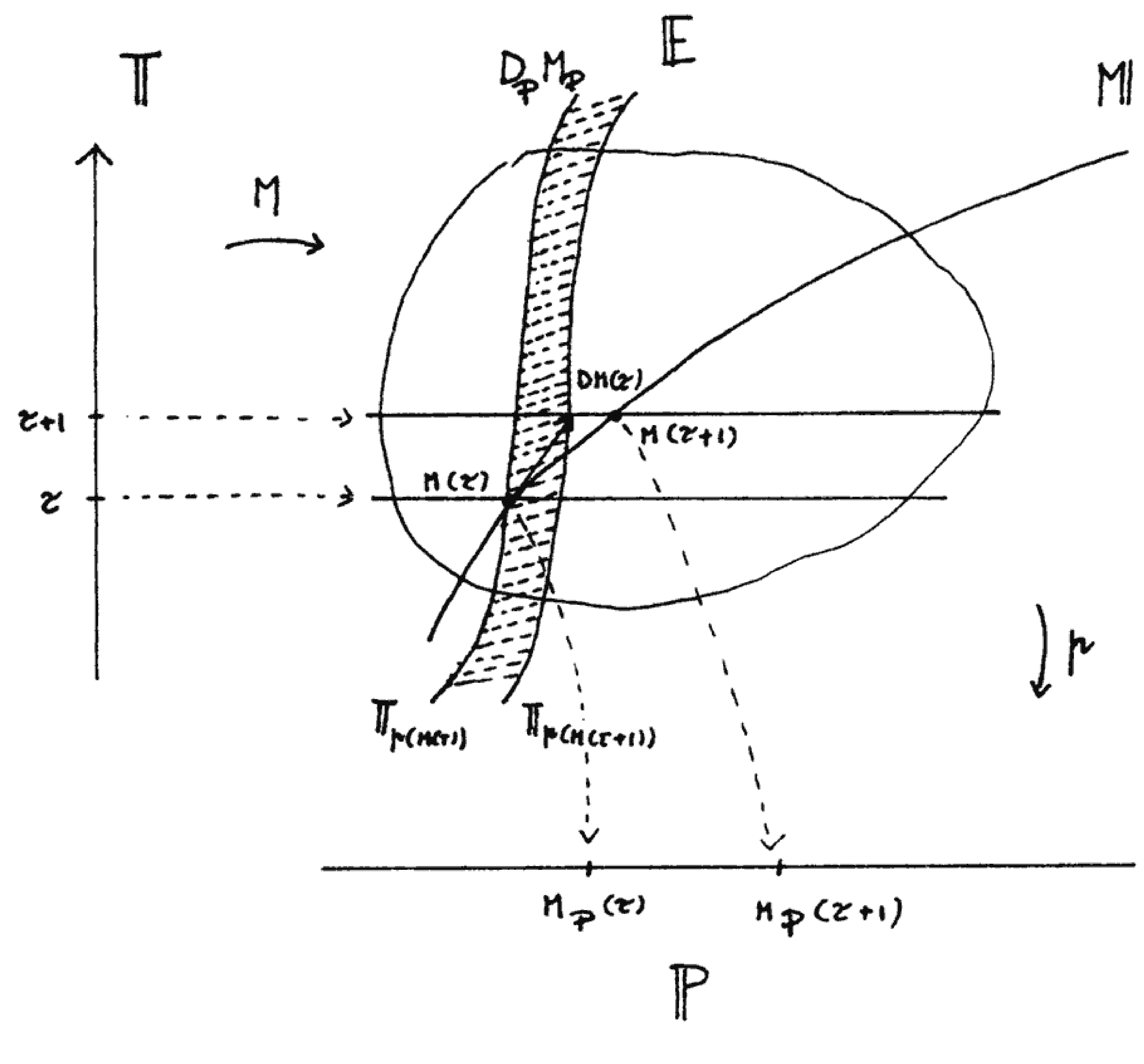
$$D^2M = D^2_{\mathcal{P}}M_{\mathcal{P}} \qquad \dot{-}$$

Physical description.

The observed motion $M_{\mathcal{P}}$ is the map that associates, with each time $\tau\in\mathbb{T}$, the position constituted by the world-line of the frame, passing throung $M(\tau)$.

The observed velocity and acceleration are the map that associate with each time $\tau\in\mathbb{T}$, the strips touched by the absolute velocity and acceleration.

The difference between the observed acceleration and the acceleration of the observed motion takes into account the variation, during the time, of the affine properties of $\mathbb{TP}$ and of the projection $T\,\mathbb{E} \to \mathbb{TP}$.



2 RELATIVE KINEMATICS.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two fixed frames and let the subfixes "1" an "2" denote quantities relative to $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Let M be a fixed motion. We make a comparison between the kinematics observed by $\mathcal{P}_1$ and $\mathcal{P}_2$.

<u>Motion of a frame observed by a frame.</u>

1 If we consider $\mathcal{P}_1$ as a set of world-lines and $\mathcal{P}_2$ as observing $\mathcal{P}_1$, we are led naturally to the following definition by (III,1,2).

We consider only free velocity and acceleration for simplicity of notations, leaving to the reader to write them in the complete form.

Here $D_{1\mathcal{P}_2}$ and $D^2_{1\mathcal{P}_2}$ are the derivative in the sense of (III,1,1,b) with respect to P_2 and the suffix 1 denote partial derivative with respect to the first variable, i.e. the time.

DEFINITION.

a) The MOTION OF $\mathcal{P}_1$ OBSERVED BY $\mathcal{P}_2$ is the map

$$\tilde{P}_{12} \equiv p \circ \tilde{P}_1 : \mathbb{T} \times \mathbb{E} \to \mathbb{P}_2$$

The MUTUAL MOTION of (P_1,P_2) is the map

$$\tilde{P}_{(1,2)} \equiv \tilde{P}_1 - \tilde{P}_2 : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{S}} .$$

b) The (FREE) VELOCITY OF THE OBSERVED MOTION $\tilde{P}_{12}$ is the map

$$\bar{P}_{12} \equiv (D_{1\mathcal{P}_2}\tilde{P}_{12})\circ j : \mathbb{E} \to \bar{\mathbb{S}}$$

The (FREE) VELOCITY OF $\mathcal{P}_1$ OBSERVED BY $\mathcal{P}_2$ is the map

$$\bar{P}_{12} \equiv \hat{P}_2 \circ \bar{P}_1 : \mathbb{E} \to \bar{\mathbb{S}}$$

The (FREE) VELOCITY OF THE MUTUAL MOTION $\tilde{P}_{(1,2)}$ is the map

$$\bar{P}_{(1,2)} \equiv D_1\tilde{P}_{(1,2)}\circ j : \mathbb{E} \to \bar{\mathbb{S}} .$$

c) The (FREE) ACCELERATION OF THE OBSERVED MOTION $\tilde{P}_{12}$ is the map

$$\bar{\bar{P}}_{12} \equiv (D^2_{1P_2}\tilde{P}_{12})\circ j : \mathbb{E} \to \bar{\mathbb{S}} .$$

The (FREE) ACCELERATION OF $\mathcal{P}_1$ OBSERVED BY $\mathcal{P}_2$ is the map

$$\bar{\bar{P}}_{1,2} \equiv \hat{P}_2 o \bar{\bar{P}}_1 : \mathbb{E} \to \bar{\mathbb{S}} .$$

The (FREE) ACCELERATION OF THE MUTUAL MOTION $\tilde{P}_{(1,2)}$ in the map

$$\bar{\bar{P}}_{(1,2)} = D^2_1 \tilde{P}_{(1,2)}\circ j : \mathbb{E} \to \bar{\mathbb{S}}.$$

d) The (FREE) STRAIN OF THE OBSERVED MOTION $\tilde{P}_{12}$ is the map

$$\epsilon_{12} = \check{S}\, D\, \bar{P}_{12} : \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}$$

The (FREE) SPIN OF THE OBSERVED MOTION $\tilde{P}_{12}$ is the map

$$\omega_{12} = \frac{A}{2}\, \check{D}\, \bar{P}_{12} : \mathbb{E} \to \mathbb{S}^* \otimes \bar{\mathbb{S}} .$$

The (FREE) ANGULAR VELOCITY OF THE OBSERVED MOTION $\tilde{P}_{12}$ is the map

$$\Omega_{12} = * \ \frac{A}{2}\, \check{D}\, \bar{P}_{12} \quad \dot{-}$$

2 We can make the comparison between the observed entities and the entities of the observed motion, as shown by (III,1,3) .

PROPOSITION.

a) $$\tilde{P}_1 = (\pi_1, \tilde{P}_{12}) \ ; \mathbb{T} \times \mathbb{E} \to \mathbb{T} \times \mathbb{P}_2 \tilde{=} \mathbb{E} .$$

b) $$\bar{P}_{(1,2)} = \bar{P}_{1,2} = \bar{P}_1 - \bar{P}_2 = \bar{P}_{12}$$

c) $$\bar{\bar{P}}_{1,2} = \bar{\bar{P}}_1 \qquad = \bar{\bar{P}}_{12} + \epsilon_{\mathcal{P}_2}(\bar{P}_{12}) + 2\Omega_{\mathcal{P}_2} \times \bar{P}_{12} + \bar{\bar{P}}_2 .$$

d) $$\bar{\bar{P}}_{(1,2)} = \bar{\bar{P}}_1 - \bar{\bar{P}}_2, \ \epsilon_{12} = \epsilon_1 - \epsilon_2 , \ \omega_{12} = \omega_1 - \omega_2 , \ \Omega_{12} = \Omega_1 - \Omega_2 \ \dot{-}$$

3 We get an immediate comparison between the quantities "12" and "21".

COROLLARY.

a) $\tilde{P}_{(1,2)} = - \tilde{P}_{(2,1)}$, $\bar{P}_{(1,2)} = - \bar{P}_{(2,1)}$, $\bar{\bar{P}}_{(1,2)} = - \bar{\bar{P}}_{(2,1)}$

$\epsilon_{12} = - \epsilon_{21}$, $\omega_{12} = - \omega_{21}$, $\Omega_{12} = -\Omega_{21}$

b) $\bar{P}_{11} = \epsilon_{11} = \omega_{11} = \Omega_{11} = 0$ $\dot{-}$

4 We have time depending diffeomorphism between spaces concerning $\mathcal{P}_1$ and $\mathcal{P}_2$.

PROPOSITION.

Let $\tau \in T$.

The maps

$$p_{12\tau} \equiv p_2 \circ P_{1_\tau} : \mathbb{P}_1 \to \mathbb{P}_2$$

given by

$$[e]_1 \to [P_1(\tau,e)]_2,$$

and

$$T\, p_{12\tau} : T\mathbb{P}_1 \to T\mathbb{P}_2,$$

given by

$$[e,u]_1 \to [\tilde{P}_1(\tau,e),\ \check{P}_1(\tau,e)(u)]_2 ,$$

are C^∞ diffeomorphisms $\dot{-}$

Velocity addition and generalized Coriolis theorems.

5 As conclusion, we get the comparison between velocity and acceleration of the motion M observed by $\mathcal{P}_1$ and $\mathcal{P}_2$.

THEOREM. "VELOCITY ADDITION AND GENERALIZED CORIOLIS THEOREMS".

a) $$M_{\mathcal{P}_2} = p_2 \quad \bar{M}_{\mathcal{P}_1} \quad .$$

b) $$D_{\mathcal{P}_2} M_{\mathcal{P}_2} = D_{\mathcal{P}_1} M_{\mathcal{P}_1} + \bar{P}_{12} \circ M \; .$$

c) $$D^2_{\mathcal{P}_2} M_{\mathcal{P}_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} + \epsilon_{12} \circ M (D_{\mathcal{P}_1} M_{\mathcal{P}_1}) + 2\Omega_{12} \circ M \times D_{\mathcal{P}_1} M_{\mathcal{P}_1} + \bar{\bar{P}}_{12} \circ M.$$

PROOF.

It follows from (II,5,3) and (II,6,2) $\underline{\cdot}$

6 COROLLARY.

Let $\mathcal{P}_2$ be inertial. Then we get

$$D^2 M = D^2_{\mathcal{P}_2} M_{\mathcal{P}_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} + \epsilon_{\mathcal{P}_1} \circ M(D_{\mathcal{P}_1} M_{\mathcal{P}_1}) + 2\Omega_{P_1} \circ M \times D_{\mathcal{P}_1} M_{\mathcal{P}_1} + \bar{\bar{P}}_1 \circ M \quad .$$

If $\mathcal{P}_1$ is affine, we have

$$D^2 M = D^2_{\mathcal{P}_2} M_{\mathcal{P}_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} + \epsilon_{\mathcal{P}_1} (D_{\mathcal{P}_1} M_{\mathcal{P}_1}) + 2 \, \Omega_{\mathcal{P}_1} \times D_{\mathcal{P}_1} M_{\mathcal{P}_2} + \bar{\bar{P}}_1 \; ;$$

if $\mathcal{P}_1$ is rigid, we have

$$D^2 M = D^2_{\mathcal{P}_2} M_{\mathcal{P}_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} + 2 \, \Omega_{\mathcal{P}_1} \times D_{\mathcal{P}_1} M_{\mathcal{P}_2} + \bar{\bar{P}}_1 \; ;$$

if $\mathcal{P}_1$ is translating we have

$$D^2 M = D^2_{P_2} M_{P_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} + \bar{\bar{P}}_1 \quad ;$$

if $\mathcal{P}_1$ is inertial, we have

$$D^2 M = D^2_{\mathcal{P}_2} M_{\mathcal{P}_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} \quad \dot{-}$$

Physical description.

The observed motion $\tilde{P}_{12e} : \mathbb{T} \to \mathbb{P}_2$ gives the position in $\mathbb{P}_2$ touched during time, by the particle of $\mathcal{P}_1$ passing through e. The velocity and the acceleration of $\tilde{P}_{12}$ are calculed by $\mathcal{P}_2$ by its differential structure and by its time depending affine structure, in the same way of any observed motion.

The velocity and acceleration of $\mathcal{P}_1, \bar{P}_{1,2}(e)$ and $\bar{\bar{P}}_{(1,2)}(e)$, are the spatial projections, performed by $\mathcal{P}_2$, of the absolute velocity and acceleration of the particle of $\mathcal{P}_1$, passing through e.

Notice that, in all the previous quantities, $\mathcal{P}_1$ is involued only through the motion of its only particle $\hat{P}_{1e}$, while $\mathcal{P}_2$ can use also its spatial derivative, which take into account the mutual motion of its particles.

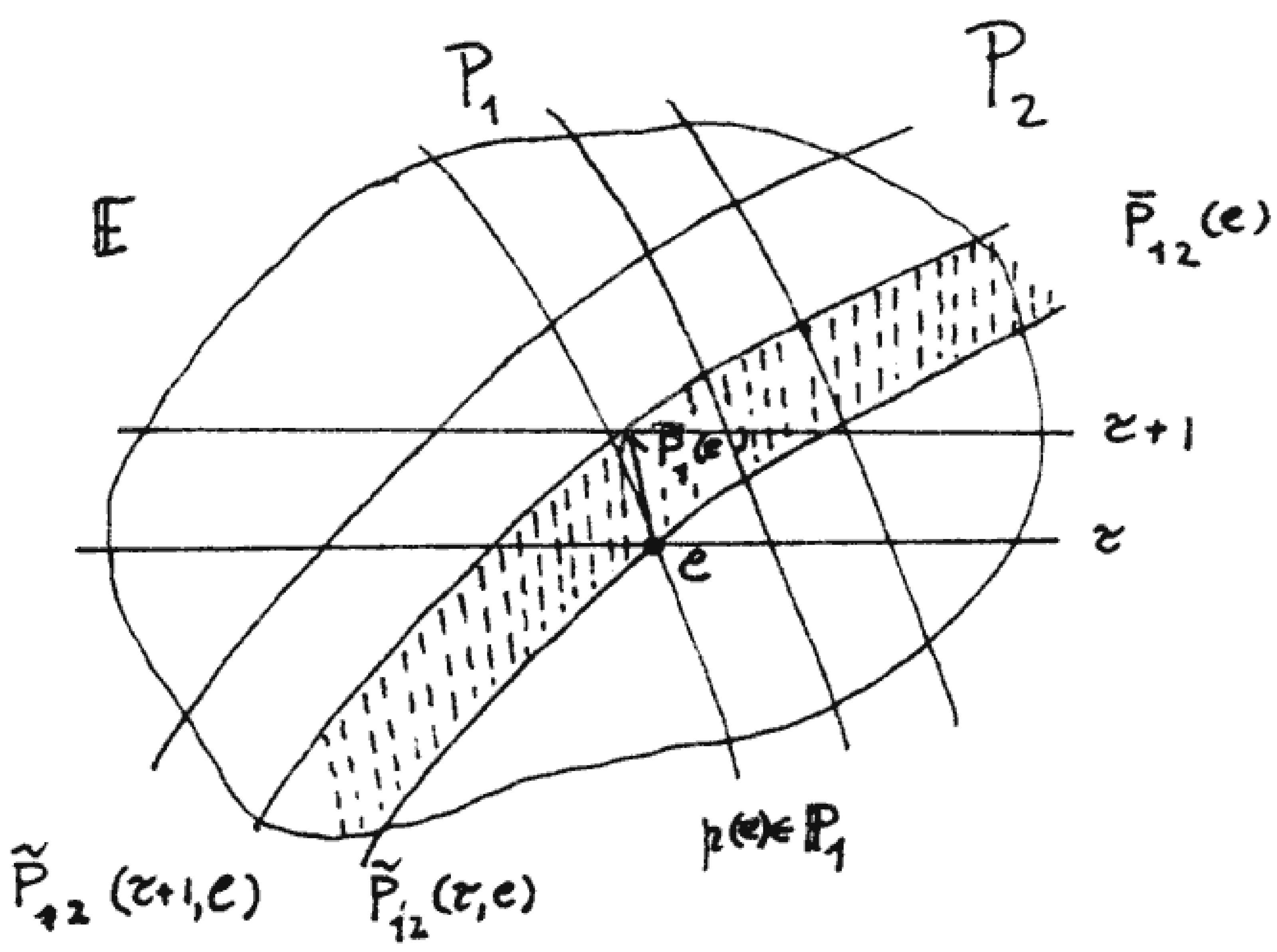
The nutual motion, velocity and acceleration $P_{(1,2)}(e) : T \to \bar{\mathbb{S}}$, $\bar{P}_{(1,2)}(e) \in \bar{\mathbb{S}}$, $\bar{\bar{P}}_{(1,2)}(e) \in \bar{\mathbb{S}}$ are the absolute spatial distance and its time first and second derivatives between the two particles, one of $\mathcal{P}_1$ and one $\mathcal{P}_2$, passing through e.

So it is not surprising if $\bar{\bar{P}}_{1,2} \neq - \bar{\bar{P}}_{2,1}$, $\bar{\bar{P}}_{12} \neq - \bar{\bar{P}}_{21}$.

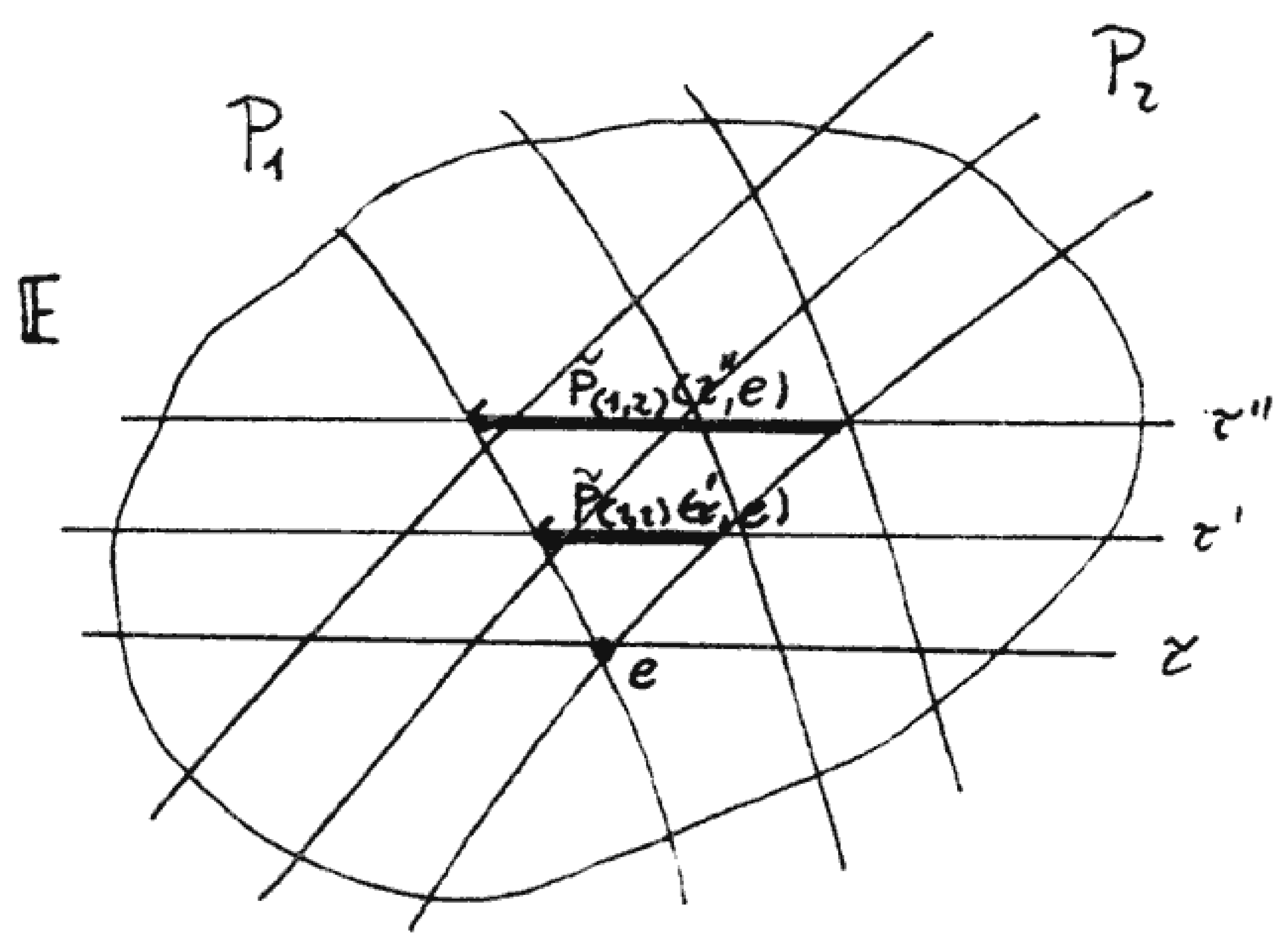
The velocity addition theorem, relative to a motion M, gives the classical result that the velocity of the observed motion by $\mathcal{P}_2$ is the sum of the velocity of the observed motion by $\mathcal{P}_1$, plus the velocity of $\mathcal{P}_1$ observed by P_2 .

The generalized Coriolis theorem says that the acceleration of the observed motion by $\mathcal{P}_2$ is the sum of the acceleration of the observed motion by $\mathcal{P}_2$, plus the acceleration of $\mathcal{P}_1$ observed by $\mathcal{P}_2$ plus the classical angular velocity term , plus a strain term. When we consider rigid frames, we get, as a particular case, the classical result.

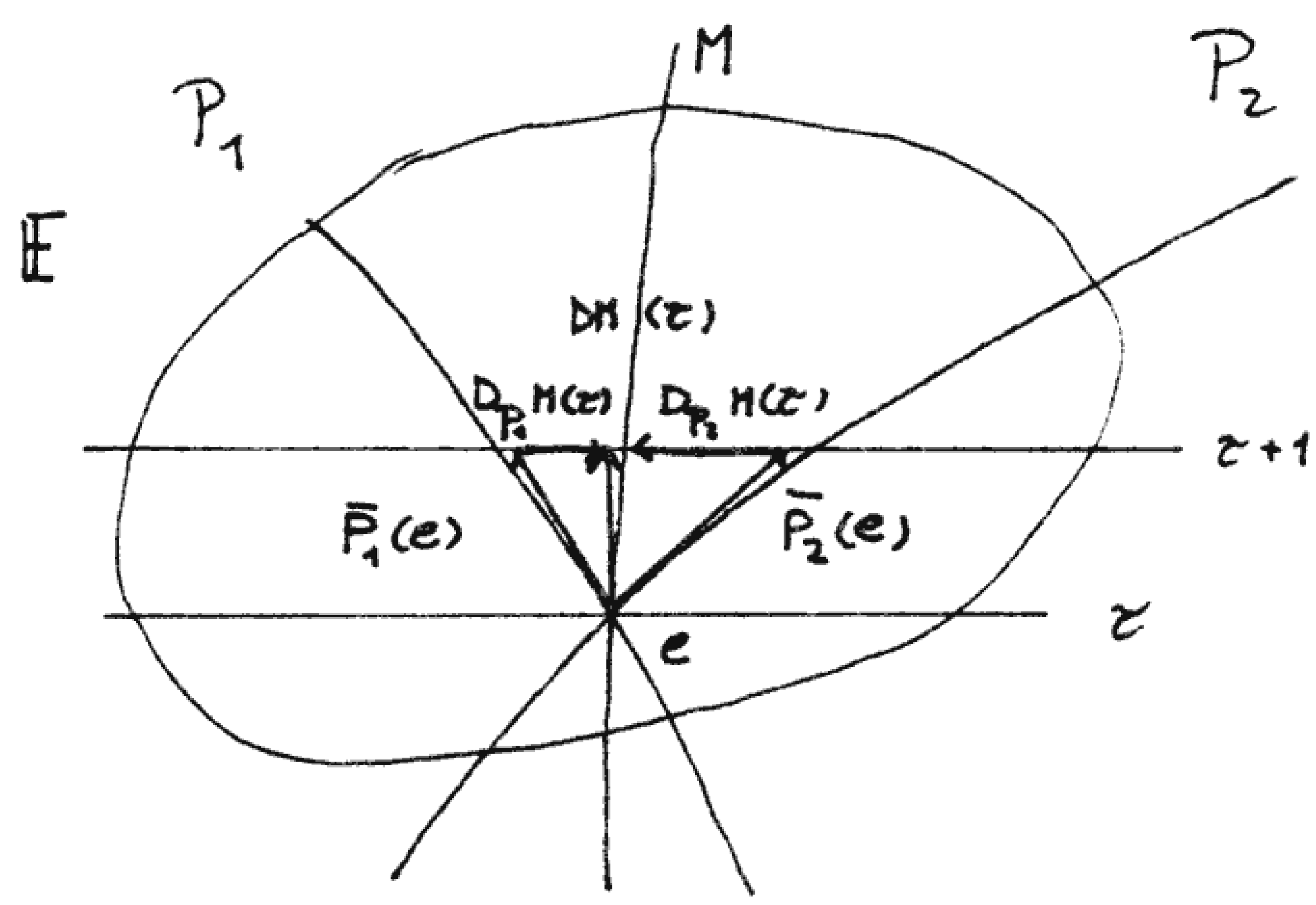
We can describe such results by a picture.



motion and velocity of $\mathcal{P}_1$ observed by $\mathcal{P}_2$



mutual motion



velocity addition