ATOMIC, NON-ATOMIC AND CONTINUOUS FINITELY ADDITIVE MEASURES: RESULTS AND APPLICATIONS(*)

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1. Introduction and notations.

The aim of this paper is to extend (and collect in a self-contained and expository account) some of our recent results concerning a finitely additive "probability"(i.e., finite) measure μ.

Among others, new topics dealt with in this note are: the construction of countably additive sequences of sets for a non-atomic μ (and not only for a continuous one, as in [13]) and also for a particular kind of atomic μ; a deeper discussion of the case of a μ which is both non-continuous and non-atomic (studied in [1]), leading to an interesting (we hope) remark concerning the existence of measurable cardinals (Ulam's problem); moreover, we touch upon non-standard methods through finitely additive measures (as treated, e.g., in [4]) , sketching out the possibility of new trends in this field.

Let Ω be an arbitrary (infinite) set, and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ a σ-algebra. A mass is a function

$$\mu : \mathcal{A} \longmapsto \mathbf{R}$$

such that

(*) The content of this paper has been also the subject of lectures given by the third author (R. Scozzafava) during his staying at the Department of Mathematics of the Karl Marx University (Budapest), in the Summer 1978. The main results were also presented as a Short Communication at the International Congress of Mathematicians (Helsinki, 1978).

(a) $\mu(E) \geqslant 0$ for any $E \in \mathcal{A}$,

(b) $E, F \in \mathcal{A}$, $E \cap F = \emptyset \Longrightarrow \mu(E \cup F) = \mu(E) + \mu(F)$,

(c) $\mu(\Omega) < +\infty$.

In particular, μ is a (finitely additive) probability measure if

$$\mu(\Omega) = 1.$$

When axiom (b) is replaced by the (stronger) condition

(b') $F_n \in \mathcal{A}$, $F_i \cap F_j = \emptyset$ for $i \neq j \Longrightarrow \mu(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n)$,

we shall call μ (in short) a <u>measure</u>.

We may (and will) assume throughout the paper that, for every $x \in \Omega$,

(1) $$\mu(\{x\}) = 0 ,$$

since (if not) we might subtract from μ the (trivial) <u>measure</u>

$$m(E) = \begin{cases} 0 & \text{if} \quad E \subseteq \Omega - A_0 \\ \mu(E \cap A_0) & \text{if} \quad E \cap A_0 \neq \emptyset \end{cases} ,$$

where

$$A_0 = \{x \in \Omega : \mu(\{x\}) > 0\}$$

is (as it is well known) a countable set.

<u>Proposition 1</u> - Let μ be a mass on a σ-algebra $\mathcal{A}$. Then, for every sequence $A_n \in \mathcal{A}$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, we have

(2) $$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty} \mu(A_n) .$$

Proof. Apply (b) to the n+1 sets $A_1, A_2, \ldots, A_n, \bigcup_{k=n+1}^{\infty} A_k$, and then apply (a) to the latter. ■

Definition 1 - A mass is null if $\mu(E) = 0$ for each $E \in \mathcal{A}$ (i.e., if $\mu(\Omega) = 0$).

Definition 2 - An atom for a mass μ is a set $A \in \mathcal{A}$ such that $\mu(A) > 0$ and $\mu(\mathcal{A} \cap \mathcal{P}(A)) = \{0, \mu(A)\}$.

Definition 3 - A mass μ on $\mathcal{A}$ is atomic if there exists an atom for μ.

We denote by $p = \{E_1, E_2, \ldots, E_n\}$ any finite partition of Ω, and by $\mathfrak{D}$ the set of all measurable (i.e., with every $E_n \in \mathcal{A}$) p's.

Definition 4 - A mass μ on $\mathcal{A}$ is continuous (or strongly non-atomic) if, given $\varepsilon > 0$, there exists $\{E_1, E_2, \ldots, E_n\} = p \in \mathfrak{D}$ such that $\mu(E_k) < \varepsilon$ $(k = 1, 2, \ldots, n)$.

Proposition 2 - If a mass μ on $\mathcal{A}$ is continuous, then μ is non-atomic.

Proof. Obvious. ■

Proposition 3 - If A is an atom for μ, then

$$\mathcal{U} = \{E \in \mathcal{A} : \mu(E \cap A) = \mu(A)\}$$

is an $\mathcal{A}$-ultrafilter over Ω, i.e. an ultrafilter over Ω whose elements belong to $\mathcal{A}$. Conversely, given any $\mathcal{A}$-ultrafilter $\mathcal{U}$ over Ω, the set function

$$\mu(E)=\begin{cases} 1 & \text{if} \quad E\in\mathcal{U} \\ 0 & \text{if} \quad E\notin\mathcal{U} \end{cases}$$

is an atomic mass on $\mathcal{A}$.

<u>Proof</u>: cfr., e.g. , [6] , p. 358.

<u>Remark</u> : According to our assumption 1), in this paper "ultrafilter", always means <u>free</u> ultrafilter, i.e. $\bigcap_{E\in\mathcal{U}} E = \emptyset$ (while a fixed, or <u>principal</u>, ultrafilter is one whose elements are the subsets of Ω containing a given point $x\in\Omega$) .

<u>Definition 5</u> - A two-valued (0 and $\mu(\Omega)$) atomic mass on $\mathcal{A}$ is called <u>ultrafilter mass</u>.

2. <u>A theorem by B. de Finetti</u>.

Given any $p\in\mathfrak{D}$, choose $E^{(p)}\in p$ such that

$$\mu(E^{(p)}) \geq \mu(E_k) \quad ,$$

for every $E_k \in p$ $(k=1,2,...,n)$, and put

(3) $$\alpha_1 = \inf_{p\in\mathfrak{D}} \mu(E^{(p)}) \; .$$

Clearly, μ is continuous if and only if $\alpha_1 = 0$.