1. Motivations. In the year 1927 Dirac [1] settled the foundations of Quantum Fields Theory by quantizing the classical Electro-Magnetic Field. Success and Failure of Quantum Electrodynamics were both immediate. The success was due to the fact that the theory leads to numerical results in agreement with experements. This was obtained by using unjustified approximations and immediately later it was descovered that "better" approximations lead to meaningless" infinite quantities". Since the theory was made of computations on "mathematical objects" that were not rigorously defined this fact might not be too much surpresing; these "mathematical objects" were treated in the computations as derivable functions defined on $\mathbb{R}^4$ and operator valued; the computations used derivation, multiplication, values at points and integration of these objects. Twventy years later procedures of extracting finite results from these "infinite quantities" were descrived and gave numerical results in perfect agreement with experiments.

After the success of Schwartz's Distribution Theory it was discovered in the fifties that the simplest mathematical object of the Teory, the free fields operators, were not functions but (vector valued) distributions. Since the classical computations begin with products of free fields, therefore products of distributions, it is generally agreed that the lack of a general product of distributions is at the origin of the mathematical difficulties of Quantum Fields Theory. This motivation for the study of multiplication of distributions is quite classical but since all our methods and ideas stem from an examination of the computations of Physics, we need to emphasize on it.

2. The problem of multiplication of distributions. In seems quite indispensable to demand that a general product of Distributions should be associative, distributive relatively to addition, that the function 1 should be unit element, that the usual formula $(uv)' = u'v + uv'$ for the derivation of a product should hold. It seems also indispensable to demand that the new product should generalize the classical product of functions.

The first computational requirements are indispensible since they are explicitely used as rules of computations in Quantum Fields Theory. The second coherence requirement is also indispensable from the viewpoint of compatibility with the classical computations.

Howewer it is a famous result of L.Schwartz [1], in 1954, that all these requirements imply the impossibility of such a product.

Since our aim was to give a mathematical sense to the computations of Physics we believed that the solution might stem from them. Anticipating on the sequel of this paper, this lead us to a general multiplication of distributions which has all the computational properties. Then from Schwartz's impossibility result we know that this new multiplication does not generalize - say exactly - the multiplication of continuous functions. But we shall check a posteriori that this new multiplication will generalize all classical products, but in a sense slightly weaker than the one leading to Schwartz's impossibility result. We shall also accertain that this weaker sense is quite good enough in practice. So that the requirement of coherence will also be satisfied.

3. Successive ideas leading to multiplication of distributions.

a. The idea to use C^∞ or holomorphic functions over $\mathcal{D}(\Omega)$.

If Ω denotes any open set in $\mathbb{R}^n$ (for some $n \in \mathbb{N}$), a natural idea is that, if T_1 and T_2 are distributions on Ω, their product might be the bilinear form on $\mathcal{D}(\Omega)$ defined by

$$(1) \qquad \phi \to \langle T_1,\phi\rangle \langle T_2,\phi\rangle$$

when ϕ ranges over $\mathcal{D}(\Omega)$. If $\mathcal{E}(\mathcal{D}(\Omega))$ denotes the space of all complex valued C^∞ functions over $\mathcal{D}(\Omega)$ (see Colombeau [1]) we notice that the elements Φ of $\mathcal{E}(\mathcal{D}(\Omega))$ admit by means of the formula

$$(2) \qquad \left(\frac{\partial}{\partial x_i}\Phi\right)(\phi) = -\Phi'(\phi)\cdot\frac{\partial\phi}{\partial x_i}$$

(that will be more strongly motivated later) a natural concept of partial derivatives in the variable $x \in \Omega$, which generalizes exatly derivation in the sense of distributions. Using also the pointwise product in $\mathcal{E}(\mathcal{D}(\Omega))$ some (small) pieces of computations of Physics make sense, and this fact attracted our attention on a possible use of C^∞ or holomorphic functions over $\mathcal{D}(\Omega)$. But this quite simple interpretation of multiplication of distributions is not convenient to explain the computations of Physics. From a purely abstract viewpoint it is also not convenient: it does not even generalize the usual multiplication of C^∞ functions: if $f_1, f_2 \in \mathcal{E}(\Omega)$ and if $\phi \in \mathcal{D}(\Omega)$ one has in general

$$(3) \qquad \int f_1(x)\ \phi(x)dx.\ \int f_2(x)\phi(x)dx \neq \int f_1(x)\ f_2(x)\phi(x)dx\ .$$

b. <u>The idea to consider a quotient.</u> Therefore if we do not abandon the above idea to use multilinear (more generally holomorphic or C^∞ functions on $\mathcal{D}(\Omega)$ it is clear tha some crucial fact is missing (for instance the two members of (3) should be identified). At this point let us notice that since $\mathcal{D}(\Omega)$ is contained and everywhere dense in $\mathcal{E}'(\Omega)$ and since $\mathcal{E}'(\Omega)$ is a Silva space it follows (Colombeau [1] 0.6.9 and 1.1.6) that the restriction map

$$\begin{array}{ccc} \mathcal{E}(\mathcal{E}'(\Omega)) & \xrightarrow{r} & \mathcal{E}(\mathcal{D}(\Omega)) \\ \Phi & & \Phi_{/\mathcal{D}(\Omega)} \end{array}$$

is injective, so that we may consider that $\mathcal{E}(\mathcal{E}'(\Omega))$ is contained in $\mathcal{E}(\mathcal{D}(\Omega))$ via this map. The elements of $L(\mathcal{E}'(\Omega), \mathbb{C}) = \mathcal{E}''(\Omega) = \mathcal{E}(\Omega)$ are the usual C^∞ functions on Ω and an element f of $L(\mathcal{E}'(\Omega), \mathbb{C})$ is identified with the function on Ω

$$(4) \qquad x \rightarrow \langle f, \delta_x \rangle = f(x)$$

if x ranges in Ω and if δ_x denotes the Dirac measure at the point x. By analogy with the preceding idea to define a multiplication of distributions we might consider the product of two elements f_1 and f_2 of $L(\mathcal{E}'(\Omega), \mathbb{C})$ as the bilinear function

(5) $$T \to \langle f_1,T\rangle \langle f_2,T\rangle$$

when T ranges in $\mathcal{E}'(\Omega)$. From (4) the product (5) coincides with the usual product $f_1.f_2$ (in $\mathcal{E}(\Omega)$) if we restrict T to range only in the set $\{\delta_x\}_{x\in\Omega} \subset \mathcal{E}'(\Omega)$. This leads to considering in $\mathcal{E}(\mathcal{E}'(\Omega))$ the equivalence relation

(6) $$\Phi_1 \sim \Phi_2 \Longleftrightarrow \phi_1(\delta_x) = \phi_2(\delta_x) \qquad \forall x \in \Omega.$$

Now if we denote by $\mathcal{A}$ the map

(7) $$\begin{array}{ccc} \mathcal{E}(\mathcal{E}'(\Omega)) & \longrightarrow & \mathcal{E}(\Omega) \\ \Phi & & \mathcal{A}\Phi : x \to \Phi(\delta_x) = (\mathcal{A}\Phi)(x) \end{array}$$

(note that $\mathcal{A}\Phi \in \mathcal{E}(\Omega)$ since it is easy to check that the map $x \to \delta_x$ is in $\mathcal{E}(\Omega, \mathcal{E}'(\Omega))$, the equivalence relation (6) is exactly

(6') $$\Phi_1 \sim \Phi_2 \Longleftrightarrow \mathcal{A}\Phi_1 = \mathcal{A}\Phi_2 .$$

Now let us consider the diagram:

$$\begin{array}{ccc} \mathcal{E}(\mathcal{E}'(\Omega)) & \searrow^{\mathcal{A}} & \\ \cup & & \\ L(\mathcal{E}'(\Omega),\mathbb{C}) & \xrightarrow[\text{identification}]{\text{natural}} & \mathcal{E}(\Omega) \end{array}$$

The two algebras $\dfrac{\mathcal{E}(\mathcal{E}'(\Omega))}{\operatorname{Ker}\mathcal{A}}$ and $\mathcal{E}(\Omega)$ are isomorphic.

We remind from this that although $\mathcal{E}(\mathcal{E}'(\Omega))$ has, concerning multiplication, defects quite similar to those of the larges space $\mathcal{E}(\mathcal{D}(\Omega))$, these defects are repaired by a suitable quotient that makes the quotient algebra isomorphic to the classical algebra $\mathcal{E}(\Omega)$ (we may also check that the map $\mathcal{A}$ changes the derivation (6) into the usual derivation in $\mathcal{E}(\Omega)$, i.e. $\frac{\partial}{\partial x_i}(\mathcal{A}\Phi) = \mathcal{A}(\frac{\partial \Phi}{\partial x_i})$).

The natural idea that stems from these considerations is that, perhaps, the ideal $\operatorname{Ker}\mathcal{A}$ of $\mathcal{E}(\mathcal{E}'(\Omega))$ might be extended as an ideal of $\mathcal{E}(\mathcal{D}(\Omega))$ and that, perhaps, the quotient algebra thus obtained might have good properties concerning multiplication and derivation.

4. General multiplication of distributions.

a. Some properties of C^∞ functions over $\mathcal{E}'(\Omega)$ and $\mathcal{D}(\Omega)$.

First we recall that if $\Phi \in \mathcal{E}(\mathcal{D}(\Omega))$ (or $\mathcal{E}(\mathcal{E}'(\Omega))$ the element $\frac{\partial}{\partial x_i}\Phi$ defined by (2) is in $\mathcal{E}(\mathcal{D}(\Omega))$ (respectively $\mathcal{E}(\mathcal{E}'(\Omega))$, and that we may define in this way $D\Phi$ for any partial derivation operator $D = \frac{\partial^{k_1+\dots+k_n}}{\partial_{x_1}^{h_1}\dots\partial_{x_n}^{k_n}}$. It is immediate to check that the usual formula for derivation of a product holds. One also may prove (Colombeau [2]).

Proposition 1. If $\Phi \in \mathcal{E}(\mathcal{E}'(\Omega))$, then $\mathcal{A}\left(\frac{\partial}{\partial x_i}\Phi\right) = \frac{\partial}{\partial x_i}(\mathcal{A}\Phi)$, i.e. the new concept of derivation in $\mathcal{E}(\mathcal{E}'(\Omega))$ corresponds via the map $\mathcal{A}$, to the usual derivation in $\mathcal{E}(\Omega)$.

This result strengthens the choice of)(2). New if $\Phi \in \mathcal{E}(\mathcal{D}(\Omega))$ δ_x is not in general in the domain of Φ and we have to seek for a characterization of $\operatorname{Ker}\mathcal{A}$ which might be extended to $\mathcal{E}(\mathcal{D}(\Omega))$.

Definition 1. if q = 1,2,.... we set

(8) $$\mathcal{A}_q = \{\phi \in \mathcal{D}(\mathbb{R}^n) \text{ such that } \int \phi(x)dx = 1 \text{ and } \int (x)^i \phi(x)dx = 0 \text{ if } i = (i_1,\dots,i_n) \in \mathbb{N}^n \text{ is such that } 1 \leq |i| = i_1+\dots+i_n \leq q\}.$$

Obwiously $\mathcal{A}_{q+1} \subset \mathcal{A}_q$ and it is easy to check that for any q the set $_q$ is non void.

Now if $\phi \in \mathcal{A}_q$, $\varepsilon > 0$ and if $x \in \mathbb{R}^n$ we set, when λ ranges in $\mathbb{R}^n$,

(9) $$\phi_{\varepsilon,x}(\lambda) = \frac{1}{\varepsilon^n}\,\phi\left(\frac{\lambda - x}{\varepsilon}\right).$$

It is immediate to check that $\phi_{\varepsilon,0} \in \mathcal{A}_q$ if $\phi \in \mathcal{A}_q$ and that for fixed ϕ and x, $\phi_{\varepsilon,x} \to \delta_x$ in $\mathcal{E}'(\mathbb{R}^n)$ when $\varepsilon \to 0$. In Colombeau [2] we prove

<u>Proposition 2.</u> Let $\Phi \in \mathrm{Ker}\mathcal{A}$ be given. Then if $\phi \in \mathcal{A}_q$ is given, for any compact subset K of Ω there are constants $\varepsilon > 0$ and $\eta > 0$ such that

$$|\Phi\,(\phi_{\varepsilon,x})| \leq c(\varepsilon)^{q+1}$$

if $0 < \varepsilon < \eta$ and $x \in K$.

The converse is obvious: letting $\varepsilon \to 0$, $\Phi(\delta_x) = 0$ since $\Phi_{\varepsilon,x} \to \delta_x$ in $\mathcal{E}'(\Omega)$. Note also that $D\Phi \in \mathrm{Ker}\mathcal{A}$ if $\Phi \in \mathrm{Ker}\mathcal{A}$ and if D is any partial derivation.

b. <u>Construction of $\mathcal{G}(\Omega)$.</u> We seek for an ideal of $\mathcal{E}(\mathcal{D}(\Omega))$ such that its intersection with $\mathcal{E}(\mathcal{E}'(\Omega))$ should be $\mathrm{Ker}\mathcal{A}$. Prop. 2 attracts our attention on the growth of $|\Phi(\phi_{\varepsilon,x})|$ when $\varepsilon \to 0$. But an arbitrary element $\Phi \in \mathcal{E}(\mathcal{D}(\Omega))$ may be such that $|\Phi(\phi_{\varepsilon,x})|$ tends to $+\infty$ very rapidly when $\varepsilon \to 0$ and therefore its product with an element of $\mathrm{Ker}\mathcal{A}$ may still have such a very rapid growth. Therefore we are led to consider elements of $\mathcal{E}(\mathcal{D}(\Omega))$ that have a "moderate" grouwth in $\frac{1}{\varepsilon}$ when $\varepsilon \to 0$.

<u>Definition 2.</u> We say that $\Phi \in \mathcal{E}(\mathcal{D}(\Omega))$ is <u>moderate</u> if for every compact subset K of Ω and every partial derivation D there is an $N \in \mathbb{N}$ such that for all $\Phi \in \mathcal{A}_N$ there are constants $c > 0$ and $\eta > 0$ such that

$$|(D\Phi)\,(\phi_{\varepsilon,x})| \leq c\left(\frac{1}{\varepsilon}\right)^N .$$

if $x \in K$ and $0 < \varepsilon < \eta$.

Equivalently one may obviouslywrite: for every K and D there are $N_1, N_2 \in \mathbb{N}$

such that $\forall \phi \in \mathcal{A}_{N_1}$ $\exists$ $c > 0$ and $\eta > 0$ such that

$$|(D\Phi)(\phi_{\varepsilon,x})| \leq c(\frac{1}{\varepsilon})^{N_2}$$

if $x \in K$ and $0 < \varepsilon < \eta$ (we obtain N above by setting $N = \max(N_1,N_2)$).

Clearly $D\Phi$ is moderate if Φ is moderate. It is obvious that the product in $\mathcal{E}(\mathcal{D}(\Omega))$ of two moderate elements is still moderate. We denote by $\mathcal{E}_M(\mathcal{D}(\Omega))$ the subalgebra of $\mathcal{E}(\mathcal{D}(\Omega))$ made of the moderate elements. Many elements of $\mathcal{E}(\mathcal{D}(\Omega))$ are moderate: any element of $\mathcal{E}(\mathcal{E}'(\Omega))$ is moderate since $\phi_{\varepsilon,x} \to \delta_x$ when $\varepsilon \to 0$; it is proved in Colombeau [2] that

Proposition 3 Any distribution is moderate

Now we are going to define an ideal $\mathcal{N}$ of $\mathcal{E}_M(\mathcal{D}(\Omega))$ such that $\mathcal{N} \cap \mathcal{E}(\mathcal{E}'(\Omega)) =$ $= \operatorname{Ker}\mathcal{A}$. We might consider the ideal of $\mathcal{E}_M(\mathcal{D}(\Omega))$ spanned by $\operatorname{Ker}\mathcal{A}$ but the following larger ideal is more convenient.

Definition 3. We set

$\mathcal{N} = \{\Phi \in \mathcal{E}_M(\mathcal{D}(\Omega))$ such that for every compact subset K of Ω and every partial derivation D there is an $N \in \mathbb{N}$ such that for all $\phi \in \mathcal{A}_q$ with $q \geq N$ there are constants $c > 0$ and $\eta > 0$ such that

$$|(D\Phi)(\phi_{\varepsilon,x})| \leq c(\varepsilon)^{q-N}$$

if $x \in K$ and $0 < \varepsilon < \eta$.

Equivalently one may write that there are N_1 and $N_2 \in \mathbb{N}$ and same statement as above but with $q \geq N_1$ and $|(D\Phi)(\phi_{\varepsilon,x})| \leq c(\varepsilon)^{q-N_2}$. This implies the above statement by choosing $N = \max(N_1,N_2)$.

Other choices of $\mathcal{N}$ are possible and even crucial for some applications; however the general picture of the theory remains quite similar.

Clearly $\mathcal{N}$ is an ideal of $\mathcal{E}_M(\mathcal{D}(\Omega))$ and $D\Phi \in \mathcal{N}$ if $\Phi \in \mathcal{N}$. One checks immediately using prop. 2 that $\mathrm{Ker}\mathcal{A} = \mathcal{N} \cap \mathcal{E}(\mathcal{E}'(\Omega))$.

Now we define our algebra $\mathcal{G}(\Omega)$ as the quotient

$$\mathcal{G}(\Omega) = \frac{\mathcal{E}_M(\mathcal{D}(\Omega))}{\mathcal{N}} .$$

Clearly any partial derivative of an element G of $\mathcal{G}(\Omega)$ is defined as the class of the corresponding partial derivative of any representative of G.

$\mathcal{E}(\Omega)$ is a subalgebra of $\mathcal{G}(\Omega)$ since $\mathcal{E}(\Omega) = \dfrac{\mathcal{E}(\mathcal{E}'(\Omega))}{\mathrm{Ker}\mathcal{A}}$.

Since from prop. 3 $\mathcal{D}'(\Omega)$ is contained in $\mathcal{E}_M(\mathcal{D}(\Omega))$, there is a canonical map from $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$, which to each distribution associates its class. In Colombeau [2] we prove

Proposition 4. The canonical map from $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ is injective (i.e. $\mathcal{N} \cap \mathcal{D}'(\Omega) = \{0\}$), and therefore we may consider that $\mathcal{D}'(\Omega)$ is contained in $\mathcal{G}(\Omega)$.

We may notice that prop. 4 becomes false if we replace in it $\mathcal{D}'(\Omega)$ by the space of all continuous multilinear forms on $\mathcal{D}(\Omega)$ of degree ≤ 2 indeed if $f_1, f_2 \in \mathcal{E}(\Omega)$ then the two following functions on $\mathcal{D}(\Omega)$

$$\phi \to \int f_1(x)\phi(x)dx . \int f_2(x)\phi(x)dx$$

and

$$\phi \to \int f_1(x)\ f_2(x)\phi(x)dx$$

are identified in $\mathcal{G}(\Omega)$ with the classical product $f_1.f_2 \in \mathcal{E}(\Omega)$ (both are in $\mathcal{E}(\mathcal{E}'(\Omega))$ and their difference is in $\mathrm{Ker}\mathcal{A}$).

c. Connection with classical products. From Schwartz's impossibility result we already know that the new product in the algebra $\mathcal{G}(\Omega)$ cannot coincide with the usual product of all continuous functions. This fact that looks rather strange and unpleasant might be a great defect of the new product in $\mathcal{G}(\Omega)$ since it is quite clear - at least from a purely mathematical viewpoint - that the new multiplication in $\mathcal{G}(\Omega)$ has to be in agreement with the classical multiplication. In order to compare the two multiplications, in this section we denote by $\odot$ the new product (in $\mathcal{G}(\Omega)$), and we set the following definition.

Definition 4. Let be given a $G \in \mathcal{G}(\Omega)$ and let $\Phi \in \mathcal{E}_M(\mathcal{D}(\Omega))$ be a representative of G. If for every $\Psi \in \mathcal{D}(\Omega)$ the complex number $\int \Phi(\phi_{\varepsilon,x})\Psi(x)dx$ has a limit when $\varepsilon \to 0$, independent on $\phi \in \mathcal{A}_q$ for q large enough and if, when Ψ ranges in $\mathcal{D}(\Omega)$, this limit defines a distribution on Ω, we say that the generalized function G admits an associated distribution. If we denote by $\tilde{G}$ this associated distribution it is defined by the formula

$$\langle \tilde{G}, \Psi \rangle = \lim_{\varepsilon \to 0} \int \Phi(\phi_{\varepsilon,x})\Psi(x)dx.$$

It is immediate to check that the above does not depend on the choice of the representative Φ of G, and that $\tilde{G}$ is unique if it exists. One may also check (Colombeau [2]) that if $T \in \mathcal{D}'(\Omega)$ then it has an associated distribution which is T itself. One may also check that the element $(\delta_0)^2$ of $\mathcal{G}(\Omega)$ has no associated distribution. It is obvious that the set of the elements of $\mathcal{G}(\Omega)$ which have an associated distribution forms a linear space, that we denote by $\tilde{\mathcal{G}}(\Omega)$, in the situation

$$\mathcal{D}'(\Omega) \subset \tilde{\mathcal{G}}(\Omega) \subset \mathcal{G}(\Omega).$$

The linear map $\sim$ is defined on $\tilde{\mathcal{G}}(\Omega)$, valued in $\mathcal{D}'(\Omega)$ and such that $\sim \circ \sim = \sim$, therefore it may be considered as a projection from $\tilde{\mathcal{G}}(\Omega)$ onto $\mathcal{D}'(\Omega)$. This concept of associated distribution is particularly relevant due to the following results proved in Colombeau [2].

Theorem 1. a) Let f and g be two continuous functions on Ω. Then their product $f \odot g$ in $\mathcal{G}(\Omega)$ admits an associated distribution (i.e. $f \odot g \in \tilde{\mathcal{G}}(\Omega)$)

and this associated distribution is the classical product f.g.

b) Let $\alpha \in \mathcal{E}(\Omega)$ and let $T \in \mathcal{D}'(\Omega)$ be given. Then their product $\alpha \odot T$ in (Ω) admits an associated distribution (i.e. $\alpha \odot T \in \tilde{\mathcal{G}}(\Omega)$) which is the classical product $\alpha \cdot T$ of Schwartz's Distribution Theory.

These results show that in some weaker sense the new product is a generalization of the classical products: the projection $\sim$ on $\mathcal{D}'(\Omega)$ of the new product is exacley the classical product. We shall ascertain in the applications that in this case the two objects $f \odot g$ and $f \cdot g$ give the same numerical results and in the developpment of the Theory that the kind of result of th 3 holds in many more general cases and reconcilies the new computations with the classical computations.

Many authors have defined a product of distributions by regularization and passage to the limit (Mikusinski [1] and other authors, see Colombeau [2]). Then, when the product $T_1 \cdot T_2$ of two distributions T_1 and T_2 exists, say in Mikusinski [1] 's sense, the element $T_1 \odot T_2$ of $\mathcal{G}(\Omega)$ is in $\tilde{\mathcal{G}}(\Omega)$ and $\widetilde{(T_1 \odot T_2)} = T_1 \cdot T_2$. See Colombeau [2] for details.

Another kind of product of distributions has been defined by Hormander [1] and Ambrose [1] using the Fourier transform. Then it follows from an easy modification of a proof in Tysk [1] that the same result as above holds also in the case of the Hormander-Ambrose product (this result was communicated to me by J. Tysk).

d. Other non linear operations on elements of $\mathcal{G}(\Omega)$. One may define much more than the multiplication in $\mathcal{G}(\Omega)$.

Definition 5 and Theorem 2. If $p = 1,2,\ldots$, if $f \in G_M(\mathbb{R}^{2p})$ (a classical notation in Schwartz [2]) and if $G_1,\ldots,G_p \in \mathcal{G}(\Omega)$ then an element of $\mathcal{G}(\Omega)$, denoted by $f(G_1,\ldots,G_p)$, is defined as the class of the function $f(\Phi_1,\ldots,\Phi_p)$ $(\in \mathcal{E}_M(\ (\Omega)))$ if, for $1 \leq i \leq p$, $\Phi_i \in \mathcal{E}_M(\mathcal{D}(\Omega))$ is an arbitrary representative of G_i.

As in the case of mulliplication the above generalizes exactly the corresponding operation on C^∞ functions. For continuous functions, as a generalization of th 1.a we have

Theorem 3. In the above conditions on f, if $G_1,\ldots G_p$ are continuous functions on Ω then the element $f(G_1,\ldots,G_p)$ of $\mathcal{G}(\Omega)$ defined above admits an associated distribution which is the usual continuous function on Ω $x \to f(G_1(x),\ldots,G_p(x))$.

The proofs are in Colombeau [2].

5. The analysis of the new generalized functions.

a- Improved and related concepts. The elements of $\mathcal{G}(\Omega)$ may be considered as generalized functions on Ω. A more detailed study shows that this concept suffers from some minor defects that may be easily repaired by rather minor modifications in definitions. This is done in Colombeau [3] chap. 1 where also several related concept, which are motivated by examples or by Physics, are introduced. It is of a particular importance for the applications to know that what we have done in section 4 is only a general pattern and that some modifications, for instance in the definition of the ideal N, may be the key of special applications (for instance "removal of divergences" in Physics).

b- The value of a generaized function at a point. For Physics we need to define the value at any point $x \in \Omega$ of the generalized function on Ω. For this we define an algebra $\bar{\mathbb{C}}$, containing $\mathbb{C}$, such that if $x \in \Omega$ and $G \in \mathcal{G}(\Omega)$ then G(x) is in $\bar{\mathbb{C}}$. In particular the gives a meaning to the value at any point of any distribution. It is important to note that $\bar{\mathbb{C}}$ depends on the dimension of the space $\mathbb{R}^n$ of which Ω is an open set. This "strange" fact is not troublesome in the development of the theory and anyway it reflects the basic fact that, in Physics, Renormalization Theory depends completely on the space-time dimension. As an obvious example $\delta_0(x) = 0$ if $x \neq 0$ whicle $\delta_0(0) \in \bar{\mathbb{C}}$. For the value at a point of a generalized function see Colombeau [3] chap. 2.

c- Integration of generalized functions. For Physics we also need the integration of generalized functions. If $G \in \mathcal{G}(\Omega)$ and if K is any compact subset of Ω then $\int_K G(x)dx$ is naturally defined as an element of $\bar{\mathbb{C}}$. As an immediate example one may check that if 0 is in the interior of K we have $\int_K \delta_0(x)dx = 1$ and that if $0 \notin K$ then $\int_K \delta_0(x)dx = 0$; if 0 is on the boundary of K then $\int_K \delta_0(x)dx$ is in $\bar{\mathbb{C}}$ (for instance in one dimension one may compute easily $\int_0^1 \delta_0(x)dx$). In some cases for G we may define $\int_{\mathbb{R}^n} G(x)dx$. All the computations on integration of generalized functions generalize exactly the integral formulas that were explained by Schwartz's Distribution Theory. For the integration of generalized functions see Colombeau [3] chap. 3.

d - Holomorphic generalized functions. If Ω is an open set in $\mathbb{C}$ there are generalized functions $G \in \mathcal{G}(\Omega)$ such that $\bar{\partial}G = 0$ ($\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$ and which are not classical holomorphic functions (therefore they are not distributions from the classical hypoellipticity of the $\bar{\partial}$ operator in the space $\mathcal{D}'(\Omega)$). They are called "generalized holomorphic functions" and, surpresingly enough, they have many properties of the usual generalized functions. See Colombeau-Galé [1].

e - Vector valued generalized functions. For Physics one needs generalized functions valued in a bornological algebra. This is a rather straightforward generalization of the scalar case and this is done in Colombeau [2,3].

f - Final coments. In short, this new concept of generalized function is at the origin of a new mathematical analysis, rather similar in its general lines to the classical analysis of C^∞ and holomorphic functions, but considerably more general.

6 - Physical applications. These concepts provide a rigorous mathematical sense to the classical computations of Physics that were our motivation. In the mathematical explanation of these computations, these concepts explain heuristic operations of "removal of divergences" as the appearance of rigorous computations on our generalized functions. However the mathematics involved are complicated by the fact that we have to deal with objects whose values are unbounded operators on a Hilbert space and the paper Colombeau. Perrot [1] may only be considered as a demonstration of the use of the new mathematical tool in a rough and uncomplete mathematical clearing: for instance an explanation of the approximation of scattering operators by renormalized perturbation series is lacking, and the new mathematical formulation of the removal of divergences in these series in only sketched. Further mathematical improvements would be welcome or indispensable. In Colombeau-Perrot [1] we only consider the Φ^4 model in a 4-dimensional space time but it seems that this work might be more a immediately adapted to other fields. However particular difficulties, such as infra-red divergences in Quantum Electrodynamics, were never considered.

7 - Contribution of J. Sebastião e Silva. The basic mathematical tool of this new theory of generalized functions is Differential Calculus and Holomorphy in locally convex spaces. The author explains these theories in the book Colombeau [1], and in this book he explains how he was lead to consider that the best definitions of C^∞ and holomorphic maps between locally convex spaces are the ones settled by J. Sebastiao e Silva already in 1956 (Sebastião e Silva [1,2,3]. So the book Colombeau [1] might be considered as some continuation of Sebastião e Silva's work in this domain. This work of the author could not also have been done without the strong impulse given by L.Nachbin (cf. [1], [2], [3], [4] and [5]) to Infinite Dimensional Holomorphy in the last fifteen years.