$$\mu(E)=\begin{cases} 1 & \text{if} \quad E\in\mathcal{U} \\ 0 & \text{if} \quad E\notin\mathcal{U} \end{cases}$$

is an atomic mass on $\mathcal{A}$ .

<u>Proof</u>: cfr., e.g. , [6] , p. 358.

<u>Remark</u> : According to our assumption 1), in this paper "ultrafilter", always means <u>free</u> ultrafilter, i.e. $\bigcap_{E\in\mathcal{U}} E = \emptyset$ (while a fixed, or <u>principal</u>, ultrafilter is one whose elements are the subsets of $\Omega$ containing a given point $x\in\Omega$) .

<u>Definition 5</u> - A two-valued (0 and $\mu(\Omega)$) atomic mass on $\mathcal{A}$ is called <u>ultrafilter mass</u>.

## 2. A theorem by B. de Finetti.

Given any $p\in\mathfrak{D}$ , choose $E^{(p)}\in p$ such that

$$\mu(E^{(p)}) \geq \mu(E_k) \quad ,$$

for every $E_k \in p$ $(k=1,2,...,n)$, and put

$$(3) \qquad \alpha_1 = \inf_{p\in\mathfrak{D}} \mu(E^{(p)}) \; .$$

Clearly, $\mu$ is continuous if and only if $\alpha_1 = 0$.

For the sake of completeness, we recall here a decomposition theorem, essentially given by B. de Finetti in [3] ; for a different proof, see also [13]. The one given here is a direct proof avoiding the use of the "coefficient of divisibility" introduced in [3] .

Theorem 1 - Let $\mu$ be a mass on a $\sigma$-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ . Then

$$\mu = \sum_{n=1}^{\infty} \beta_n + \mu_0 , \tag{4}$$

where each $\beta_n$ (if not null) is atomic and $\mu_0$ is continuous (or null).

Proof - If $\mu$ is continuous, there in nothing to prove, since (4) is true with $\mu_0 = \mu$ and with each $\beta_n$ null. Let now $\mu$ be non-continuous: then $\alpha_1 > 0$ and so, by (3), for every partition $p \in \mathfrak{P}$ the set $E^{(p)}$ is such that $\mu(E^{(p)}) \geq \alpha_1$, and there is a partition $p_0 \in \mathfrak{P}$ such that $\mu(E^{(p_0)}) < 2\alpha_1$ .

Let $\mathcal{E} = \{E \in p_0 : \mu(E) \geq \alpha_1\}$: there exists (again by (3), and remembering (1)) a set $E_0 \in \mathcal{E}$ such that $\mu(A) \geq \alpha_1$ for at least a proper subset $A \subset E_0$ . It follows then easily that

$$\mathcal{U}_1 = \{E \in \mathcal{A} : \mu(E \cap E_0) \geq \alpha_1\}$$

is an $\mathcal{A}$-ultrafilter over $\Omega$ (the only thing which may not be completely trivial is that $A,B \in \mathcal{U}_1$ implies $A \cap B \in \mathcal{U}_1$; but, since only one of the four subsets into which $A$ and $B$ divide $E_0$ (i.e., $(A-B) \cap E_0, (B-A) \cap E_0, A \cap B \cap E_0, E_0 - (A \cup B)$) can have a mass $\geq \alpha_1$, it is not difficult to see that such subset must necessarily be $A \cap B \cap E_o$). So the mass

$$\beta_1(E) = \begin{cases} 0 & \text{if} \quad E \notin \mathcal{U}_1 \\ \alpha_1 & \text{if} \quad E \in \mathcal{U}_1 \end{cases}$$

is atomic. Put $\mu_1 = \mu - \beta_1$; if the mass $\mu_1$ is non-continuous, then

$$\alpha_2 = \inf_{p \in \mathfrak{D}} \mu_1(E^{(p)}) > 0,$$

and so it is possible to go on in the same fashion.

After $n$ steps, we get

$$\mu_n = \mu - \sum_{k=1}^{n} \beta_k$$

and, if $\mu_n$ is continuous, eq.(4) holds with $\mu_o = \mu_n$ and with each $\beta_k$ null for $k > n$. If $\mu_n$ is non-continuous for any $n$, we get a sequence $(\beta_n)$ such that the corresponding series $\sum_{n=1}^{\infty} \beta_n(E)$ converges for every $E \in \mathcal{A}$ (since $\mu(E) < +\infty$). Then $\lim_{k\to\infty} \alpha_k = 0$, and it follows that $\mu_o = \lim_{n\to\infty} \mu_n$ is continuous. ■

3. Non atomic masses.

In the classical case of a measure, non-atomicity is equivalent to