ATOMIC, NONATOMIC AND CONTINUOUS FINITELY ADDITIVE MEASURES: RESULTS AND APPLICATIONS

E. BARONE - A. GIANNONE - R. SCOZZAFAVA

Sommario - *Scopo di questo lavoro è di estendere (e di riunire in una esposizione completa) alcuni dei nostri recenti risultati sulle misure finite semplicemente additive.*

ATOMIC, NON-ATOMIC AND CONTINUOUS FINITELY ADDITIVE MEASURES: RESULTS AND APPLICATIONS(*)

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1. Introduction and notations.

The aim of this paper is to extend (and collect in a self-contained and expository account) some of our recent results concerning a finitely additive "probability"(i.e., finite) measure μ.

Among others, new topics dealt with in this note are: the construction of countably additive sequences of sets for a non-atomic μ (and not only for a continuous one, as in [13]) and also for a particular kind of atomic μ; a deeper discussion of the case of a μ which is both non-continuous and non-atomic (studied in [1]), leading to an interesting (we hope) remark concerning the existence of measurable cardinals (Ulam's problem); moreover, we touch upon non-standard methods through finitely additive measures (as treated, e.g., in [4]) , sketching out the possibility of new trends in this field.

Let Ω be an arbitrary (infinite) set, and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ a σ-algebra. A mass is a function

$$\mu : \mathcal{A} \mapsto \mathbf{R}$$

such that

(*) The content of this paper has been also the subject of lectures given by the third author (R. Scozzafava) during his staying at the Department of Mathematics of the Karl Marx University (Budapest), in the Summer 1978. The main results were also presented as a Short Communication at the International Congress of Mathematicians (Helsinki, 1978).

(a) $\mu(E) \geq 0$ for any $E \in \mathcal{A}$,

(b) $E, F \in \mathcal{A}$, $E \cap F = \emptyset \implies \mu(E \cup F) = \mu(E) + \mu(F)$,

(c) $\mu(\Omega) < +\infty$.

In particular, μ is a (finitely additive) probability measure if

$$\mu(\Omega) = 1.$$

When axiom (b) is replaced by the (stronger) condition

(b') $F_n \in \mathcal{A}$, $F_i \cap F_j = \emptyset$ for $i \neq j \implies \mu(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n)$,

we shall call μ (in short) a measure.

We may (and will) assume throughout the paper that, for every $x \in \Omega$,

$$\mu(\{x\}) = 0 \quad , \tag{1}$$

since (if not) we might subtract from μ the (trivial) measure

$$m(E) = \begin{cases} 0 & \text{if} \quad E \subseteq \Omega - A_0 \\ \mu(E \cap A_0) & \text{if} \quad E \cap A_0 \neq \emptyset \end{cases} ,$$

where

$$A_0 = \{x \in \Omega : \mu(\{x\}) > 0\}$$

is (as it is well known) a countable set.

Proposition 1 - Let μ be a mass on a σ-algebra $\mathcal{A}$. Then, for every sequence $A_n \in \mathcal{A}$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, we have

(2) $$\mu(\bigcup_{n=1}^{\infty} A_n) \geq \sum_{n=1}^{\infty} \mu(A_n) .$$

Proof. Apply (b) to the n+1 sets $A_1, A_2, \dots, A_n, \bigcup_{k=n+1}^{\infty} A_k$, and then apply (a) to the latter. ■

Definition 1 - A mass is null if $\mu(E) = 0$ for each $E \in \mathcal{A}$ (i.e., if $\mu(\Omega) = 0$).

Definition 2 - An atom for a mass μ is a set $A \in \mathcal{A}$ such that $\mu(A) > 0$ and $\mu(\mathcal{A} \cap \mathcal{P}(A)) = \{0, \mu(A)\}$.

Definition 3 - A mass μ on $\mathcal{A}$ is atomic if there exists an atom for μ.

We denote by $p = \{E_1, E_2, \dots, E_n\}$ any finite partition of Ω, and by $\mathfrak{D}$ the set of all measurable (i.e., with every $E_n \in \mathcal{A}$) p's.

Definition 4 - A mass μ on $\mathcal{A}$ is continuous (or strongly non-atomic) if, given $\varepsilon > 0$, there exists $\{E_1, E_2, \dots, E_n\} = p \in \mathfrak{D}$ such that $\mu(E_k) < \varepsilon$ $(k = 1, 2, \dots, n)$.

Proposition 2 - If a mass μ on $\mathcal{A}$ is continuous, then μ is non-atomic.

Proof. Obvious. ■

Proposition 3 - If A is an atom for μ, then

$$\mathcal{U} = \{E \in \mathcal{A} : \mu(E \cap A) = \mu(A)\}$$

is an $\mathcal{A}$-ultrafilter over Ω, i.e. an ultrafilter over Ω whose elements belong to $\mathcal{A}$. Conversely, given any $\mathcal{A}$-ultrafilter $\mathcal{U}$ over Ω, the set function

$$\mu(E) = \begin{cases} 1 & \text{if} \quad E \in \mathcal{U} \\ 0 & \text{if} \quad E \notin \mathcal{U} \end{cases}$$

is an atomic mass on $\mathcal{A}$.

Proof: cfr., e.g. , [6] , p. 358.

Remark : According to our assumption 1), in this paper "ultrafilter", always means free ultrafilter, i.e. $\bigcap_{E \in \mathcal{U}} E = \emptyset$ (while a fixed, or principal, ultrafilter is one whose elements are the subsets of Ω containing a given point $x \in \Omega$) .

Definition 5 - A two-valued (0 and $\mu(\Omega)$) atomic mass on $\mathcal{A}$ is called ultrafilter mass.

2. A theorem by B. de Finetti.

Given any $p \in \mathcal{D}$, choose $E^{(p)} \in p$ such that

$$\mu(E^{(p)}) \geq \mu(E_k) \quad ,$$

for every $E_k \in p$ $(k=1,2,\ldots,n)$, and put

(3) $$\alpha_1 = \inf_{p \in \mathcal{D}} \mu(E^{(p)}) \ .$$

Clearly, μ is continuous if and only if $\alpha_1 = 0$.

$$\mu(E) = \begin{cases} 1 & \text{if} \quad E \in \mathcal{U} \\ 0 & \text{if} \quad E \notin \mathcal{U} \end{cases}$$

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$$\alpha_1 = \inf_{p \in \mathfrak{D}} \mu(E^{(p)}) \quad . \tag{3}$$

Clearly, μ is continuous if and only if $\alpha_1 = 0$.

For the sake of completeness, we recall here a decomposition theorem, essentially given by B. de Finetti in [3] ; for a different proof, see also [13]. The one given here is a direct proof avoiding the use of the "coefficient of divisibility" introduced in [3] .

Theorem 1 - Let μ be a mass on a σ-algebra $\mathcal{A} \subseteq \wp(\Omega)$. Then

$$\mu = \sum_{n=1}^{\infty} \beta_n + \mu_0 , \tag{4}$$

where each β_n (if not null) is atomic and μ_0 is continuous (or null).

Proof - If μ is continuous, there in nothing to prove, since (4) is true with $\mu_0 = \mu$ and with each β_n null. Let now μ be non-continuous: then $\alpha_1 > 0$ and so, by (3), for every partition $p \in \mathfrak{P}$ the set $E^{(p)}$ is such that $\mu(E^{(p)}) \geq \alpha_1$, and there is a partition $p_0 \in \mathfrak{P}$ such that $\mu(E^{(p_0)}) < 2\alpha_1$.

Let $\mathcal{E} = \{E \epsilon p_0 : \mu(E) \geq \alpha_1\}$: there exists (again by (3), and remembering (1)) a set $E_0 \epsilon \mathcal{E}$ such that $\mu(A) \geq \alpha_1$ for at least a proper subset $A \subset E_0$. It follows then easily that

$$\mathcal{U}_1 = \{E \in \mathcal{A} : \mu(E \cap E_0) \geq \alpha_1\}$$

is an $\mathcal{A}$-ultrafilter over Ω (the only thing which may not be completely trivial is that $A,B \in \mathcal{U}_1$ implies $A \cap B \in \mathcal{U}_1$; but, since only one of the four subsets into which A and B divide E_0 (i.e., $(A-B) \cap E_0, (B-A) \cap E_0, A \cap B \cap E_0, E_0 - (A \cup B))$ can have a mass $\geq \alpha_1$, it is not difficult to see that such subset must necessarily be $A \cap B \cap E_0$). So the mass

$$\beta_1(E) = \begin{cases} 0 & \text{if} \quad E \notin \mathcal{U}_1 \\ \alpha_1 & \text{if} \quad E \in \mathcal{U}_1 \end{cases}$$

is atomic. Put $\mu_1 = \mu - \beta_1$; if the mass μ_1 is non-continuous, then

$$\alpha_2 = \inf_{p \in \mathfrak{D}} \mu_1(E^{(p)}) > 0,$$

and so it is possible to go on in the same fashion.

After n steps, we get

$$\mu_n = \mu - \sum_{k=1}^{n} \beta_k$$

and, if μ_n is continuous, eq.(4) holds with $\mu_o = \mu_n$ and with each β_k null for $k > n$. If μ_n is non-continuous for any n, we get a sequence (β_n) such that the corresponding series $\sum_{n=1}^{\infty} \beta_n(E)$ converges for every $E \in \mathcal{A}$ (since $\mu(E) < +\infty$). Then $\lim_{k \to \infty} \alpha_k = 0$, and it follows that

$\mu_o = \lim_{n \to \infty} \mu_n$ is continuous. ■

3. Non atomic masses.

In the classical case of a measure, non-atomicity is equivalent to continuity. This can be seen, for example, as an easy consequence of the following

Theorem 2 (Saks) - Let μ be a measure on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Given any $\varepsilon > 0$, there exists $p \in \mathfrak{D}$ such that each $E_k \in p$ is either an atom or $\mu(E_k) < \varepsilon$.

Proof : see [2], p. 308 . ■

This equivalence cannot be carried over to the general case of a mass: in [1] it is shown (cfr. also the following Theorem 5) that, if $\mu=\nu+\lambda$, where ν is atomic and λ is continuous, with $\nu\ll\lambda$ (i.e. ν is absolutely continuous with respect to λ), then μ is both non-atomic and non-continuous.

In other words, while atomic masses are necessarily non-continuous (see Proposition 2), non-atomic ones can be either continuous or not.

For continuous masses, the following result has been established in [13]:

Theorem 3 - Let μ be a continuous mass on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Then there exists a sequence (F_n) of mutually disjoint measurable sets, with $\mu(F_n) > 0$, such that $\sum_{n=1}^{\infty}\mu(F_n)$ equals any preassigned α , with $0 < \alpha < \mu(\Omega)$, and

$$(5) \qquad \alpha = \sum_{n=1}^{\infty}\mu(F_n) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) .$$

So to say, the mass μ "behaves" like a measure on each collection (F_n) : then it would seem interesting a deeper investigation of the family (or of some suitable subfamily) of all such sets obtained when α ranges in the open interval from 0 to $\mu(\Omega)$.

A simple corollary of Theorem 3 is the following: the range of a continuous μ is the whole interval $[0,\mu(\Omega)]$. The latter statement is no langer true if μ is non-continuous: a counterexample is given in [1]; moreover, there it is shown that the range of μ need not even be a closed subset of $\mathbb{R}$, contrary to the classical case of a measure.

As far as continuous masses are concerned, let us quote also a recent result obtained through non-standard methods: a necessary and sufficient condition for the existence of a continuous mass, which is invariant for a transformation of Ω into itself, is given in [16].

We want now to extend Theorem 3 to the more general case of a non-atomic μ: the previous remarks show that we can hope, at most, in countable additivity on a suitable sequence of sets (and not also, as in eq. (5), in a beforehand given value of α).

Theorem 4 - Let μ be a non-atomic mass on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Then there exists a sequence (A_n) of mutually disjoint measurable sets, with $\mu(A_n) > 0$, such that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \ . \qquad (6)$$

Proof - Let $B \in \mathcal{A}$, with $0 < \mu(B) < \mu(\Omega)$: then also $B' = \Omega - B$ satisfies $0 < \mu(B') < \mu(\Omega)$. At least one of them (call it B_1) is such that $\mu(B_1) \leq \frac{1}{2}\mu(\Omega)$. Now, let $B_2 \in \mathcal{A}$, $B_2 \subset B_1$, be such that $0<\mu(B_2)<\mu(B_1)$ and $\mu(B_2) \leq \frac{1}{2}\mu(B_1)$. In general, we define $B_n \subset B_{n-1}$, with $0<\mu(B_n)<\mu(B_{n-1})$ and $\mu(B_n) \leq \frac{1}{2}\mu(B_{n-1}) \leq \frac{1}{2^n}\mu(\Omega)$.

Put $A_n = B_n - B_{n+1}$: we have $\mu(A_n) = \mu(B_n) - \mu(B_{n+1}) > 0$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Moreover

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{k} \mu(A_n) + \mu(\bigcup_{n=k+1}^{\infty} A_n) =$$

$$= \sum_{n=1}^{k} \mu(A_n) + \mu(B_{k+1}) \leq \sum_{n=1}^{k} \mu(A_n) + \frac{1}{2^{k+1}} \mu(\Omega) .$$

As $k \to \infty$, we get

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n) ,$$

which, taking into account Proposition 1, gives (6). ■

The next theorem will enable us to extend (in Section 4) the previous result also to a particular class of atomic masses.

<u>Theorem 5</u> - Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a σ-algebra, ν an atomic mass on $\mathcal{A}$, and λ a continuous mass on $\mathcal{A}$ such that $\lambda(A) > 0$ for any atom A of ν (e.g., such that $\nu \ll \lambda$). Then $\mu = \nu + \lambda$ is non atomic and non-continuous.

<u>Proof</u> : see [1]. ■

<u>Remark</u> - Put $\nu = \sum_n \beta_n$ in Theorem 1, eq.(4): ν need not be atomic (an example is given in [10] , p. 47), and so we may have masses which are at the same time non-atomic and non-continuous, but <u>not</u> of the form given by Theorem 5.

Corollary 1 - Let β be an ultrafilter mass and λ a continuous mass on $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, with $\beta \ll \lambda$. Then $\mu = \beta + \lambda$ is non-atomic and non-continuous.

Corollary 2 - Let β be an ultrafilter mass on $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ such that $\beta \ll \lambda$, where λ is a continuous measure on $\mathcal{A}$. Then β cannot be a measure on $\mathcal{A}$.

Remark - It is interesting also to look at Theorem 5 as another counterexample to known results for measures: in [7] it is shown that, given two measures λ and ν, with $\nu \ll \lambda$ and λ non atomic (i.e. continuous), then ν also is non-atomic. Actually, this need not be true if ν is only a mass (and not a measure), for example if it is an ultrafilter mass β, as that of Corollary 2. The existence of such a mass (given λ) can be proved (cfr. [1]) taking an $\mathcal{A}$-ultrafilter containing the filter

$$\mathcal{F} = \{E \in \mathcal{A} : \lambda(E) = \lambda(\Omega)\} .$$

4. Atomic masses and measurable cardinals.

Since the mass μ occurring in Theorem 5 is non-atomic (and non-continuous), Theorem 4 can be suitably applied to it, giving easily a countably additive sequence of sets also for the atomic mass ν.

Theorem 6 - Let ν be an atomic mass on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, such that $\nu \ll \lambda$, where λ is a continuous measure on $\mathcal{A}$. Then there exists a sequence (A_n) of mutually disjoint measurable sets, such that

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n) .$$

Proof - Use Theorem 4 for $\mu = \nu + \lambda$, taking into account the countable additivity of λ. ∎

Now, in order to deal with the so-called "Ulam's measure problem", we recall some known facts about ultrafilter over a set Ω; we limit owrselves to free ultrafilters(cfr. the remark following Proposition 3).

Definition 6 - An ultrafilter $\mathcal{U}$ over Ω is δ-complete if, given any sequence of sets $A_n \in \mathcal{U}$, one has $\bigcap_{n=1}^{\infty} A_n \in \mathcal{U}$.

Proposition 4 - Let β be an ultrafilter mass on $\mathcal{A} = \mathcal{P}(\Omega)$, and let $\mathcal{U}$ be the corresponding (free) ultrafilter. Then β is a measure if and only if $\mathcal{U}$ is δ-complete.

Proof - Countable additivity of β implies that, given any sequence of sets $A_n \in \mathcal{U}$, for $A'_n = \Omega - A_n \notin \mathcal{U}$ we must have $\bigcup_{n=1}^{\infty} A'_n \notin \mathcal{U}$. Therefore $\Omega - \bigcup_{n=1}^{\infty} A'_n = \bigcap_{n=1}^{\infty} A_n \in \mathcal{U}$, i.e. $\mathcal{U}$ is δ-complete. The converse is also easily seen, since β is two-valued.

Definition 7 - Let Ω be a set: card Ω is said measurable when there exists a δ-complete free ultrafilter over Ω.

Corollary 3 - An ultrafilter measure exists on $\mathcal{A} = \mathcal{P}(\Omega)$ if and only if card Ω is measurable.

(Notice that the latter measure is finite,defined for all subsets of Ω, and zero on singletons).

The question concerning the existence of measurable cardinals (known also under the name of Ulam's measure problem) cannot be settled in ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice).

It was shown that a measurable cardinal (assuming its existence) must be very large and, in fact, must be an inaccessible cardinal: really, if k is a measurable cardinal, then there are k inaccessible cardinals preceding it (cfr., e.g., [11], p. 26 and [14], p. 26).

Moreover, the existence of a measurable cardinal settles many mathematical problems: see [8].

On the other hand, if we assume that "all" sets are constructible (the so-called "axiom of constructibility" V = L), no measurable cardinal exists: in fact, if there is a measurable cardinal, then V = L " is as false as it possibly can be" (cfr. [14], p. 31).

Notice that, by Corollary 3, the existence of an ultrafilter measure on $\mathcal{P}(\Omega)$ is equivalent to the statement that card Ω is measurable, while an ultrafilter mass always exists, by a classical result due to Tarski [15].

Proposition 5 - Let card $\Omega = \mathfrak{c}$ and assume the continuum hypothesis (CH). Then no ultrafilter measure exists on $\mathcal{P}(\Omega)$ (i.e., under CH, $\mathfrak{c}$ is not a measurable cardinal).

Proof - See [17] or [11]. ■

We point out that Corollary 2 (cfr. Section 3) gives non-existence of a particular class of ultrafilter measures, without any assumption on the cardinality of Ω.

We end this Section with a necessary condition for a cardinal to be measurable, which gives an interesting remark to Ulam's measure problem; we state first the following obvious

Lemma - Let Ω be a set such that card Ω is measurable, and let β be the corresponding ultrafilter measure. Then $\beta(E)>0$ implies card $E > \aleph_0$.

Theorem 7 - Let β be an ultrafilter measure on $\mathcal{P}(\Omega)$. Then, given any continuous measure λ on $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, necessarily $\beta \perp \lambda$ (i.e., β is singular with respect to λ) and there are sets $E \subset \Omega$, with card $E>\aleph_0$, such that $\lambda(E) = 0$.

Proof - It is essentially a reformulation of Corollary 2, taking into account the preceding Lemma. ■

Remark - Theorem 7 can be looked at to give some grounds for the acceptance or not of the axiom concerning the existence of measurable cardinals: for example, if we assume that, given a set Ω, there exists at least a continuous measure on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, vanishing only on countable (*) sets, then card Ω is not measurable.

This result is also a partial converse to a theorem given by Ulam (cfr. Satz 2, p. 147) in [17]: he proved that, if card Ω is not measurable and there exists a measure on Ω, then this measure is necessarily continuous.

(*) Here it would be possible to replace "countable sets" by "sets of cardinality less than card Ω", just using a suitable definition of measure, in which countable additivity is replaced by the "natural" stronger requirement (cfr. [14], p. 20).

5. Non-standard methods.

It is well-known that ultrafilter masses (or measures: this stronger terminology may be used when the cardinality of the index set is measurable) are a tool in non-standard analysis for the construction of the relevant superstructure.

In [4] it is shown how some fundamental ideas in this field may be introduced through a finitely additive probability measure μ (i.e., a mass) on the index set, extending the concept of ultrapower to that of "μ-power".

To facilitate the exposition, the attention was confined to a structure $\mathcal{E} = \langle E, \mathbb{R} \rangle$, consisting of a non-empty set E and a set $\mathbb{R}$ of relations on E. Given an index set J, let μ be a mass on $\mathcal{P}(J)$, with $\mu(J) = 1$: the superstructure ${}^*\mathcal{E} = \langle {}^*E, {}^*\mathbb{R} \rangle$ consists of the set ${}^*E$ of all functions $f : J \to E$ (modulo μ-null sets) and any relation $R \in {}^*\mathbb{R}$ "is true" (loosely speaking) in the "model" ${}^*\mathcal{E}$ if and only if it is true in $\mathcal{E}$ for almost all $j \in J$ (here each value of one of the "equivalent" functions, with domain J, defining an element of ${}^*E$, is a point of E; ${}^*E$ is a proper extension of E, by virtue of condition (1) : cfr.[4]) .

Let us consider, to be definite, the structure given by the ordered field $\mathbb{R}$. It is clear that, using μ-powers instead of ultrapowers (i.e., arbitrary masses on J instead of ultrafilter ones), ${}^*\mathbb{R}$ is not necessarily a field (and, moreover, it is only partially ordered): take, for example, $A \subset J$ with $0 < \mu(A) < 1$. Its characteristic function χ_A is not equivalent to the null function 0, and so gives rise, in ${}^*\mathcal{E}$, to an element (i.e., an equivalence class modulo μ-null sets) $[\chi_A]$ which is <u>not</u> $^*0 = [0]$; the same is true for χ_{J-A} . But $\chi_A \cdot \chi_{J-A} \equiv 0$, and so in $^*\mathcal{E}$ the product $[\chi_A] \cdot [\chi_{J-A}]$ of this two non-null elements is null (i.e., $^*R$ has zero divisors).

This fact, from the usual point of view adopted in the costruction of non-standard models of the reals, should be considered a "defect", since it is possible (as it is well known) to build up an enlargement $^*R$ which is an ordered field (though, of course, a non-archimedean one). But it is possible to look at the question from different viewpoints, similar (apart from the dropping out of the condition of σ-additivity for the measure on the index set) to those sketched by D . Scott in [12] .

A problem of interest in probability theory is the following: if we take μ to be a <u>measure</u>, it does <u>not</u> exist a denumerable "uniform" (and measurable) partition of the index set $J = \bigcup_{n=1}^{\infty} E_n$, with $\mu(E_n)=0$ for each n, otherwise

$$0 = \sum_{n=1}^{\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n) = 1 \quad \text{(impossible)}.$$

On the other hand, if μ is only a <u>mass</u> and such a partition exists, the latter inequality is consistent with Prop. 1; moreover we show the possibility of looking at it as a sort of "non-standard" countable additivity.

To begin with, we may give a meaning to $\sum_{n=1}^{\infty} {}^*a_n$ (with $^*a_n \in {}^*R$) by choosing <u>suitable</u> representatives a_n of $^*a_n$ (n = 1,2,...) such that $\sum_{n=1}^{\infty} a_n(i)$ converges for almost all $i \in J$, and by putting

(7) $$\sum_{n=1}^{\infty}{}^{*}a_n = \left[\left(\sum_{n=1}^{\infty}a_n(i)\right)_{i\in J}\right] \quad .$$

Now, we remark that, since $\chi_{E_n} = 0$ for almost all $j \in J$, we have $\left[\chi_{E_n}\right] = {}^{*}0$; on the other hand, $\sum_{n=1}^{\infty}\chi_{E_n} = \chi_J \equiv 1$, and so $[\chi_J] = {}^{*}1$.

If we apply (7) to $\sum_{n=1}^{\infty}{}^{*}0 = \sum_{n=1}^{\infty}[\chi_{E_n}]$, choosing χ_{E_n} as representative of $[\chi_{E_n}]$, we get

$$\sum_{n=1}^{\infty}{}^{*}0 = \sum_{n=1}^{\infty}[\chi_{E_n}] = \left[\sum_{n=1}^{\infty}\chi_{E_n}\right] = [\chi_J] = {}^{*}1 \ .$$

So an uniform probability distribution on a countable set does not conflict, in $^{*}\mathbb{R}$, with "countable additivity" of μ .

We point out that this approach differs from the well-know one (see, e.g., [5] , [9]) through $*$-finite sets: is there some hope that such "models"would open new trends in this field?

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