$= \emptyset$, *then there exists a vicinity* $W \in \mathcal{W}$ *such that* $X_1, \ldots, X_n$ *are* W-*enlargable.*

Proof. - We suppose all the sets X_i are non-empty, otherwise the proposition is trivial. Since S is compact, $\forall i = 1, \ldots, n$, the family $\{W(\overline{X}_i)\}$, $\forall W \in \mathcal{W}$, constitute a basis of the neighbourhoods filter of $\overline{X}_i$ (see [2], Cap. 2, §4, n° 3); moreover, since S is normal, the neighbourhoods filter of $\overline{X}_i$ is closed. Consequently, $\{W(\overline{X}_1) \cap \ldots \cap W(\overline{X}_n)\}$ $\forall W \in \mathcal{W}$ is the basis of a closed filter $\mathfrak{F}$. Now, if $\mathfrak{F}$ is the null filter, there exists $W \in \mathfrak{F}$ such that $W(\overline{X}_1) \cap \ldots \cap W(\overline{X}_n) = \emptyset = W(X_1) \cap \ldots \cap W(X_n)$, i.e. $X_1, \ldots, X_n$ are W-enlargable. Otherwise, since S is compact, there exists a point x adherent to $\mathfrak{F}$, and since $\mathfrak{F}$ is a closed filter, $x \in W(\overline{X}_1) \cap \ldots \cap W(\overline{X}_n)$, $\forall W \in \mathcal{W}$. Then it is $x \in W(\overline{X}_i)$, $\forall W \in \mathcal{W}$, $i = 1, \ldots, n$. As the sets $W(\overline{X}_i)$ constitute a basis of the neighbourhoods filter of $\overline{X}_i$, it follows $x \in \overline{X}_i$, $i = 1, \ldots, n$, i.e. $x \in \overline{X}_1 \cap \ldots \cap \overline{X}_n$. Contradiction

COROLLARY 2. - *Let* S *be a compact metric space and* $X_1, \ldots, X_n$ *subsets of* S *such that* $\overline{X}_1 \cap \ldots \cap \overline{X}_n = \emptyset$, *then it is* $enl(X_1, \ldots, X_n) > 0$. □

2) *The second normalization theorem.*

DEFINITION 4. - *Let* A *be a non-empty set,* G *a finite graph and* $P = \{X_j\}$, $j \in J$, *a partition of* A. *A function* $f: A \to G$ *is called* quasi constant with respect to P (w.r.t. P) *or* P-constant *if the restrictions of* f *to each* X_j *are constant functions. Moreover , if* A *is a topological space ,* $f: A \to G$ *is called* weakly quasi-constant w.r.t. P *or* **weakly** P-costant *if the restrictions of* f *to the interior of every* X_j *are constant.*

REMARK. - If $P' = \{X'_k\}$, $k \in K$, is a partition of A finer than P, i.e. if all the $X_i \in P$ are the union of elements $X'_k \in P'$, then the function f is obviously quasi-contant also w.r.t. P'.

DEFINITION 5. - *Let* $(S, \mathcal{W})$ *be a uniform space and* W *a vicinity of* $\mathcal{W}$. *A subset* X *of* S *is called* small of order W *or a* W-subset *if* $X \times X \subseteq W$. *Moreover a family* $\mathcal{X} = \{X_j\}$, $j \in J$, *is called* small of order W *or a* W-family *if* $X_j \times X_j \subseteq W$, $\forall j \in J$.

REMARK 1. - If W is closed and $\{X_j\}$, $j \in J$, is a W-family, $\{\overline{X}_j\}$, $j \in J$, is a W-family .

REMARK 2. - If S is metric, small of order W^{ε} is the same as saying that the diameter of X is $< \varepsilon$ and, respectively, the mesh of the family $\mathcal{X}$ is $< \varepsilon$.

THEOREM 3. - (The second normalization theorem). *Let S be a compact space, the filter $\mathcal{W}$ the uniformity of S, G a finite directed graph and $f: S \to G$ a completely o-regular function from S to G. Then there exists a vicinity $W \in \mathcal{W}$ such that, for all the W-partitions $P = \{X_j\}$, $j \in J$, there exists a function $h: S \to G$ which is completely o-regular, weakly P-constant and completely o-homotopic to f.*

Proof. - Consider all the n-tuples $a_1, \dots, a_n$, $n \geqslant 2$, non-headed in G. Since f is c. o-regular, it follows $\overline{A}_1 \cap \dots \cap \overline{A}_n = \emptyset$. By Proposition 1, for every n-tuple $a_1, \dots, a_n$, there exists a vicinity $V^{(a_1, \dots, a_n)} \in \mathcal{W}$ such that $A_1, \dots, A_n$ are $V^{(a_1, \dots, a_n)}$-enlargable. Then we put $V = \bigcap V^{(a_1, \dots, a_n)}$ and consider a simmetric vicinity $W \in \mathcal{W}$ such that $W \circ W \subseteq V$. Now, if $P = \{X_j\}$, $j \in J$, is a W-partition, we can define a relation $g: S \to G$, by putting, as constant value, for every $X_j \in P$, any vertex of $H(\{f(X_j)\})$. We prove that g satisfies the following conditions:

i) g is a function. We have only to state that, for all X_j, the set $\{f(X_j)\} = \{a_1, \dots, a_n\}$ is headed. Suppose it is non-headed, and let $x_1, \dots, x_n \in X_j$ be, such that $f(x_1) = a_1, \dots, f(x_n) = a_n$. Since $X_j \times X_j \subseteq W$ it follows $(x_r, x_s) \in W$, $r, s = 1, \dots, n$, and also $x_1 \in W(x_1) \cap \dots \cap W(x_n) \subseteq V(A_1) \cap \dots \cap V(A_n)$. Contradiction.

ii) g is completely quasi-regular, i.e. $\forall x \in S$ the image-envelope $\langle g(x) \rangle$ is totally-headed. Suppose there exists $x \in S$ and a n-tuple $a_1, \dots, a_n \in \langle g(x) \rangle$ non-headed. Then it results $x \in \overline{A}_1^g \cap \dots \cap \overline{A}_n^g$ and so $W(x) \cap A_i^g \neq \emptyset$, $\forall i = 1, \dots, n$. Hence in $W(x)$ there are n points $x_1, \dots, x_n$ such that $x_i \in A_i^g$, $\forall i = 1, \dots, n$. But, from the definition of g, there exist n elements $X_i \in P$ and n points y_i such that $g(x_i) = a_i = f(y_i)$, $\forall i = 1, \dots, n$ where $x_i, y_i \in X_i$. Since P is a W-partition, we have $(x_i, y_i) \in W$. Therefore by $(x, x_i) \in W$, $(x_i, y_i) \in W$ and $W \circ W \subseteq V$, $\forall i = 1, \dots, n$, it results $x \in V(y_1) \cap \dots \cap V(y_n) \subseteq V(A_1) \cap \dots \cap V(A_n)$. Contradiction.

iii) The function $F: S \times I \to G$, given by:

$$F(x,t) = \begin{cases} f(x) & \forall x \in S, \quad \forall t \in [0, \tfrac{1}{2}[\\ g(x) & \forall x \in S, \quad \forall t \in [\tfrac{1}{2}, 1] \end{cases}$$

is completely quasi-regular. This is true $\forall x \in S$, $\forall t \neq \frac{1}{2}$, since f and g are completely quasi-regular functions. We have to prove this also $\forall x \in S$, $t = \frac{1}{2}$, i.e. that $\langle F(x,t) \rangle = \langle f(x) \rangle \cup \langle g(x) \rangle$ is totally headed. Suppose $x \in S$ and let $a_1, \dots, a_n \in \langle f(x) \rangle \cup \langle g(x) \rangle$ be a n-tuple non-headed.

We can order the a_i in such a way that $a_1,\dots,a_p \in \langle f(x)\rangle$ and $a_{p+1},\dots,a_n \in \langle f(x)\rangle - \langle g(x)\rangle$. Therefore it is $x \in \bar{A}_1^f \cap \dots \cap \bar{A}_p^f$, and so $W(x) \cap A_i^f \neq \emptyset$, $\forall i = 1,\dots,p$. Hence there are in $W(x)$ p points $x_1,\dots,x_p$ such that $x_i \in A_i^f$, $\forall i = 1,\dots,p$. Then it is $x \in W(x_1) \cap \dots \cap W(x_p) \subseteq V(A_1^f) \cap \dots \cap V(A_p^f)$. Moreover it is $x \in \bar{A}_{p+1}^g \cap \dots \cap \bar{A}_n^g$, and by *ii)* it follows $x \in V(A_{p+1}^f) \cap \dots \cap V(A_n^f)$. Hence we obtain the contradiction $x \in V(A_1^f) \cap \dots \cap V(A_n^f)$.

Now if we consider any o-pattern h of g, we obtain the sought functio In fact we have:

i') $h: S \to G$ is completely o-regular (see [5], Proposition 7).

ii') h is weakly p-constant by the definition of o-pattern of a quasi-constant function.

iii') h is completely o-homotopic to f. Since the homotopy F is completely quasi-regular by iii), there exists an o-pattern E of F (which is completely o-regular by [5], Proposition 7). Moreover we can choose E such that $E(x,0) = f(x)$, $E(x,1) = h(x)$, $\forall x \in S$, since f and h are completely o-regular i.e. $f(x) \in H(\langle f(x)\rangle) = H(\langle F(x,0)\rangle)$ and $h(x) \in H(\langle g(x)\rangle) = H(\langle F(x,1)\rangle)$, $\forall x \in S$. Then h is completely o-homotopic to f by E. □

REMARK 1. If W is a closed set, we can give the function g, by choosing as constant image of $X_j \in P$ any vertex of $H(\{f(\bar{X}_j)\})$.

REMARK 2. - *If S is a compact metric space, we can determine a real positive number r and choose partitions P with mesh $< r$.* In fact, we have just to calculate $enl(A_1,\dots,A_n)$, $\forall n$-tuple $a_1,\dots,a_n$ non-headed; so the real number r is given by $\frac{1}{2}\, inf(enl(A_1,\dots,A_n))$.

REMARK 3. - If G is an undirected graph, the function g can be choosen quasi-constant. Moreover if S is a compact metric space, by Remark to Definition 2, we have just to consider the couples of non-adjacent vertices a_h, a_k and then to find the distances $d(A_h,A_k)$ rather than the enlargabilities $enl(A_h,A_k)$. Consequently, if we put $r' = inf(d(A_h,A_k))$ and $r = \frac{1}{2} inf(enl(A_h,A_k))$, since by Remark 3 to Definition 3 it follows $r' \leqslant 4r$, we can choose a covering $P = \{X_j\}$, $j \in J$, with mesh $< \frac{r'}{4}$. So we obtain again Property 7 of [8].

3) The third normalization theorem.

By comparing the second normalization theorem for directed and