NEAR-RINGS OF GROUP MAPPINGS

by

C.J. Maxson

Texas A & M University
College Station, Texas

I. Introduction

It is a basic result of ring theory that the set of endomorphisms of an abelian group is a ring under function addition and composition and furthermore every ring is isomorphic to a subring of a ring of this type. If the group is not abelian then the set of endomorphisms is no longer closed under addition. This leads one to the study of near-rings. It is the purpose of this paper to present a survey of some of the more recent results in the area of near-rings of group mappings. We start with some basic definitions and concepts to be used throughout the paper. For further details about these concepts and other results in near-ring theory we refer the reader to the books of Meldrum, [14] and Pilz, [17].

We recall that a near-ring $N := (N,+,\cdot)$ is a set N with binary operations of addition + and multiplication $\cdot$ such that

(i) $(N,+)$ is a group (not necessarily abelian) with neutral element 0;

(ii) $(N,\cdot)$ is a semigroup;

(iii) $(a+b)c = ac + bc, \forall a,bc \in N$.

More precisely we have defined a right near-ring. Using

(iii)' $a(b+c) = ab + ac, \forall a,b,c \in N$

one gets a left near-ring. Henceforth we consider only right near-rings and refer to them as "near-rings". Examples of near-rings are abundant. They arise in a natural manner when one deals with "non-linear" mappings.

Examples: Let (G,+) be a group with neutral element 0, let T be a topological group, V a vector space and R a commutative ring. With respect to function addition, +, and function composition, $\cdot$, the following are near-rings:

(a) $M(G) := \{f:G \to G\}$;

(b) $M_0(G) := \{f \in M(G) \mid f(0) = 0\}$;

(c) $M_{cont}(T) := \{f \in M(T) \mid f \text{ is continuous on } T\}$;

(d) $M_{aff}(V) := \{f \in M(V) \mid f \text{ is an affine map on } V\}$;

(e) $R[x] := \{f \mid f \text{ is a polynomial over } R \text{ in a single indeterminant, } x\}$.

Further every ring is a near-ring and if we define $*$ on any group (G,+) by $a*b = a$, $a,b\in G$ then we get a near-ring $(G,+,*)$, i.e., every group can be made into a near-ring.

A near-ring N is said to be zero-symmetric if $a \cdot o = o \cdot a = o$ $\forall a \in N$. A near-ring N is a near-ring with identity if $\exists i \in N$ such that $i \cdot a = a \cdot i = a$, $\forall a \in N$. In the sequel all near-rings will be zero-symmetric with identity.

Let G be a group, End G the monoid of endomorphisms of G and let $S \subseteq$ End G be any semigroup of endomorphisms of G such that the zero map and identity map are in S. We discuss two ways of associating near-rings with the pair (G,S).

Distributively generated near-rings. Let dg S denote the subgroup of M(G) generated by S. Thus dg $S = \{f = \sum_{i=1}^{n} \pm \sigma_i \mid \sigma_i \in S\}$. It is straightforward to verify that dg S is a near-ring, zero-symmetric and with identity. We call dg S the near-ring distributively generated by S.

Centralizer near-rings. Let $M_S(G) = \{f \in M(G) \mid f\sigma = \sigma f, \forall \sigma \in S\}$. Since S contains the zero map we see that $M_S(G)$ is a zero-symmetric near-ring with identity. We call $M_S(G)$ the centralizer near-ring determined by (G,S).

Our main focus in the remainder of this paper will be on various centralizer near-rings although distributively generated near-rings will reappear.

A near-field is a near-ring N with the property that $(N^* := N - \{0\}, \cdot)$ is a group. Historically near-fields were the first class of near-rings investigated. In 1905, L.E. Dickson gave the first example of a near-field which is not a field. In 1936, H. Zassenhaus determined all finite fields. He found that, except for seven isomorphism types, all finite near-fields can be constructed by a method going back to Dickson.

Subnear-rings and homomorphisms are defined in the usual manner. The ideals of a near-ring N are defined as kernels of near-ring homomorphisms. This gives rise to the internal characterization that a subset I of a near-ring N is an ideal of N if

(i) (I,+) is a normal subgroup of (N,+);

(ii) $\forall a \in I, \forall n,m \in N$, $n(a+m) \cdot nm \in I$;

(iii) $\forall a \in I, \forall n \in N$, $an \in I$.

A subset A of N satisfying (i) and (ii) of the above definition is called a left ideal of N and a subgroup (B,+) of (N,+) is an N-subgroup if $nb \in B$, $\forall n \in N$, $\forall b \in B$.

We define the J_2 radical of a near-ring N as the intersection of all ideals of N which are maximal as N-subgroups and we denote this radical by $J_2(N)$. When N is a ring the J_2 radical corresponds to the Jacobson radical of the ring.

A near-ring N is simple when the only ideals of N are {0} and N. A near-ring is 2-semisimple when $J_2(N) = \{0\}$. When N is finite, $J_2(N)$ is the intersection of all maximal ideals of N and N is 2-semisimple if and only if N is the direct sum of simple near-rings.

The interest in centralizer near-rings stems from the following result which shows that such near-rings are general in the sense that every near-ring (as usual, zero-symmetric with identity) arises as a centralizer near-ring.

Theorem I.1. Let N be a near-ring. Then there exists a group G and a semigroup S of endomorphisms of G such that $N \cong M_S(G)$.
Proof. For each $a \in N$ the map $\beta_a : N \to N$ defined by $\beta_a(x) = xa$ $\forall x \in N$ is an endomorphism of $(N,+)$. Then, for $S = \{\beta_a \mid a \in N\}$, one finds $N \cong M_S(N)$.

Therefore, since $M_S(G)$ is as general as possible, in order to obtain specific structural results, one must put some restrictions on the pair (G,S). In the next section we indicate structural results for certain choices of (G,S).

II. Structure of the centralizer near-ring $M_S(G)$

When S is a group of automorphisms of G one can make use of the theory of groups acting on sets. This situation has received a great deal of attention. Hence we first consider (G,A) where A is group of automorphisms of G with zero adjoined.

Recall that in this situation, for each $a \in G$, we have a subgroup $st(a) := \{\alpha \in A \mid \alpha(a)=a\}$, the A-stabilizer of a. Also for $a \in G$, the orbit Aa of a is defined by $Aa := \{\alpha(a) \mid \alpha \in A\}$. The next result, due to G. Betsch and known as Betsch's Lemma is fundamental to the study of $M_A(G)$.

Lemma II.1. Let A be a group of automorphisms of the group G and let $x,y \in G$. There exists a function $f \in M_A(G)$ such that $f(x) = y$ if and only if $st(x) \subseteq st(y)$.

When G is finite several definitive structural results can be given.

Theorem II.2. [10] Let G be a finite group and A an automorphism group of G.
1. The following are equivalent:
    a) $M_A(G)$ is a near-field;
    b) A acts transitively on $G^* = G - \{0\}$;
    c) G^* is a single orbit under the action of A on G.
2. $M_A(G)$ is a simple near-ring if and only if all A-stabilizers of non-zero elements of G are A-conjugate, i.e., for $a,b \in G^*$ there exists $\gamma \in A$ such that $st(a) = \gamma\, st(b)\gamma^{-1}$.
3. $M_A(G)$ is 2-semisimple if and only if all A-stabilizers of elements in G^* are maximal, i.e., for $a,b \in G^*$, $st(a) \subseteq st(b)$ implies $st(a) = st(b)$.

In particular, if A is a group of fixed point free automorphisms (only the identity of A has more than one fixed point) then $st(a) = \{id\}$ for each $a \in G^*$ so in this case $M_A(G)$ is simple.

Much more is known. When G is finite and $M_A(G)$ is not 2-semisimple the J_2 radical has been characterized and the structure of $M_A(G)/J_2(M_A(G))$ determined ([10]).

We now consider the case in which G is an infinite group. Recall that a near-ring N is regular if for every $a \in N$, $a = aba$ for some $b \in N$. In [15], Meldrum and Oswald obtain a very nice characterization for regular centralizer near-rings.

Theorem II.3. [15] Let A be a group of automorphisms of a group G. The near-ring $M_A(G)$ is regular if and only if for $a,b \in G^*$, $st(a) \subseteq st(b)$ implies $st(a) = st(b)$.

We remark that in the finite case regularity coincides with 2-semisimple. Further, if A is fixed point free then $M_A(G)$ is regular. If the pair (G,A) satisfies the condition of Theorem II.3 then we say (G,A) is regular.

When G is infinite, it seems to be a rather difficult problem to determine in general whether or not $M_A(G)$ is a simple near-ring. If $A = \{0, id\}$ (recall the groups of automorphisms have zero adjoined) then it is a classical result of Berman and Silverman (see [14] or [17]) that $M_0(G)$ is a simple near-ring. The investigation of the general situation was initiated by Meldrum and Oswald [15] and continued in [16] and [2]. When dealing with regular pairs, Meldrum and Zeller [16] showed that it suffices to restrict A to be fixed point free. They prove the following result

Theorem II. 4. [16] If (G,A) is regular and the stabilizers in A form a single conjugacy class then there exists a subgroup H of G and a fixed point free group of automorphisms, B, of H such that $M_B(H) \cong M_A(G)$.

Thus one focuses on fixed point free automorphism groups A. Let $\{w_\lambda \mid \lambda \in \Lambda\}$ be a complete set of A-orbit representatives in G and define for $v \in G$,
$\Lambda_v = \{\lambda \in \Lambda \mid Aw_\lambda + v \not\subseteq Aw_\lambda\}$.

Lemma II.5. [16] Let A be fixed point free on G. If there exists $v \in G^*$ such that $|\Lambda_v| = |\Lambda|$, then $M_A(G)$ is a simple near-ring.

Using this result Meldrum and Zeller then prove

Theorem II.6. [16] If A is fixed point free on G and $|A| < |G|$ then $M_A(G)$ is a simple near-ring.

Given a function $f \in M_A(G)$, define the rank of f, rk(f), to be the cardinality of the set of A-orbits in the range of f. For a nonempty subset B of G, define the rank of B, rk(B), to be the cardinality of the set of A-orbits in G which intersect B nontrivially. For each cardinal $\aleph_\alpha$, define $R_\alpha = \{f \in M_A(G) \mid rk(f) < \aleph_\alpha\}$. It was proven by Meldrum and Zeller [16] that these sets R_α are the only candidates for ideals in $M_A(G)$.

Theorem II.7. [16] Let A be fixed point free on G. If I is an ideal of $M_A(G)$ then $I = R_\alpha$ for some ordinal α.

This result was recently improved.

Theorem II.8. [2] Let A be fixed point free on G. Then MA(G) has at most one nontrivial ideal I. Specifically, $I = \{f \in M_A(G) \mid rk(f) < |A|\}$ is the only possible nontrivial ideal of $M_A(G)$.

In [2] several conditions on the pair (G,A) are given which force $M_A(G)$ to be a simple near-ring. Moreover it is shown that if a nonsimple near-ring $M_A(G)$ exists then A and G have rather unusual properties. But that is where the matter now stands. It remains an open question if $M_A(G)$ is simple.

Question: If A is fixed point free on G, is $M_A(G)$ a simple near-ring?

We leave the case of automorphisms and return to the situation in which S is a monoid of endomorphisms with zero. We discuss a particular situation.

Definition II.8. [12] A semigroup S of endomorphisms of a group G is fixed point free if

(a) $\bigcap_{\alpha \in S} \operatorname{Ker} \alpha = \{0\}$;

(b) $\forall \beta \in S$, $\operatorname{Ker} \beta = \operatorname{Ker} \beta^2 = \ldots$;

(c) $\forall \alpha, \beta \in S$, $\forall a \in G$, if $\alpha a = \beta a \neq 0$ then $\alpha = \beta$.

It is clear that if S is a group of automorphisms then this concept agrees with the previous use of fixed point free.

Theorem II.10. [12] Let N be a finite near-ring. Then N is 2-semisimple near-ring with its simple summands being non-rings or fields if and only if $N \cong M_S(G)$ for some finite group G and S a semigroup of fixed point free endomorphisms of G.

If S is a fixed point free semigroup of endomorphisms of a finite group G then S is a completely regular inverse semigroup, [12]. Thus the previous theorem suggests a study of near-rings of the form $M_S(G)$ where S is a completely regular inverse semigroup. In [12] it was determined for finite groups when such a near-ring is 2-semisimple. There are also other isolated results on the structure of $M_S(G)$ when S has certain properties (see e.g., [7]). However much more work needs to be done in this area.

We mentioned above that $M_S(G)$ is indeed general. However, one has been able to characterize those pairs (G,S) such that $M_S(G)$ is a near-field. Not surprisingly, the discussion breaks into the cases in which S is a group and when it is not.

Theorem II.11. [6] Let A be a group of automorphisms of a group G. The following are equivalent:

(i) $M_A(G)$ is a near-field;

(ii) $G = \{0\} \cup Ax$ and (G,A) is regular;

(iii) $G = \{0\} \cup Ax$ and (G,A) satisfies the property (F.C.): If $st(x) \subseteq st(\alpha x)$, $x \in G$, $\alpha \in A$, then $st(x) = st(\alpha x)$.

When G is finite, (F.C.) is always satisfied so we obtain Theorem II.2, (1). Moreover, if the action of A on G is fixed point free then regularity is equivalent to (F.C.) and in this case both conditions hold trivially.

Corollary II.12. [6] If A is a group of fixed point free automorphisms of G then $M_A(G)$ is a near-field if and only if $G = \{0\} \cup Ax$.

We mention here that we know of no example of a group G and a group A of automorphisms of G such that $G = \{0\} \cup Ax$ but (G,S) does not satisfy (F.C.).

Now let S be a semigroup of endomorphisms of G as usual with zero and identity. For any $x \in G$, $x \in Sx$ so we have $G = \bigcup_{i \in I} Sy_i$. We call $Y = \{y_i \mid i \in I\}$ a generating set. Henceforth we take $Y = \{y_i \mid i \in I\}$ as an arbitrary but fixed generating set and we consider I well ordered by the relation "$\leq$". Hence we consider Y as an I-sequence $\{y_i\}$.

For $u,v \in G$ define the relation $F(u,v) := \{(\alpha,\beta) \in S \times S \mid \alpha u = \beta v\}$. Further let $H = \{\text{I-sequences } \{x_i\} \mid x_i \in G, F(y_i,y_j) \subseteq F(x_i,x_j), i \leq j\}$. If π_i is the i-th projection map then clearly $\pi_i(H) \subseteq G$. We define another relation R on G^* by $(x,y) \in R$ if there exists $\alpha \in S$ such that $\alpha(x) = y$. Let $\tilde{R}$ denote the equivalence relation generated by R. We call the equivalence classes of $\tilde{R}$ the connected components of G and we say G is S-connected provided G^* is a connected component.

Equivantly $u,v \in G^*$ are S-connected if and only if there exist $x_1,x_2,\ldots,x_{n-1} \in G^*$, $\sigma_1,\ldots,\sigma_n,\rho_1,\ldots,\rho_n \in S$ such that

$\sigma_1 u = \rho_1 x_1 \neq 0$

$\sigma_2 x_1 = \rho_2 x_2 \neq 0$

$\vdots$

$\sigma_n x_{n-1} = \rho_n v \neq 0.$

We now introduce a concept needed in the next theorem but also used very much in the following section.

Definition II.13. Let G be a group and $\mathcal{F} = \{G_\alpha\}$ a collection of subgroups of G such that

(i) $\{0\} \subsetneq G_\alpha \subsetneq G$;

(ii) $\cup G_\alpha = G$;

(iii) $G_\alpha \cap G_\beta = \{0\}$ if $\alpha \neq \beta$.

Then $\mathcal{F}$ is called a fibration of G and $(G,\mathcal{F})$ is called a fibered group.

If $\mathcal{F} = \{G_\alpha\}$ is a fibration of G, we say $\sigma \in S$ is a $\mathcal{F}$-isomorphism if for each $G_\alpha \in \mathcal{F}$, $\sigma(G_\alpha) = \{0\}$ or $\text{Ker}\ \sigma \cap G_\alpha = \{0\}$ and $\sigma(G_\alpha) = G_\beta$ for some $G_\beta \in \mathcal{F}$. Thus $\sigma \in S$ is a $\mathcal{F}$-isomorphism if and only if for each $G_\alpha \in \mathcal{F}$, σ is the zero map on G_α or σ is an isomorphism on G_α with image in $\mathcal{F}$. The characterization result is as follows.

Theorem II.14. [6] Let S be a semigroup of endomorphisms of a group G. Then $M_S(G)$ is a near-field if and only if

(i) G is S-connected,

(ii) G has a fibration, say $\mathcal{F} = \{H_j \mid j \in J\}$ and each $\sigma \in S$ is an $\mathcal{F}$-isomorphism,

(iii) if $y_i \in H_j^*$ then $\pi_i(H) = H_j$.

III. Geometry and Near-rings

From the time of Descartes, early in the 17th century , mathematicians have been interested in associating algebraic structures with geometric structures and investigated the transfer of information. In this section we introduce a geometric structure, associate two near-rings to the geometry and indicate how the geometry influences the algebra. We start with a definition due to André, [1].

Definition III.1. [1] Let $\Sigma = (\mathcal{P},\mathcal{L},\|)$ where $\mathcal{P}$ is a set of points, $\mathcal{L}$ a collection of subsets of $\mathcal{P}$ called lines, with the incidence relation "belongs to", and a parallelism relation $\|$ defined on $\mathcal{L}$ such that

(A1) Every two points in $\mathcal{P}$ determine a unique line;

(A2) $|\mathcal{L}| \geq 2$ and for each $A \in \mathcal{L}$, $|A| \geq 2$;

(A3) Parallelism is an equivalence relation;

(A4) $\forall x \in \mathcal{P}, \forall A \in \mathcal{L}$, there exists a unique $B \in \mathcal{L}$ such that $x \in B$ and $B \| A$.

Further there exists a one-one map $\Phi: \mathcal{P} \to \text{Coll}\ \Sigma$ such that $\Phi(\mathcal{P})$ is a point transitive group of fixed point free collineations. We say (Σ,Φ) is a translation structure. (See [1] and [4].)

Let $(G,\mathcal{F} = \{G_i\})$ be a fibered group (see Definition II.13). By taking $\mathcal{P}(G) = G$, $\mathcal{L}(G) = \{x + G_i \mid G_i \in \mathcal{F}, x \in G\}$ and setting $a + G_i \| b + G_j$ if and only if i=j one gets an incidence structure $\Sigma(G) = (\mathcal{P}(G), \mathcal{L}(G), \|)$ satisfying (A1) - (A4). Further define $\Phi(G) : \mathcal{P}(G) \to \text{Coll}\ \Sigma\ (G)$ by $\Phi(G) : a \to \lambda_a$ where λ_a denotes the left translation of G determined by $a \in G$. We then find we have a translation structure $(\Sigma(G), \Phi(G))$.

Conversely, every translation structure arises in this manner. That is, if (Σ,Φ), $\Sigma = (\mathcal{P},\mathcal{L},\|)$ is a translation structure, then there is a fibered group $(G,\mathcal{F})$ such that $\mathcal{P} = \mathcal{P}(G)$,

$\mathcal{L} = \mathcal{L}(G)$, | | is as defined above and $\Phi = \Phi(G)$. Hence a translation structure may be considered as a fibered group and we henceforth do so.

If the translation structure $(G, \mathcal{F} = \{G_i\})$ has the property that $G_i + G_j = G$ for each $G_i, G_j \in \mathcal{F}$, $i \neq j$ then $\mathcal{F}$ is called a congruence fibration and in this case one obtains the classical translation planes.

Thus congruence fibrations tighten the structure of the geometry. We next tighten the structure in an alternative fashion. Let $(G, \mathcal{F} = \{G_i\})$ be a translation structure and let S be a semigroup of endomorphisms of G such that

(01) The identity map and zero map are in S;

(02) For each $\sigma \in S$, for each $G_i \in \mathcal{F}$, $\exists G_j \in \mathcal{F}$ such that $\sigma(G_i) \subseteq G_j$.

Then S is called a semigroup of operators for $(G, \mathcal{F})$ and $(G, \mathcal{F}, S)$ is a translation structure with operators, TSO. We mention that operators can also be defined in a geometric manner, ([4]).

We now show how to associate near-rings with TSO's, $(G, \mathcal{F}, S)$. First we consider the set $\mathrm{Dil}(G, \mathcal{F}) = \{\sigma \in \mathrm{End}\, G \mid \sigma(G_i) \subseteq G_i, \forall G_i \in \mathcal{F}\}$. (Note that the operators play no role here.) Under function composition $\mathrm{Dil}(G, \mathcal{F})$ is a semigroup with zero and identity, called the semigroup of dilitations of $(G, \mathcal{F}, S)$. Our first near-ring is d.g. $\mathrm{Dil}(G, \mathcal{F})$ called the kernel of $(G, \mathcal{F}, S)$. For our second associated near-ring we take $M_S(G, \mathcal{F}) = \{f \in M_0(G) \mid f(G_i) \subseteq G_i, \forall G_i \in \mathcal{F}, f\sigma = \sigma f, \forall \sigma \in S\}$, a near-ring under funtion addition and composition called the centralizer of $(G, \mathcal{F}, S)$.

We restrict now to the case in which G is a finite group and look at various properties of these associated near-rings.

III.A: Kernel of $(G, \mathcal{F}, S)$.

The structure of $\mathrm{Dil}(G, \mathcal{F})$ is well-known, ([3], [9]).

Theorem III.2. For a finite fibered group $(G, \mathcal{F})$, $\mathrm{Dil}(G, \mathcal{F}) \backslash \{0\}$ is a cyclic group of fixed point free automorphisms of G.

Proof. To illustrate some of the ideas we show that each $0 \neq \sigma \in \mathrm{Dil}(G, \mathcal{F})$ is a monomorphism. Hence, since G is finite σ is an automorphism. Suppose $\sigma \in \mathrm{Dil}(G, \mathcal{F})$ and $\sigma(x) = 0$ for some $x \in G$, say $x \in G_i$. Let $y \in G_j$, $j \neq i$. Then $x+y \in G_k$, $i \neq k \neq j$. Now $\sigma(y) \in G_j$ and $\sigma(y) = \sigma(x+y) \in G_k$. Hence $\sigma(y) = 0$. For any $w \in G_i$, use w and y to get $\sigma(w) = 0$. Thus σ is the zero map.

A classical result states that when $\mathcal{F}$ is a congruence fibration, G is an abelian group, therefore $\mathrm{Dil}(G, \mathcal{F})$ is a finite field. Thus when G is an abelian group dg $\mathrm{Dil}(G, \mathcal{F}) = \mathrm{Dil}(G, \mathcal{F})$ is a field. We now turn to the non-abelian case.

Theorem III.3. [3] If $(G,\mathcal{F})$ is a finite fibered group with $Dil(G,\mathcal{F}) \neq \{0,id\}$ then G is a p-group for some prime p, is of exponent p and of nilpotency class at most 2.

Using this result and the known structure of $Dil(G,\mathcal{F})$ the following rather surprising result has been obtained.

Theorem III.4. [9] If $(G,\mathcal{F})$ is a finite fibered group then dg $Dil(G,\mathcal{F})$ is a commutative ring. If further, $Dil(G,\mathcal{F}) \neq \{0,id\}$ then dg $Dil(G,\mathcal{F})$ is a field.

Clearly if $Dil(G,\mathcal{F}) = \{0,id\}$ then dg $Dil(G,\mathcal{F}) = Z_n$ where n is the exponent of G. The above theorem shows that whether or not G is abelian, whenever $Dil(G,\mathcal{F}) \neq \{0,id\}$ there is a field associated with the geometry $(G,\mathcal{F})$ is a natural manner. We also mention that in the abelian case the field has geometric significance. The significance of the field dg $Dil(G,\mathcal{F})$ in the non abelian case is still unknown.

III.B. Centralizer of $(G,\mathcal{F},S)$.

As above, to obtain definitive structural results one places some restrictions on the semigroup of operators. One first considers the case where S is a group of automorphisms (with 0). As one might expect from the previous discussion on centralizer near-rings, the orbits of the action and the stabilizers play an important role. For results in this situation see [8].

Next one considers the situation in which S is a cyclic semigroup, say $S = \langle\alpha\rangle \cup \{0,id\}$. We write $M_\alpha(G,\mathcal{F})$ for $M_S(G,\mathcal{F})$. We are mainly interested as to when $M_\alpha(G,\mathcal{F})$ is a simple near-ring. If α is an automorphism, using the results in [8], one notes when $M_\alpha(G,\mathcal{F})$ is simple. In other cases we have the following.

Theorem III.5. [8] If $S = \langle\alpha\rangle \cup \{0,id\}$, α not invertible and α not nilpotent, then $M_\alpha(G,\mathcal{F})$ is not a simple near-ring.

Proof. Since α is not invertible, Ker $\alpha \neq \{0\}$. Thus there is some fiber, G_i, of the fibration such that $G_i \cap \text{Ker } \alpha \neq \{0\}$. For $f \in M_\alpha(G,\mathcal{F})$, $f(\text{Ker } \alpha \cap G_i) \subseteq \text{Ker } \alpha \cap G_i$ so $A = (\{0\}: \text{Ker } \alpha \cap G_i)$ is an ideal in $M_\alpha(G,\mathcal{F})$. Using the fact that α is not nilpotent one gets $A \neq \{0\}$, hence $M_\alpha(G,\mathcal{F})$ is not simple.

We henceforth restrict our attention to nilpotent endomorphisms. We recall the concepts of generating set and connected components as discussed after Corollary II.12.

Lemma III.6. There are k-1 connected components of G^* where $|\text{Ker } \alpha| = k$.

Proof. Let C_i be a connected component and let $y \in C_i$. Since α is nilpotent there exists some s such that $\alpha^s(y) \in \operatorname{Ker} \alpha$ and since $\alpha^s(y) \in C_i$, $\alpha^s(y) \in \operatorname{Ker} \alpha \cap C_i$. Thus there exists a kernel element in each connected component. Suppose $x_1, x_2 \in \operatorname{Ker} \alpha \cap C_i$. We then find $x_1 = \alpha^k(x_2)$ for some integer $k \geq 0$. If $k \neq 0$, $x_1 = 0$, a contradiction. Thus each connected component has a unique kernel element.

If we let $\{0\}$ be a connected component then we say the number of connected components is the cardinality of Ker α. In particular $(G, \mathcal{F})$ is S-connected if and only if $\operatorname{Ker} \alpha = \{0, x\}$.

Suppose $M_\alpha(G, \mathcal{F})$ is a simple near-ring. We know there is some fiber G_i such that $\operatorname{Ker} \alpha \cap G_i \neq \{0\}$. If $\operatorname{Ker} \alpha \cap G_j \neq \{0\}$, $i \neq j$ then one finds there exists a component with more than one kernel element which contradicts the above lemma. This gives the following result.

Lemma III.7. If $M_\alpha(G, \mathcal{F})$ is simple and α is nilpotent then Ker α is contained in a single fiber of $\mathcal{F}$, say G_0.

Lemma III.8. If $M_\alpha(G, \mathcal{F})$ is a simple near-ring and α is nilpotent there is a unique generating set $Y = G \setminus \operatorname{Ker} \alpha^{n-1}$ where $\alpha^n = 0$ but $\alpha^{n-1} \neq 0$.

When G is S-connected much can be said.

Theorem III.9. [8] Let α be a nilpotent operator on $(G, \mathcal{F})$ and let G be S-connected, $S = \langle\alpha\rangle \cup \{0, id\}$ with $\operatorname{Ker} \alpha \subseteq G_0$. Let Y be any generating set for G. The following are equivalent.

(i) $Y \cap G_0 = \emptyset$;
(ii) $M_\alpha(G, \mathcal{F})$ is a near-field;
(iii) $M_\alpha(G, \mathcal{F})$ is a simple near-ring;
(iv) $M_\alpha(G, \mathcal{F})$ is a 2-semisimple near-ring;
(v) $M_\alpha(G, \mathcal{F}) \cong Z_2$.

When G is not S-connected necessary and sufficient conditions, in terms of the geometry, are known for $M_\alpha(G, \mathcal{F})$ to be simple, [8]. Instead of stating these we give an external characterization.

Theorem III.10. [8] Let α be a nilpotent operator on $(G, \mathcal{F})$. Then $M_\alpha(G, \mathcal{F})$ is a simple near-ring if and only if $M_\alpha(G, \mathcal{F}) \cong M_0(\operatorname{Ker} \alpha)$.

In [8] an example is given where $G := (F)^6$, F a finite field $\mathcal{F}$ a fibration of G and α a nilpotent operator such that $M_\alpha(G,\mathcal{F}) \cong M_D(F \oplus F)$. Thus simple near-rings, not rings, actually arise.

IV. Rings and Near-rings

Let R be a ring with identity and let G be a (right) unitary R-module. Then R determines a semigroup of endomorphisms of G so we have a centralizer near-ring $M_R(G) = \{f \in M_D(G) \mid f(xr) = (fx)r, \forall x \in G, \forall r \in R\}$. In this section we discuss some of the interplay between the properties of the ring R, the R-module, G_R, and the near-ring $M_R(G)$.

We recall that a <u>cover</u> for an R-module G is a collection $\mathcal{C} = \{G_\alpha\}$ of submodules of G sucht that

(i) $\{0\} \subsetneq G_\alpha \subsetneq G$;

(ii) $G_\alpha \nsubseteq G_\beta$ for $\alpha \neq \beta$;

(iii) $\cup\, G_\alpha = G$.

Let $R := Z$ and $G := Z^2$ and let $\mathcal{C}$ be a cover by maximal cyclic submodules. Further let $f \in M_Z(Z^2)$ be determined on $G_\alpha = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} Z$ by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Since G_α is a maximal submodule we have $\gcd(x_1,x_2) = 1$ so $\exists h,k \in Z$, $hx_1 + kx_2 = 1$. But then f can be represented on G_α by the matrix $\begin{bmatrix} c_1h & c_1k \\ c_2h & c_2k \end{bmatrix}$, i.e., f/G_α can be extended to an endomorphism of G. Equivalently, every $f \in M_Z(Z^2)$ is piecewise an endomorphism of Z^2 in the sense that for each $G_\alpha \in \mathcal{C}$, $\exists \varphi \in End_Z(Z^2)$ with $f/G_\alpha = \varphi$.

In general, let $\mathcal{C} = \{G_\alpha\}$ be a cover of G by maximal cyclic submodules of G and let $N := \{f \in M_R(G) \mid f/G_\alpha$ can be extended to an endomorphism of G$\}$, a subnear-ring of $M_R(G)$ which we call the near-ring of piecewise endomorphisms determined by $(R,G,\mathcal{C})$. We ask, "When is $N = M_R(G)$?". The next example shows that in general, $N \neq M_R(G)$.

<u>Example IV.1.</u> [5] Let $R := Z[x]$, $G := R^2$ and let $\mathcal{C}$ be a cover by maximal cyclic submodules. One verifies that $\begin{bmatrix} x \\ x+2 \end{bmatrix} R \in \mathcal{C}$. Further, $\exists f \in M_R(G)$ with

$$f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{cases} \begin{bmatrix} 1 \\ 1 \end{bmatrix} r, \text{ if } \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ x+2 \end{bmatrix} r, r \in R, \\ \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{otherwise,} \end{cases}$$

However, there is no $\varphi \in End_R(G)$ with $\varphi \begin{bmatrix} x \\ x+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence $N \neq M_R(G)$.

Note that in the above example R is not a PID. For PID's the situation is quite different. In fact we have the next rather interesting result.

Theorem IV.2. [5] Let G be a finitely generated module over a PID, D, let $C = \{G_\alpha\}$ be a cover by maximal cyclic submodules and let $N = \{f \in M_R(G) \mid f/G_\alpha$ can be extended to an endomorphism of G$\}$. Then $N = M_D(G)$.

We mention that it is an open question whether or not the requirement that G be finitely generated can be omitted.

In the next theorem we present some further relationships between the ring module G_R and the near-ring $M_R(G)$.

Theorem IV.3. [13] (a) If D is an integral domain, not necessarily commutative then $M_D(D^2)$ is a near-ring, not a ring.

(b) Let R be a commutative ring. $M_R(R^2)$ is a simple near-ring if and only if R is an integral domain.

(c) Let R be a left Artinian ring. Then $M_R(R^2)$ is 2-semisimple if and only if R is semisimple.

It should be pointed out that rings do arise as $M_R(G)$. In fact if D is a commutative integral domain and Q(D) its field of fractions, then $M_D(Q(D))$ is a ring. Further, if R is a complete nxn matrix ring over a ring S then for each R-module, G, $M_R(G)$ is a ring, in fact $M_R(G) = End_R(G)$.

On the other hand if R is the field of real numbers, for G := R, $M_R(R)$ is a ring while for G := R^2, $M_R(R^2)$ is not a ring.

This raises the questions:

(Q1): Which rings R have the property that $M_R(G)$ is a ring for each R-module G ?

(Q2): Which rings R have the property that $M_R(G) = End_R(G)$?

For finite rings R the above questions have been shown to be equivalent and those rings R such that $M_R(G)$ is a ring for each R-module have been characterized, (see [11]). However the general problem remains open.

REFERENCES

1. André, J., Über Parallelstrukturen, II: Translationsstrukturen, Math. Z., 76 (1961), 155-163.
2. Fuchs, P., Maxson, C.J., Pettet, M.R. and Smith, K.C., Centralizer near-rings determined by fixed point free automorphism groups, Proc. Royal Soc. Edin., 107 (1987), 321-337.
3. Herzer, A., Endliche nichtkommutative Gruppen mit Partition Π und fixpunktfreie Π-Automorphismen, Arch. Math., 34 (1980), 385-392.
4. Maxson, C.J., Near-rings associated with generalized translation structures, Journ. of Geom., 24 (1985), 175-193.
5. Maxson, C.J., Piecewise endomorphisms of PID-modules, (submitted).
6. Maxson, C.J. and Meldrum, J.D.P., Centralizer representations of near-fields, Journ. of Alg., 89 (1984), 406-415.
7. Maxson, C.J. and Oswald, A., On the centralizer of a semigroup of group endomorphisms, Semigroup Forum, 28 (1984), 29-46.
8. Maxson, C.J. and Oswald, A., Kernels of fibered groups with operators, Arch. Math., 9 (1987), 453-486.
9. Maxson, C.J. and Pilz, G.F., Near-rings determined by fibered groups, Arch. Math., 44 (1985), 311-318.
10. Maxson, C.J. and Smith, K.C., The centralizer of a set of group automorphisms, Comm. in Alg., 8 (1980), 211-230.
11. Maxson, C.J. and Smith, K.C., Centralizer near-rings that are endomorphism rings, Proc. Amer. Math. Soc., 80 (1980), 189-195.
12. Maxson, C.J. and Smith, K.C., Centralizer near-rings determined by completely regular inverse semigroups, Semigroup Forum, 22 (1981), 47-58.
13. Maxson, C.J. and Van der Walt, A.P.J., Centralizer near-rings over free ring modules, (submitted).
14. Meldrum, J.D.P., Near-rings and their links with groups, Pitman (Research Notes Series No. 134), 1985.
15. Meldrum, J.D.P. and Oswald, A., Near-rings of mappings, Proc. Royal Soc. Edin., 83 (1979), 213-223.
16. Meldrum, J.D.P. and Zeller, M., The simplicity of near-rings of mappings, Proc. Royal Soc. Edin., 90 (1981), 185-193.
17. Pilz, G., Near-rings, revised ed., North-Holland, Amsterdam, 1983.

Department of Mathematics
Texas A & M University
College Station, Texas 77843
U.S.A.

CONGRUENCES ON REGULAR SEMIGROUPS

H. Mitsch

University of Vienna/Austria

1. Generalities.

Let $(S,.)$ be a semigroup; an element $a \in S$ will be called <u>regular</u> if there is some $x \in S$ such that $axa = a$. S is called regular if each element of S is regular. Notice that if $a = axa$, then for $y = xax$ we have that

$$a = aya \text{ and } y = yay.$$

For every $a \in S$, denote $V(a) = \{ x \in S \mid a = axa,\ x = xax \}$; hence S is regular iff $V(a) \neq \emptyset$ for all $a \in S$. Furthermore, if $a = axa$, then clearly ax and xa are idempotents; denote by E_X the set of all idempotents in X for any subset X of S.

<u>Examples</u> for regular semigroups are:

Idempotent semigroups (bands); groups; unions of groups (completely regular semigroups); inverse semigroups (i.e. $|V(a)| = 1$ for all $a \in S$); $(T_X, \circ)$ the semigroup of all mappings of the set X into itself with respect to composition of functions; $(P_X, \circ)$ the semigroup of all partial mappings of X into itself; $(L_V, \circ)$ the semigroup of all linear mappings of the vector space V into itself; $(M_n(F),.)$ the semigroup of all $(n\times n)$-matrices over the field F; direct products and homomorphic images of regular semigroups, but not subsemigroups (the subsemigroup of all natural numbers of the additive group of all integers is not regular), hence the class of all regular semigroups does not form a variety.

Not only since appearance of the book of M.Petrich [22] on "Inverse Semigroups", the theory of regular semigroups has attracted wide attention. This is particularly true for the study of congruences. They play a central role in many of the structure theorems and various considerations of semigroups in general. The efficient handling of the congruences is a basic prerequisite for their useful application. For this reason, the most important facts concerning congruences on regular semigroups are collected here, with particular emphasis on

- the construction of general congruences, and
- the explicite form of special types of congruences.

An equivalence relation ρ on a semigroup $(S,.)$ is called a congruence if

$$a\,\rho\,b\,(a,b\in S) \text{ implies that } ac\rho bc \text{ and } ca\rho cb \text{ for all } c\in S.$$

The set S/ρ of all congruence classes $a\rho$ $(a\in S)$ of ρ forms a semigroup with respect to the multiplication

$$(a\,\rho) * (b\,\rho) = (ab)\rho$$

and is a homomorphic image of $(S,.)$. Conversely, every homomorphic image of $(S,.)$ is obtained by a congruence on $(S,.)$.

With respect to the partial ordering

$$\rho \leq \tau \text{ iff } a\rho b \text{ implies that } a\tau b \ (a,b \in S),$$

the set C(S) of all congruences on (S,.) forms a complete lattice with least element ε, the identity relation, and greatest element ω, the universal relation. It is easily seen that if $\rho,\tau \in \mathcal{C}(S)$ such that $\rho \leq \tau$ then $(S/\tau, *)$ is a homomorphic image of $(S/_\rho, *)$.

In general, particular homomorphic images of a given semigroup S are of special interest; thus particular congruences on S have to be found. If $\mathcal{C}$ denotes any class of semigroups, then a congruence ρ is called a <u>$\mathcal{C}$- congruence</u> if the semigroup $(S/_\rho, *)$ belongs to the class $\mathcal{C}$. For example, let $\mathcal{C}$ the class of all groups, semilattices, bands,resp.; then a $\mathcal{C}$ - congruence is called a group congruence, semilattice congruence, band congruence, respectively. In particular, we will be interested in the least or the greatest congruence for some given class $\mathcal{C}$ (with respect to the partial order $\leq$ above):

β .... the least band congruence

σ .... the least group congruence

γ .... the least right-group congruence

η .... the least semilattice congruence

Y .... the least inverse congruence

ν ...... the least semilattice of groups congruence

π .... the greatest idempotent-pure congruence (i.e. $a\pi e,\ a\in S,\ e\in E_S \rightarrow a\in E_S$)

μ .... the greatest idempotent separating congruence (i.e. $e\mu f, e,f\in E_S \rightarrow e=f$)

For details see section 3. below. Note that for general semigroups not all of these congruences exist; but they do exist for any regular semigroup (see Howie-Lallement [8]).
If, for example, we consider the least group congruence on a semigroup (S,.) - if exists, then we have for every $\tau \in C(S)$ satisfying $\tau \geq \sigma$ that $(S/_\tau, *)$ is again a group, a homomorphic image of the group $(S/_\sigma, *)$. Thus, one can say that the least group congruence on S gives the greatest group homomorphic image of S.

A very useful result on congruences on regular semigroups is the following

Lemma 1.1. (Lallement [9]) Let (S,.) be a regular semigroup and ρ any congruence on S. If $a\rho \in S/_\rho$ is idempotent then there is some $e \in E_S$ such that $a\rho = e\rho$.

2. General congruences.

Our first aim will be the description of an arbitrary congruence on a regular semigroup. For this, let us consider first the special case of a group.
If G is a group then it is known that for every normal subgroup N of G, the relation ρ_N on G defined by

$$a \ \rho_N \ b \leftrightarrow ab^{-1} \in N$$

is a congruence on G, and conversely that for a congruence ρ on G the

ρ-class containing the identity e of G is a normal subgroup N of G such that $\rho_N = \rho$. Furthermore, all of ρ can be reconstructed from any one of its classes, in particular the class $e\rho = N$. For semigroups, such a reconstruction of a congruence ρ from a single ρ-class is not possible, in general. If S is regular, we have at least the following result:

<u>Lemma 2.1.</u> (Clifford-Preston [1]). For a regular semigroup S, any congruence ρ on S is uniquely determined by the ρ-classes containing idempotents.

Note that this result does not tell us how to reconstruct all of the congruence ρ from the set of all idempotent ρ-classes. Various attempts have been made to find an analog of the connection between congruences on groups and normal subgroups. For <u>inverse</u> semigroups, G.B.Preston abstractly characterised the set of all idempotent classes of a congruence on S and gave a construction of the congruence associated with such a <u>kernel normal system</u> (see Clifford-Preston [1]). Meakin [14]generalized this result to regular semigroups:

<u>Definition.</u> A set $A = \{A_i \mid i \in I\}$ of disjoint subsets of a regular semigroup S is called a <u>kernel system on S</u> if

(1) $A_i \cap A_j = \emptyset$ for all $i \neq j$ in I

(2) each A_i contains an idempotent of S and each idempotent of S belongs to some $A_j (j \in I)$

(3) $x A_i y \cap A_j \neq \emptyset$ implies that $xA_i y \subseteq A_j$ for $x, y \in S^1, i, j \in I$.

The construction of the unique congruence ρ having every $A_i \in A$ as idempotent ρ-class, is now the following:

Theorem 2.2. (Meakin [14]) Let (S,.) be a regular semigroup and $A = \{A_i | i \in I\}$ a kernel system on S. Then the unique congruence on S with all A_i ($i \in I$) as idempotent classes is given by

$$a \rho_A b \leftrightarrow a' \in V(a),\ b' \in V(b):\ aa', ba' \in A_i,\ b'a,\ b'b \in A_j \text{ for some } i,j \in I.$$

This result suffers - as in the inverse case - from the disadvantage that the conditions imposed on a kernel system are very difficult to utilize. For inverse semigroupsanother approach proved very useful: it is possible to reconstruct any congruence ρ from the set-theoretical union of all the idempotent ρ-classes (the kernel of ρ) taking into account the partition on the set of all idempotents induced by ρ (the trace of ρ); see M.Petrich [23]. Following this idea, Pastijn-Petrich [19] introduced the concept of congruence pair for a regular semigroup generalizing the corresponding notion for inverse semigroups. The exact definitions are the following:

Definition. Let S be a regular semigroup; then for any congruence ρ on S consider the following two characteristic concepts:

1) $\operatorname{tr} \rho = \rho|E_S$ is called the trace of ρ,
2) $\ker \rho = \{a \in S | a\rho e \text{ for some } e \in E_S\}$ is called the kernel of ρ.

Note that $\operatorname{tr}\rho$ is the restriction of ρ to the subset E_S of S and thus yields a certain partition of E_S. Furthermore, $\ker\rho$ is the set-theoretical union of all idempotent ρ-classes.

It was shown by R.Feigenbaum [4] that every congrüence ρ can be reconstructed from its trace and kernel. We give this result in the formulation of Pastijn-Petrich [18] which uses Green s relation $\mathcal{L}$ and $\mathcal{R}$; recall that for a regular semigroup S, $a\mathcal{L}b$ iff Sa = Sb and $a\mathcal{R}b$ iff aS = bS.

Theorem 2.3. (Pastijn-Petrich [19]. Any congruence ρ on a regular semigroup S with ker ρ = K and tr $\rho = \tau$ can be described in the following way:

$a\rho b \leftrightarrow a(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})b$, $ab' \in K$ for some (all) $b' \in V(b)$.

Note. It was proved by G. Gomes [5] that ρ can be obtained also in the following way:

$a\rho b \leftrightarrow aa'\rho\, bb'aa'$, $b'b\rho\, b'ba'a$, $ab' \in K$ for some (all) $a' \in V(a)$, $b' \in V(b)$.

As it was observed above, to every congruence ρ on S there can be associated the pair (ker ρ, tr ρ). But the problem is to find conversely all congruences on S. In general, for a pair (K, τ) with $K \subseteq S$ and τ an equivalence on E_S, there is not always a congruence ρ on S such that K=ker ρ and τ= tr ρ. Thus, the pairs (ker ρ, tr ρ), $\rho \in C(S)$, have to be characterized abstractly in order to give all pairs (K, τ) by means of which a congruence on S can be defined.

For inverse semigroups S this attempt was successful in the following way (Petrich [23]): if $K \subseteq S, \tau \in C(E_S)$, then the pair (K, τ) is called a congruence-pair if

1) K satisfies: (i) $E_S \subseteq K$; (ii) $a \in K \rightarrow a^{-1} \in K$, (iii) $a^{-1}Ka \subseteq K\ \forall a \in S$

   (where a^{-1} denotes the unique element $a' \in V(a)$)

2) τ satisfies: $e\tau f$, $a \in S$ imply that $a^{-1}ea\ \tau\ a^{-1}fa$

3) $ae \in K$, $a^{-1}a\tau e$ $(a \in S,\ e \in E_S) \rightarrow a \in K$

4) $aa^{-1}\tau a^{-1}a$ for all $a \in S$.

Then for any inverse semigroup S and every congruence ρ on S the pair $(\ker \rho, \operatorname{tr}\rho)$ is a congruence pair, and conversely, for every congruence pair (K,τ) the relation

$$a\ \rho_{(K,\tau)}\ b \leftrightarrow a^{-1}a\ \tau\ b^{-1}b,\ ab^{-1} \in K$$

is a congruence on S such that $\ker \rho_{(K,\tau)} = K$, $\operatorname{tr}\rho_{(K,\tau)} = \tau$.

For regular semigroups S, Pastijn-Petrich [19] found an abstract characterization of those pairs (K,τ) for which a congruence ρ on S can be defined in an analogous way. The first trivial observation is that K has to be the kernel of some congruence, which is equivalent to say that $K = \ker \pi_K$, where π_K is defined on S by

$$a\ \pi_K\ b \leftrightarrow xay \in K \text{ is equivalent } xby \in K (x,y \in S^1).$$

Also, τ has to be the trace of some congruence, which is equivalent to the requirement that $\tau = \operatorname{tr} \tau^*$ (where τ^* denotes the congruence on S generated by the equivalence τ on E_S): see Pastijn-Petrich [19] .

The key to the theory-similar to the inverse case-is the following concept.

Definition. (Pastijn-Petrich [19]). Let S be a regular semigroup, $K \subseteq S$, τ an equivalence on E_S; then a pair (K,τ) is called a congruence-pair if

(i) K is a normal subset of S(i.e. K is the kernel of some congruence on S)

(ii) τ is a normal equivalence on E_S (i.e. τ is the trace of some congruence on S)

(iii) $K \subseteq \ker (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^\circ$ (where for any equivalence ξ on S, ξ° denotes the greatest congruence on S contained in ξ)

(i) $\tau \leq \operatorname{tr} \pi_K$.

Note that in case that S is an inverse semigroup, this definition of congruence-pair reduces to that given above.

With this concept we are ready for the construction of all congruences on a regular semigroup, which is completely analogue to the situation in the inverse case.

Theorem 2.4. (Pastijn-Petrich [19]). If (K,τ) is a congruence-pair of the regular semigroup S, then the relation $\rho_{(K,\tau)}$ defined as in Theorem 2.3 is the unique congruence ρ on S for which $\ker\rho = K$, $\operatorname{tr}\rho = \tau$. Conversely, if ρ is a congruence on S then $(\ker\rho, \operatorname{tr}\rho)$ is a congruence pair of S and $\rho_{(\operatorname{Ker}\rho,\operatorname{tr}\rho)} = \rho$.

An obvious, but very useful consequence is the following

Corollary 2.5. (Pastijn-Petrich [19]). Let $(C(S),\leq)$ be the lattice of all congruences and $Cp(S)$ the set of all congruence-pairs of a regular semigroup S, partially ordered by: $(K,\tau) \leq (K',\tau')$ iff $K \subseteq K'$, $\tau \leq \tau'$. Then the mappings $\rho \to (\ker\rho, \operatorname{tr}\rho)$, $(K,\tau) \to \rho_{(K,\tau)}$ are mutually inverse isomorphisms of the lattices $C(S)$ and $Cp(S)$.

The special case of orthodox semigroups is worthy of note. A regular semigroup $(S,.)$ is called orthodox if E_S forms a subsemigroup of S. Note that an inverse semigroup S can be characterized as a regular semigroup, in which all idempotents commute; thus every inverse semigroup is orthodox. The concept of congruence-pair reduces in this case to a set of axioms which is strongly reminiscent to the inverse case.

Gomes [5] called a pair (K,τ) a <u>congruence-pair</u> of the orthodox semigroup $(S,.)$ if

1) K satisfies: (i) $E_S \subseteq K$; (ii) $a \in K \rightarrow a' \in K$ for some [all] $a' \in V(a)$; (iii) $a'Ka \subseteq K$ for all $a \in S$, $a' \in V(a)$

2) τ satisfies: $e \tau f$, $a \in S$ imply that $a'ea \tau a'fa$ (note that $a'ea \in E_S$ for all $a \in S$, $a' \in V(a)$, $e \in E_S$)

3) $ae \in K$, $a'a \tau e$ $(a \in S, e \in E_S) \rightarrow a \in K$

4) $a'ea \tau a'a'eaa$ for all $a \in S$, $e \in E_S$.

Then Gomes [5] showed that for any <u>orthodox</u> semigroup S, if (K,τ) is a congruence-pair of S then $\rho(K,\tau)$ defined by:

$$a \, \rho_{(K,\tau)} b \leftrightarrow \exists a' \in V(a),\ b' \in V(b):\ aa' \tau bb'\ \ aa', b'b \tau b'b\ a'a\ ,\ ab' \in K$$

is a congruence on S with kernel K and trace τ. Conversely, if ρ is a congruence on S then $(\ker\rho, \operatorname{tr}\rho)$ is a congruence-pair of S and $\rho = \rho(\ker\rho, \operatorname{tr}\rho)$. Also, the mappings $\rho \rightarrow (\ker\rho, \operatorname{tr}\rho)$, $(K,\tau) \rightarrow \rho_{(K,\tau)}$ are mutually inverse lattice isomorphisms between $(C(S), \leq)$ and $(Cp(S), \leq)$.

In order to illustrate the construction of all congruences on a regular semigroup S, some <u>special cases</u> will be considered. Compare also with the explicite form of certain congruences given in section 3. below.

<u>a) $K = E_S$, $\tau = \varepsilon$</u>

It is easily seen that (E_S, ε) is a congruence-pair of S defining the identity relation on S: $\rho_{(E_S,\varepsilon)} = \varepsilon$.

b) $K = S, \tau = \omega$

It is immediate that (S, ω) is a congruence-pair of S defining the universal relation on S: $\rho_{(S,\omega)} = \omega$.

c) $K = E_S, \tau = \omega$

(E_S, ω) is a congruence-pair of S iff tr $\pi_{E_S} = \omega$, i.e. iff for all $e, f \in E_S$

$$xey \in E_S \text{ is equivalent to } xfy \in E_S \quad (x, y \in S^1).$$

In this case, S is orthodox (put $x = e$, $y = 1$). Note that conversely, if S is orthodox then (E_S, ω) is not necessarily a congruence-pair. In fact, consider $S = T_2$, the semigroup of all transformations on the set $X = \{1,2\}$.
Then S is orthodox, but for $x = \alpha$, $e = \alpha_1$, $f = y = id$ (where $\alpha(1) = 2$, $\alpha(2) = 1$; $\alpha_1(x) = 1$; $\alpha_2(x) = 2$ for all $x \in X$), $\alpha \circ \alpha_1 \circ id = \alpha_2 \in E_S$ and $\alpha \circ id \circ id = \alpha \notin E_S$.
Furthermore, if (E_S, ω) is a congruence-pair then by 2.5.
$\rho_{(E_S,\omega)} = \sigma$ – the least group congruence on S (since for every group congruence ρ on S, tr $\rho = \omega$); it is given explicitely by

$$a \sigma b \leftrightarrow ab' \in E_S \text{ for some (all) } b' \in V(b).$$

Also, in this case $\rho(E_S, \omega) = \pi$, the greatest idempotent-pure congruence on S (since for every such congruence ρ, ker $\rho = E_S$).

d) K normal, $\tau = \omega$

(K, ω) is a congruence pair of S iff tr $\pi_K = \omega$, i.e. iff for all $e, f \in E_K$

$$xey \in K \text{ is equivalent to } xfy \in K (x, y \in S^1).$$

In this case, $\rho_{(K,\omega)}$ is a group congruence given by

$$a\,\rho_{(K,\omega)}\,b \leftrightarrow ab' \in K \text{ for some (all) } b' \in V(b).$$

Thus, by 2.5, the least group congruence σ on S is defined by the least normal subset K of S satisfying the condition at the beginning of this paragraph.

e) $K = E_S, \tau$ normal

(E_S, τ) is a congruence-pair of S iff $\tau \leq tr\pi_{E_S}$, i.e. iff for all $e,f \in E_S$,

$$e\,\tau f \rightarrow xey \in E_S \text{ is equivalent to } xfy \in E_S \ (x,y \in S^1).$$

In this case, $\rho_{(E_S,\tau)}$ is an idempotent-pure congruence on S (since for every such congruence ρ on S, $\ker \rho = E_S$). Thus by 2.5, the greatest idempotent-pure congruence π on S is defined by the greatest normal equivalence τ on E_S satisfying the condition at the beginning of this paragraph.

f) $K = S, \tau = \varepsilon$

(S,ε) is congruence-pair of S iff $\ker \mathcal{H}^\circ = S$ (where $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$).
We shall see that this is the case iff S is a band of groups (i.e. S is a union of groups and $\mathcal{H}$ is a congruence on S; see Petrich [21], IV.1.7).

In fact, if S is a band of groups, then $\mathcal{H}^\circ = \mathcal{H}$ and for every $a \in S$, $a \in H_e$ for some $e \in E_S$; thus $a \in \ker \mathcal{H} = \ker \mathcal{H}^\circ$, i.e. $\ker \mathcal{H}^\circ = S$. Conversely, suppose that $\ker \mathcal{H}^\circ = S$. Then for every $a \in S$ there is some $e \in E_S$ such that $a\,\mathcal{H}^\circ\,e$, hence $a\,\mathcal{H}\,e$ and S is the union of the groups H_e (see Clifford-Preston [1], 2.16). Let $a\mathcal{H}b$ $(a,b \in S)$; by Lallement [9], $\mathcal{H}^\circ = \mu$ hence $\ker \mu = S$ (μ the greatest idempotent-separating congruence on S). Thus, $a\,\mu\,e$ for some $e \in E_S$, and $b\ \mu\ f$ for some $f \in E_S$. Consequently, $a \mathcal{H} e$ and $b \mathcal{H} f$, thus $e \mathcal{H} f$ and by Clifford-Preston [1], 2.16, $e = f$. Hence, $a\ \mu\ e$ and $b \mu e$, thus $a\ \mu\ b$. Since μ is a congruence, it follows that $ac\ \mu\ bc$, $ca\ \mu\ cb$ for all $c \in S$. Now by Lallement [9] $\mu \leq \mathcal{H}$, hence $ac \mathcal{H} bc$ and $ca \mathcal{H} cb$ for all $c \in S$. Consequently, $\mathcal{H}$ is a congruence and S is a band of groups.

In this case, $\rho_{(S,\varepsilon)} = \beta = \mu = \mathcal{H}$, where β denotes the least band congruence on S. In fact, $\rho_{(S,\varepsilon)} = \beta$ since $\ker \mathcal{H}^\circ = S$ implies that for every $a \in S$ there is $e \in E_S$ such that $a\, \rho_{(S,\varepsilon)} e$, thus each $\rho_{(S,\varepsilon)}$ - class is idempotent. This means, that $\rho_{(S,\varepsilon)}$ is a band congruence. It is the least such (by 2.5), since for every band congruence ρ, $\ker \rho = S$ (by 1.1.). Furthermore, by Lallement [9], $\mathcal{H}^\circ = \mu$. Hence by hypothesis, $\ker \mu = S$ and $\mu = \beta$ by 2.5 (since $\ker \beta = S = \ker \mu$ and $\operatorname{tr} \beta = \operatorname{tr} \rho_{(S,\varepsilon)} = \varepsilon = \operatorname{tr} \mu$). But by Howie-Lallement [8], 1.3, $\mu = \mathcal{H}^\circ \leq \mathcal{H} \leq \beta$, so that $\mathcal{H} = \beta = \mu$ (which again implies that $\mathcal{H}$ is a congruence).

<u>g) K normal, $\tau = \varepsilon$</u>

(K, ε) is a congruence-pair of S iff $K \subseteq \ker \mathcal{H}^\circ = \ker \mu$. Recall (Latorre [10], 12) that for every regular semigroup S,

$$\ker \mu = \{ a \in S \mid \exists\, a' \in V(a):\ a'\ ea = e \text{ for each idempotent } e \leq aa' \}.$$

Hence, (K, ε) is a congruence-pair iff for every $a \in K$ there is some $a' \in V(a)$ such that $a'ea = e$ for each idempotent $e \leq aa'$. In this case, $\rho_{(K,\varepsilon)}$ is an idempotent separating congruence on S (since for every such congruence ρ on S, $\operatorname{tr} \rho = \varepsilon$), explicitely given by

$$a\ \rho_{(K,\varepsilon)}\ b \leftrightarrow a \mathcal{H} b \text{ and } ab' \in K \text{ for some (all) } b' \in V(b).$$

Thus by 2.5, the greatest idempotent separating congruence μ on S is defined by the greatest normal subset K of S satisfying the above condition.

3. Particular congruences

The knowledge of a simple, explicite form of a particular congruence on a semigroup is of special importance when applying it in certain considerations. For regular semigroups, useful descriptions of some important congruences are known. A survey of these will be given including several different methods of characterization, which have been found up to now. Note that in section 2. some particular congruences have appeared already, given explicitely by their kernel and their trace .

a) Group congruences

Croisot [2] found a description of all group congruences on an arbitrary semigroup S by means of particular subsemigroups of S. For every subset H of S and any $a \in S$ denote

$$a:H = \{ (x,y) \in S \times S \mid xay \in H \} .$$

Theorem 3.1. (Croisot [2]). Let S be a semigroup and H be a subsemigroup of S such that (i) $a:H \neq \emptyset$ for all $a \in S$, and (ii) $a:H \cap b:H \neq \emptyset$ $(a,b \in S) \rightarrow$ $\rightarrow a:H = b:H$. Then the relation $a \ \rho_H \ b \leftrightarrow a:H = b:H$ is a group congruence on S. Conversely, if ρ is any group congruence on S then the identity class E of S/ρ is a subsemigroup of S satisfying (i),(ii) and $\rho_E = \rho$.

For regular semigroups, numerous other characterizations of group congruences are known. Note first that every cancellative congruence ρ on a regular semigroup S is a group congruence, and conversely (since S/ρ is a regular, cancellative semigroup, thus a group). Since the universal congruence on S is cancellative and since the intersection of all cancellative congruences on S is again a cancellative congruence on S, the least group-congruence can be described in the following way.

Theorem 3.2. (Masat [12]). Let S be a regular semigroup; then the least group congruence σ on S is given by $\sigma = \rho^t$ where

$$a \rho b \leftrightarrow e a e = e b e \text{ for some } e \in E_S$$

and ρ^t means the transitive closure of ρ.

Note. If S is a conventional semigroup (i.e. S is regular and $a E_S a' \subseteq E_S$ for all $e \in S$, $a' \in V(a)$) then the unpleasant transitive closure of ρ can be omitted and $\sigma = \rho$ (Masat [12]). In particular, this is true for every orthodox semigroup which was proved already by Meakin [14].

A more convenient description of σ on a general regularsemigroup was given by Masat[12] by means of the reflexive subsemigroup of S generated by its idempotents E_S. A subset T of S is called reflexive if $ab \in T$ $(a,b \in S)$ implies $ba \in T$.

Theorem 3.3. (Masat [12]). Let S be a regular semigroup; denote by T the reflexive subsemigroup of S generated by E_S and by $T\omega = \{a \in S \mid ta \in T$ for some $t \in T\}$. Then the least group congruence σ on S is given by

$$a \sigma b \leftrightarrow xa,\ xb \in T\omega \text{ for some } x \in S.$$

Note. If S is conventional, then $T\omega = \{ a \in S \mid ea \in E_S \text{ for some } e \in E_S \}$ (Masat [12]). In particular, if S is E-unitary and regular (i.e. $ea, e \in E_S \rightarrow a \in E_S$) then S is orthodox (see Howie-Lallement [8], 2.1) and $T\omega = E_S$. Hence, in this case

$$a \sigma b \leftrightarrow xa, xb \in E_S \text{ for some } x \in S.$$

An other approach to the characterization of all group congruences on a regular semigroup S was found by Feigenbaum [4] using full and self-conjugate subsemigroups of S: a subset T of S is called full if $E_S \subseteq T$, and self-conjugate if $a'Ta \subseteq T$ for all $a \in S$, $a' \in V(a)$. Let C denote the set of all full and self-conjugate subsemigroups of S and let U be the intersection of all semigroups in C.

Theorem 3.4. (Feigenbaum [4]). For each $H \in C$, the relation

$$a \rho_H b \leftrightarrow xa = by \text{ for some } x, y \in H$$

is a group congruence on the regular semigroup S. The least group congruence σ on S is given by $\sigma = \rho_U$.

Defining the closure of a subset H of a regular semigroup S as the set

$$H\omega = \{a \in S \mid h\, a \in H \text{ for some } h \in H\},$$

Feigenbaum [4] showed that also for each $H \in C$

$$a \rho_H b \leftrightarrow ab' \in H\omega \text{ for some (all) } b' \in V(b).$$

Further details for the description of ρ_H can be found in Latorre [10] .

In particular, he showed that

$$a \sigma b \leftrightarrow aub' \in U \text{ for some } u \in U \text{ and some (all) } b' \in V(b).$$

Now let $\bar{C}$ be the set of all <u>closed</u> subsemigroups in C, i.e. consider those full and self-conjugate subsemigroups Hof S such that $H\omega = H$. Note that a closed subsemigroup H of a regular semigroup is necessarily regular. Feigenbaum [4] proved that the mapping

$$\bar{H} \to \rho_{\bar{H}}, \text{ where } a \rho_{\bar{H}} b \leftrightarrow ab' \in \bar{H} \text{ for some } b' \in V(b),$$

is a bijektive and inclusion preserving function of $\bar{C}$ onto the set of all group congruences on S. (For every group congruence ρ on S, $\rho = \rho_{\bar{H}}$ with $\bar{H} = \ker\rho$). It is easily seen that the intersection $\bar{U}$ of all semigroups $\bar{H}$ in $\bar{C}$ is again closed. Consequently, we obtain that $\sigma = \rho_{\bar{U}}$ and

$$a \sigma b \leftrightarrow ab' \in \bar{U} \text{ for some (all) } b' \in V(b).$$

<u>Note.</u> For the much larger class of <u>E-inversive</u> semigroups an explicite description of all group congruences was given by Mitsch[17]. A semigroup S is called E-inversive if for every $a \in S$ there is some $x \in S$ such that $ax \in E_S$. This is equivalent to the condition that $I(a) = \{x \in S \mid ax, xa \in E_S\} \neq \emptyset$ for all $a \in S$. The characterization is strongly reminiscent to that given by Feigenbaum for the regular case (see Theorem 3.4).

<u>b) Right group congruences</u>

A group is right- and left-simple and also right- and left-cancellative. Weakening these properties one may ask for those homomorphic images of a semigroup S which are <u>right groups</u>, i.e. which are right-simple and left-cancellative (for several equivalent definitions see Clifford-Preston [1]).

A description of all right-group congruences on an arbitrary semigroup S was given by Massant [13] by means of group-congruences and left-zero congruences on S. Numerous characterizations of group congruences were given in a). Concerning right-zero congruences ρ (i.e. such that S/ρ is a right-zero semigroup: $xy = y \ \forall x,y \in S/\rho$) a description for <u>arbitrary</u> semigroups S can be found in Petrich[22] , III. 1:

Let L_S be the set of all left ideals $L \neq S$ of S such that $ab \in L$ implies $b \in L (a,b \in S)$; denoting by L_x the least left ideal of S in L_S containing $x \in S$, the following characterization of right-zero congruences on S holds:

Let S be a semigroup and $\emptyset \neq A \subseteq L_S$; then the relation

$$a \rho_A b \leftrightarrow \text{for every } L \in A : \text{either } a,b \in L \text{ or } a,b \notin L$$

is a right-zero congruence on S. The least right-zero congruence ξ on S is given by $\xi = \rho_{L_S}$, or equivalently by $a \xi b \leftrightarrow L_a = L_b$.

<u>Theorem 3.5.</u> (Massat[13). Let S be a semigroup; then a congruence ρ on S, which is not a group- nor a right-zero congruence, is a right-group congruence iff is the intersection of a non-trivial group congruence on S and a right-zero congruence on S.

Since on <u>regular</u> semigroup S the least group congruence exists (see a) above) we obtain the following

Corollary 3.6. Let S be a regular semigroup; then the least right-group congruence on S is given by

$$a \,\gamma\, b \leftrightarrow L_a = L_b \text{ and } xa = by \text{ for some } x,y \in U,$$

where U is the intersection of all full and self-conjugate subsemigroups of S.

For the special case that S is regular with E_S a rectangular band (i.e. $e f e = e$ for all $e,f \in E_S$), Massat [12] gave the following description of the least right-group congruence on S:

$$a \rho b \leftrightarrow ea = eb \text{ for all } e \in E_S.$$

Conversely, he showed that if the congruence ρ so defined on a regular semigroup S is a right-group congruence on S then E_S is a rectangular band.

c) The least inverse congruence

Reducing the condition that the homomorphic image of S has to be a group one can ask for those congruences ρ on S, for which S/ρ is an inverse semigroup. In the general case, there is no description of such congruences similar to the group case. Even the characterization of the least inverse congruence Y is not very satisfactory. It is based on the fact that a regular semigroup S is inverse iff the idempotents of S commute (see Petrich [22]).

Theorem 3.7. (Hall [6]). Let S be a regular semigroup; then the least inverse congruence Y on S is given by $Y = \rho^*$, where

$$a \rho b \longleftrightarrow a = ef, \quad b = fe \quad \text{for } e,f \in E_S$$

and ρ^* denotes the congruence on S generated by ρ .

In the particular case that S is orthodox, Hall [6] gave the following explicite description of Y: $a \, Y \, b \leftrightarrow V(a)=V(b)$.
Also, he showed conversely that if for a regular semigroup S, Y is an inverse congruence on S then S is orthodox.

Using the concept of congruence pair G.Gomes (R-unipotent congruences on regular semigroups, Semigroup Forum 31 (1985), 265-280) found a description of all inverse congruences on an arbitrary regular semigroup S. She called a pair (K,τ) an inverse congruence pair of S if

a) K satisfies: (i) K is a regular subsemigroup of S; (ii) $E_S \subseteq K$;
(iii) $a'Ka \subseteq K$ for all $a \in S$, $a' \in V(a)$;

b) τ is a congruence on $\langle E_S\rangle$, the subsemigroup of S generated by E_S, such that (i) $\langle E_S\rangle/\tau$ is a semilattice, (ii) $x \, \tau \, y$, $x,y \in \langle E_S\rangle \rightarrow$
$\rightarrow a'xa \, \tau \, a'ya$, whenever $a'xa, a'ya \in \langle E_S\rangle$ for $a \in S$, $a' \in V(a)$;

c) (i) $ax \in K$, $a'a \, \tau \, x$ $(a \in S, a' \in V(a), x \in \langle E_S\rangle \rightarrow a \in K$
(ii) $ab \in K (a,b \in S) \rightarrow axb \in K$ for all $x \in \langle E_S\rangle$
(iii) $axa' \tau aa'x$, whenever $axa' \in \langle E_S\rangle$ for $a \in S$, $a' \in V(a)$, $x \in \langle E_S\rangle$

Given such an inverse congruence pair the unique inverse congruence on S, whose kernel is K and whose restriction to $\langle E_S\rangle$ is τ, is given by

$$a \, \rho_{(K,\tau)} b \Longleftrightarrow \exists \, a' \in V(a), \, b' \in V(b) : aa' \tau \, bb', \, a'b \in K.$$

Conversely, if ρ is an inverse congruence on S then $(\ker\rho, \tau)$ with $\tau = \rho|_{\langle E_S\rangle}$ is an inverse congruence pair of S and $\rho_{(\ker\rho, \tau)} = \rho$.

Remark. As a consequence, the particular case of group congruences on a general regular semigroup S now can be described in the following way (G. Gomes, loc.cit.):

If $K \subseteq S$ satisfies (a) above and (d) $ax \in K (a \in S,\ x \in \langle E_S \rangle) \rightarrow a \in K$, then the relation

$$a\ \rho_K b \Leftrightarrow \exists b' \in V(b) \text{ such that } ab' \in K$$

is a group congruence on S with kernel K. Conversely, if ρ is a group congruence on S then ker ρ satisfies (a) and (d) above and $\rho_{\ker\rho} = \rho$.

d) The least semilattice of groups congruence

A semilattice of groups (or: Clifford semigroup) can be defined as a regular semigroup with central idempotents (i.e. ea = ae for every $a \in S$ and every $e \in E_S$). Thus, such a semigroup is a special inverse semigroup and also a particular union of groups (see Clifford-Preston [1]). An explicite form of the least congruence ρ on a regular semigroup S such that S/ρ is a semilattice of groups was found by Latorre [11] . It is a characterization by means of the least group congruence σ on S (see Theorem 3.4 above) and the least semilattice congruence η on S (see paragraph f) below).

Theorem 3.8. (Latorre [11]). Let S be a regular semigroup; then the least semilattice of groups congruence on S is given by

$$a\ \nu\ b \rightarrow a\ \eta\ b \text{ and } xa = by \text{ for some } x,y \in U \cap (a\eta),$$

where U is the intersection of all full and self-conjugate subsemigroups of S.

In the particular case that S is orthodox, J. Mills [16] showed that

$$a\ \nu\ b \Leftrightarrow a\ \eta\ b \text{ and } eae = ebe \text{ for some } e \in E_S \cap (a\eta).$$

Latorre [11] described ν on an orthodox semigroup S in a slightly different way:

$$a \nu b \leftrightarrow a \eta b \quad \text{and } ea = bf \text{ for some } e,f \in E_S \cap (a\eta).$$

e) Orthodox congruences

An inverse congruence ρ on a regular semigroup S yields a (regular) homomorphic image S/ρ, in which the idempotents commute. Generalizing, one may ask for those congruences ρ on S, for which the idempotents of S/ρ form a subsemigroup, only. Gomes [5] gave a description of all these orthodox congruences by means of so called orthodox congruence-pairs, specializing the general concept of congruence-pair on a regular semigroups defined by Pastijn-Petrich [18] (see section 2, above).

Definition (Gomes [5]). Let S be a regular semigroup.

1) A subset K of S is said to be a normal subsemigroup of S if K is a regular subsemigroup of S such that $E_S \subseteq K$ and $aKa' \subseteq K$ for every $a \in S$, $a' \in V(a)$.
2) A congruence ξ on $\langle E_S \rangle$, the subsemigroup of S generated by E_S, is called normal if $x \xi y \rightarrow a'xa \; \xi \; a'ya$ for all $a \in S$, $a' \in V(a)$, whenever $a'xa, a'ya \in \langle E_S \rangle$.
3) The restriction of a congruence ρ on S to $\langle E_S \rangle$ is called the hypertrace (core) of ρ, denoted by $\text{htr}\,\rho$.

Those congruence-pairs, which yield all the orthodox congruences on a regular semigroup, are characterized abstractly in the following

Definition (Gomes [5]). Let S be a regular semigroups, K a normal subsemigroup of S and ξ a normal congruence on $<E_S>$ such that $<E_S>/\xi$ is a band. Then the pair (K,ξ) is called an orthodox congruence-pair of S if for all $a,b \in S$, $a' \in V(a)$, $x \in <E_S>$ and $f \in E_S$,

(i) $xa \in K$, $x \,\xi\, aa' \rightarrow a \in K$

(ii) $ab \in K$, $a'a \,\xi\, bb'\, a'a \rightarrow axb \in K$

(iii) $a \in K$, $aa' \,\xi\, f \rightarrow fxf \,\xi\, fa'xaf$, whenever $fa'xaf \in <E_S>$.

Theorem 3.9 (Gomes [5]). Let S be a regular semigroup. If (K,ξ) is an orthodox congruence-pair of S then the relation

$$a\, \rho_{(K,\xi)} b \leftrightarrow aa' \xi\, bb'aa',\ b'b \xi\, b'ba'a,\ ab' \in K \text{ for some (all) } a' \in V(a),\ b' \in V(b)$$

is an orthodox congruence on S such that $\ker \rho_{(K,\xi)} = K$, $\operatorname{htr} \rho_{(K,\xi)} = \xi$. Conversely, if ρ is an orthodox congruence on S, then $(\ker \rho, \operatorname{htr} \rho)$ is an orthodox congruence-pair of S and $\rho_{(\ker\rho,\ \operatorname{htr}\rho)} = \rho$.
Furthermore, the mappings $\rho \rightarrow (\ker\rho, \operatorname{htr}\rho)$, $(K,\xi) \rightarrow \rho_{(K,\xi)}$ are mutually inverse order-preserving between the lattice of all orthodox congruences on S and the set of all orthodox congruence-pairs of S partially ordered by

$$(K,\xi) \leq (K',\xi') \text{ iff } K \subseteq K',\ \xi \leq \xi'.$$

Remark 1. For the special case that S is orthodox itself, this result yields a description of all congruences on S (see section 2. above).

2. The least orthodox congruence λ on a regular semigroup S can be descripted also in the following evident way: $\lambda = \rho^*$, where

$$a \,\rho\, b \leftrightarrow a = ef,\ b = efef \text{ for } e,f \in E_S.$$

f) Semilattice congruences

A semilattice is defined as a commutative and idempotentsemigroup, i.e. as a special band. Band congruences on a semigroup S are of particular interest, because all the congruence classes form subsemigroups of S. For general semigroups a construction of all band congruences is known, as are descriptions of all rectangular band congruences and of all (right,left) normal band congruences (see Petrich[22], III, IV). The least band congruences on a regular semigroup satisfies

$$\mathcal{H} \leq \beta \leq \mathcal{R}^* \cap \mathcal{L}^*,$$

$\mathcal{H}, \mathcal{R}, \mathcal{L}$ are Greens's relations (see Howie-Lallement [8]).

For the important special case of semilattice congruences, the construction found by Petrich [20] for arbitrary semigroups will be given now. Recall that a filter F of a semigroup S is a subsemigroup of S such that ab$\in$ F implies that a,b $\in$F. Note that $\emptyset \neq F \subseteq S$ is a filter of S iff $I = S \setminus F$ is empty or a completely prime ideal of S (i.e. an ideal I of S such that ab $\in$I implies that a $\in$I or b $\in$I).Denote by $\mathfrak{F}$ the set of all filters of S and F_x the least filter of S containing x $\in$S.

Theorem 3.10 (Petrich [20]) Let S be a semigroup and $A \subseteq \mathfrak{F}$ be a set of filters of S. Then the relation

$$a \rho_A b \quad \longleftrightarrow \quad \text{for every } F \in A \text{ either } a,b \in F \text{ or } a,b \notin F$$

is a semilattice congruence on S. Conversely, for every such congruence ρ

on S there is some $A \subseteq \mathfrak{F}$ such that $\rho = \rho_A$. The least semilattice congruence η on S is given by $\eta = \rho_{\mathfrak{F}}$, or equivalently by

$$a \eta b \leftrightarrow F_a = F_b \leftrightarrow \text{for every filter } F: a \in F \text{ iff } b \in F.$$

For regular semigroups S, η can be described by means of Green's relation $\mathcal{D}$ or $\mathcal{J}$ on S (where $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ and $\mathcal{J}$ is defined by : $a \mathcal{J} b$ iff $SaS = SbS$):

Theorem 3.11 (Howie-Lallement [8] Let S be a regular semigroup. Then the least semilattice congruence η on S is given by $\eta = \mathcal{D}^* = \mathcal{J}^*$ (where $\mathcal{D}^*$ denotes the congruence on S generated by $\mathcal{D}$).

g) The greatest idempotent-pure congruence

A congruence ρ on a semigroup S is called idempotent-pure (also: idempotent-determined) if

$$a \rho e, \quad a \in S, \quad e \in E_S \rightarrow a \in E_S,$$

i.e. each ρ-class containing an idempotent consists entirely of idempotents. Evidently, the identity relation on S is an idempotent-pure congruence. For general semigroups, the greatest such congruence can be described in the following way.

Theorem 3.12. (Theissier [24]). If S is a semigroup, then the relation

$$a \pi b \longleftrightarrow xay \in E_S \text{ if and only if } xby \in E_S (x, y \in S^1)$$

is the greatest idempotent-pure congruence on S.

h) Idempotent-separating congruences

In a certain sense opposite to the idempotent-pure congruences are those congruences ρ for which each congruences class contains at most one idempotent, i.e.

$$e \rho f, \quad e,f \in E_S \rightarrow e = f.$$

Clearly, the identity relation on S is always idempotent-separating. It was noted by Lallemenet [9] that for a regular semigroup every such congruence is contained in Green's relation $\mathcal{H}$.

Theorem 3.13. (Lallement [9]) Let S be a regular semigroup. Then a congruence ρ on S is idempotent-separating iff $\rho \leq \mathcal{H}$. Thus, the greatest idempotent-separating congruence on S is given by $\mu = \mathcal{H}^\circ$ (the greatest congruence contained in $\mathcal{H}$), i.e.

$$a \mu b \leftrightarrow xay \mathcal{H} xby \; \forall x,y \in S^1 .$$

Note that the hypothesis of the regularity of S cannot be removed: if $S = \{0,a\}$ is the two-element zero semigroup ($a^2 = a0 = 0a = 00 = 0$) then $\mathcal{H} = \varepsilon$, the identity relation, and $\mu = \omega$, the universal relation, hence $\mu \nsubseteq \mathcal{H}$.

Another characterization of μ on a regular semigroup was given by Hall [7] and Meakin [15] , independently:

Theorem 3.14. (Hall [7]) Let S be a regular semigroup; then

$a \mu b \leftrightarrow \exists\, a' \in V(a), b' \in V(b): aa'=bb',\ a'a=b'b,\ a'ea=b'eb$ for each idempotent $e \leq aa'$.

For the special case that S is orthodox, Meakin [15] found the following description of μ:

$a \mu b \leftrightarrow \exists\, a' \in V(a),\ b' \in V(b): a'ea=b'eb,\ aea'=beb'$ for all $e \in E_S$.

Note. For the much larger class of eventually regular semigroups an explicite description of μ was found by Edwards [3] . A semigroup S is called eventually regular if for every $a \in S$ there is some positive integer n such that $a^n \in S$ is regular. The greatest idempotent-separating congruence on such a semigroup is given by

$a\ \mu\ b \leftrightarrow$ if $x \in S$ is regular then each of $x \mathcal{R} xa$, $x \mathcal{R} xb$ implis $xa \mathcal{H} xb$, and each of $x \mathcal{L} ax$, $x \mathcal{L} bx$ implies $ax \mathcal{H} bx$.

It is noted also, that the hypothesis on S to be eventually regular cannot be removed. An example of a semigroup is given for which the greatest idempotent-separating congruence is different from μ described above (see Edwards [3], Ex. 3).

Remark. There is still another approach of characterizing particular congruences on a regular semigroup S. Since every congruence on S is uniquely determined by its kernel and trace, one can define the following equivalence relations on the lattice C(S) of all congruences on S:

$$\rho K \tau \leftrightarrow \ker\rho = \ker\tau\ ;\ \rho T \tau \leftrightarrow \operatorname{tr}\rho = \operatorname{tr}\tau\ .$$

Then each K-class and each T-class is an interval in $(C(S), \leq)$. Using these two relations, P. Alimpić-D.Krgović(Some congruences on regular semigroups, Proceedings Oberwolfach 1986, Lect. Notes Math. 1320(1988), 1-10) gave an alternative description of some special congruences; for example:

(i) the least band of groups congruence on S is the least element of the T-class containing β;

(ii) the least semilattice of groups congruence on S is the least element of the T-class of η ;

(iii) the least E-unitary congruence on S is the least element of the K-class containing σ.

REFERENCES

[1] Clifford, A.H. and G.B. Preston: The algebraic theory of semigroups. Math. Surveys 7, Amer. Math. Soc., Providence (Vol I: 1961, Vol II: 1967)

[2] R.Croisot: Équivalences principales bilatères définies dans un demi-groupe. J. Math. Pures Appl. 36(1957), 373-417.

[3] P. Edwards: Eventually regular semigroups.Bull. Austral. Math. Soc. 28(1983), 23-38.

[4] R. Feigenbaum: Kernels of regular semigroup homomorphisms. Doctoral Dissertation, University of South Carolina (1975).

[5] G.Gomes: Orthodox congruences on a regular semigroup. Semigr. Forum 37(1988), 149-166.

[6] T.Hall: On regular semigroups whose idempotents form a subsemigroup. Bull.Austral.Math.Soc.1(1969), 195-208.

[7] T.Hall: On regular semigroups. J.of Algebra 24(1973), 1-24.

[8] Howie, J. and G. Lallement: Certain fundamental congruences on a regular semigroup. Glasgow Math.J.7 (1965/66), 145-159.

[9] G.Lallement: Congruences et équivalences de Green sur un demi-groupe régulier. C.R.Acad.Sci.Paris 262(1966), 613-616.

[10] D.Latorre: Group congruences on a regular semigroup. Semigr. Forum 24(1982), 327-340.

[11] D.Latorre: The least semilattice of groups congruence on a regular semigroup. Semigr. Forum 27(1983), 319-330.

[12] F.Masat: Right group and group congruences on a regular semigroup. Duke Math. J. 40(1973), 393-402.

[13] F.Masat: Right group congruences on a semigroup. Proc. Amer.Math. Soc. 50(1975), 107-114.

[14] J.Meakin: Congruences on a regular semigroups. Semigr.Forum 1 (1970), 232-235.

[15] J. Meakin: Congruences on orthodox semigroups II. J. Austral. Math. Soc. 13(1972), 259-266.

[16] J. Meakin: The maximum idempotent-separating congruence on a regular semigroup. Proc. Edinburgh Math. Soc. 18(1972/3), 159-163.

[17] J.Mills: Certain congruences on orthodox semigroups. Pac. J. Math. 64(1976), 217-226.

[18] H.Mitsch: Subdirect products of E-inversive semigroups. J. Austral. Math. Soc. - to appear.

[19] Pastijn,F. and M. Petrich: congruences on regular semigroups. Trans.Amer.Math.Soc. 295(1986), 607-633.

[20] M.Petrich: The maximal semilattice decomposition of a semigroup. Math. Zeitschr. 85(1964), 68-82.

[21] M.Petrich: Introduction to semigroups. Merill Publ.Comp., Columbus/Ohio (1973).

[22] M.Petrich: Lectures in semigroups. J. Wiley, London (1977).

[23] M.Petrich: Inverse semigroups. J. Wiley, New York (1984).

[24] M. Theissier: Sur les équivalences réguliers dans les demigroupes. C.R.Acad.Sci.Paris 232(1951), 1987-1989.

STRUCTURE, CONGRUENCES AND VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

NORMAN R. REILLY

1. LOCAL AND GLOBAL STRUCTURE

One area of research in the field of Semigroup Theory in which there have been significant successes in recent years has been the subject of completely regular semigroups. The aim of these lectures is to give a brief review of some of the achievements in the theory of completely regular semigroups. We will start with some familiar and well known results and concepts.

An element a of a semigroup S is regular if there exists an element x in S such that $a = axa$ and a semigroup S is regular if every element of S is regular.

If $a, x \in S$, a semigroup, are such that $a = axa$ and $y = xax$, then a simple calculation will verify that $a = aya$ and $y = yay$. Such an element y is called an inverse of a.

An element a of a semigroup S is completely regular if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. In particular, x must be an inverse of a.

LEMMA 1.1. For any element a in a semigroup S, the following statements are equivalent.

(i) a is completely regular.

(ii) a has an inverse with which it commutes.

(iii) H_a is a subgroup.

We say that a semigroup S is completely regular if every element of S is completely regular.

LEMMA 1.2. For any semigroup S the following statements are equivalent.

(i) S is completely regular.

(ii) S is a union of (disjoint) groups.

(iii) Every $\mathcal{H}$ - class of S is a group.

NOTATION Let $\mathcal{CR}$ denote the class of all completely regular semigroups and for any $a \in S \in \mathcal{CR}$, let a^{-1} denote the inverse of a in the (group) $\mathcal{H}$ - class H_a and let a^0 denote the element $aa^{-1} = a^{-1}a$, the identity of the group H_a.

It is not hard to see that the class $\mathcal{CR}$ is closed with respect to products and homomorphic images. However, the additive group of integers is completely regular but has the infinite cyclic semigroup of positive integers, which is not completely regular, as a subsemigroup. Thus the class $\mathcal{CR}$ is not closed under subsemigroups. On the other hand, any subsemigroup of a completely regular semigroup which is closed under inverses ($a \longrightarrow a^{-1}$) is also completely regular.

These observations suggest considering completely regular semigroups not simply as semigroups but as semigroups endowed with a unary operation ($a \longrightarrow a^{-1}$). This has now become the accepted viewpoint from which to study the class $\mathcal{CR}$. When we do this the class $\mathcal{CR}$ becomes a variety of algebras endowed with a binary and a unary operation satisfying the following identities:

$$x(yz) = (xy)z, \quad x = xx^{-1}x, \quad (x^{-1})^{-1} = x, \quad xx^{-1} = x^{-1}x.$$

In this context, consistent with earlier notation, we shall write $x^0 = xx^{-1} = x^{-1}x$.

The manipulation of inverses in completely regular semigroups can present quite a problem. One observation that is sometimes helpful is the following.

LEMMA 1.3. (Petrich and Reilly [19], Lemma 2.8) The variety $\mathcal{CR}$ satisfies the identity

$$(xy)^{-1} = (xy)^0y^{-1}(yx)^0x^{-1}(xy)^0.$$

Recall that a simple semigroup is one without proper ideals. A completely simple semigroup is one which is both completely regular and simple.

Let S be the disjoint union of the semigroups S_α $(\alpha \in Y)$, where Y is a semilattice and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Then S is said to be a semilattice of the semigroups S_α, $\alpha \in Y$, and we write $S = (Y;S_\alpha)$. The importance of this concept in the theory of completely regular semigroups was revealed by the following theorem.

THEOREM 1.4. (Clifford [2] and [4], Theorem 4.6) Let S be a completely regular semigroup. Then $\mathcal{D} = \mathcal{J}$ is a congruence, each $\mathcal{J}$-class is a completely simple semigroup and $S/\mathcal{J}$ is a semilattice. Thus S is a semilattice of its $\mathcal{J}$-classes.

This theorem focusses the attention on the class of completely simple semigroups, not just as an interesting special class of completely regular semigroups but as an essential component of the structure of all completely regular semigroups. That the class of completely simple semigroups is an interesting class is also attested to by the fact that it can be characterized in so many different ways, as illustrated in the next theorem.

We adopt the notation E(S) for the set of idempotents of a semigroup S.

THEOREM 1.5. The following conditions on a semigroup S are equivalent.

(i) S is completely simple.

(ii) S is completely regular and satisfies the identity $(axb)^0 = (ab)^0$.

(iii) S is completely regular and satisfies the identity $(axa)^0 = a^0$.

(iv) S is completely regular and, for all $a,b,x \in S$, $ab \mathcal{H} axb$.

(v) S is completely regular and, for all $a,x \in S$, $a \mathcal{R} ax$.

(vi) S is regular and, for all $a,b \in S$, aSb is a maximal subgroup of S.

(vii) S is regular and weakly cancellative (that is, $ax = bx$ and $xa = xb$ implies that $a = b$).

(viii) S is regular and $a = axa$ implies that $x = xax$.

(ix) S is regular and every idempotent is primitive in E(S) ($e \in E(S)$ is primitive if $f \in E(S)$ and $ef = fe = e$ implies that $e = f$).

(x) S is simple and E(S) contains a primitive element.

It follows immediately from Theorem 1.5(ii) and (iii) that $\mathcal{CS}$ is a subvariety of $\mathcal{CR}$.

For any 4-tuple $(I,G,\Lambda;P)$ where G is a group, I and Λ are non-empty sets and $P: (\lambda,i) \longrightarrow p_{\lambda i}$ is a function from $\Lambda \times I$ to G, let

$\mathcal{M}(I,G,\Lambda;P) = I\times G\times\Lambda$ together with the multiplication

$$(i,g,\lambda)(j,h,\mu) = (i,gp_{\lambda j}h,\mu).$$

It is a straightforward exercise to show that $\mathcal{M}(I,G,\Lambda;P)$ is a completely simple semigroup. This construction is due to Rees and such semigroups are therefore called <u>Rees matrix semigroups</u>. However, Rees matrix semigroups are much more than examples of completely simple semigroups.

<u>THEOREM</u> 1.6. (Rees [31] and [4], Theorem 3.5) Every completely simple semigroup is isomorphic to a Rees matrix semigroup.

The Rees Theorem is tremendously important in the study of completely regular semigroups in general and completely simple semigroups in particular. Congruences and homomorphisms can be effectively studied in terms of the Rees matrix representations following from Theorem 1.6. Indeed, the construction of Rees matrix semigroups is so simple, it would almost seem as if any problem concerning completely simple semigroups could be resolved by the simple expedient of representing all completely simple semigroups as Rees matrix semigroups and then performing the appropriate arithmetic. While many problems are indeed amenable to such an approach it is not universally true as we shall see later.

We can view Clifford's Theorem as giving a global structure to any completely regular semigroup while Rees's Theorem provides a local structure. However, much of the complexity in the study of completely regular semigroups arises in going from the local to the global picture. This is perhaps best illustrated by the following general structure theorem for completely regular semigroups where the "simple" local components interact by means of factors and mappings.

<u>THEOREM</u> 1.7.(Petrich, [17]) For every $\alpha \in Y$ a semilattice, let $S_\alpha = \mathcal{M}(I_\alpha,G_\alpha,\Lambda_\alpha;P_\alpha)$ be normalized at $\alpha \in I_\alpha\cap\Lambda_\alpha$. For $\alpha \geq \beta$, let

(1) $\langle\ ,\ \rangle: S_\alpha\times I_\beta \longrightarrow I_\beta$,

(2) $S_\alpha \longrightarrow G_\beta$, denoted by $a \longrightarrow a_\beta$,

(3) $[\ ,\ ]:\Lambda_\beta\times S_\alpha \longrightarrow \Lambda_\beta$

be functions such that, for $a \in S_\alpha$, $b \in S_\beta$

(i) if $i \in I_\beta$ and $\lambda \in \Lambda_\beta$, then

$$p_{\lambda<a,i>}a_\beta p_{[\beta,a]i} = p_{\lambda<a,\beta>}a_\beta p_{[\lambda,a]i}$$

(ii) if $i \in I_\alpha$ and $\lambda \in \Lambda_\alpha$, then

$$a = (<a,i>,\ a_\alpha,\ [\lambda,a]).$$

On $S = \bigcup_{\alpha\in Y} S_\alpha$ define a multiplication by

(4) $$a\circ b = (<a,<b,\alpha\beta>>, a_{\alpha\beta}p_{[\alpha\beta,a]<b,\alpha\beta>}b_{\alpha\beta}, [[\alpha\beta,a],b]).$$

Suppose that

(iii) for $\gamma \le \alpha\beta$, $i \in I_\gamma, \lambda \in \Lambda_\gamma$,

$$(<a,<b,i>>,\ a_\gamma p_{[\gamma,a]<b,\gamma>}b_\gamma, [[\lambda,a],b]) = (<a\circ b,i>, (a\circ b)_\gamma, [\lambda,a\circ b]).$$

Then S is a completely regular semigroup whose multiplication restricted to each S_α coincides with the given multiplication. Conversely, every completely regular semigroup is isomorphic to one so constructed.

This result is remarkable for its complete generality. A special case of particular importance arises as follows.

Let $S = (Y;S_\alpha)$ and, for all α, $\beta \in Y$ with $\alpha \ge \beta$, let $\varphi_{\alpha,\beta}: S_\alpha \longrightarrow S_\beta$ be a homomorphism such that

(1) $\varphi_{\alpha,\alpha} = 1_\alpha$,

(2) for $\alpha \ge \beta \ge \gamma$, $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$.

If, in addition, for any $a \in S_\alpha$, $b \in S_\beta$, we have $ab = a\varphi_{\alpha,\alpha\beta}b\varphi_{\beta,\alpha\beta}$, then we say that S is a <u>strong semilattice of the semigroups</u> S_α and write $S = [Y;S_\alpha, \varphi_{\alpha,\beta}]$. Clearly, any strong semilattice of completely simple semigroups is completely regular. There are various nice characterizations of the semigroups that arise in this way. We require a few preliminary concepts.

Recall that a <u>normal band</u> is a band which satisfies the identity $axyb = ayxb$ and that a semigroup is a <u>normal cryptogroup</u> if $\mathcal{H}$ is a congruence on S and $S/\mathcal{H}$ is a normal band.

For any completely regular semigroup S, let the relation $\le$ be defined in S by: for $a,b \in S$

$$a \le b \iff a = eb = bf, \quad \text{for some } e,f \in E(S).$$

Let S be a completely regular semigroup with completely simple components S_α, $\alpha \in Y$. If S is such that, for α, $\beta \in Y$ with $\alpha \ge \beta$, and any idempotent e in S_α there exists a unique idempotent f in S_β

with $e \geq f$, then S is said to satisfy $\mathcal{D}$-majorization.

Let

$\mathcal{CS}$ = the variety of completely simple semigroups

$\mathcal{S}$ = the variety of semilattices.

We can now provide a number of different characterizations of normal cryptogroups.

THEOREM 1.8. For any semigroup S the following statements are equivalent.

(i) S is a normal cryptogroup.

(ii) S is completely regular and, for all $e \in E(S)$, eSe is an inverse semigroup.

(iii) S is completely regular and, for all $e \in E(S)$, $E(eSe)$ is a semilattice.

(iv) S is completely regular and satisfies $\mathcal{D}$-majorization.

(v) $S = (Y;S_\alpha)$ is completely regular with completely simple components S_α and for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and for all $a \in S_\alpha$, there exists a unique element $a^* \in S_\beta$ with $a^* \leq a$.

(vi) $S = (Y;S_\alpha)$ is a strong semilattice of the completely simple semigroups S_α, $\alpha \in Y$.

(vii) S is regular and a subdirect product of completely simple semigroups with, possibly, a zero adjoined.

(viii) $S \in \mathcal{CS} \vee \mathcal{S}$.

2. CONGRUENCES

We begin our treatment of congruences with congruences on completely simple semigroups. With the aid of the Rees Theorem, congruences on completely simple semigroups can be described fairly completely. The details of the following treatment can be found in Howie [10].

Let $S = \mathcal{M}(I,G,\Lambda;P)$. A triple $(\mathcal{S},N,\mathcal{T})$, where $\mathcal{S}$ is an equivalence relation on I, $\mathcal{T}$ is an equivalence relation on Λ and N is a normal subgroup of G, is said to be admissible if

$$(i,j) \in \mathcal{S} \text{ or } (\lambda,\mu) \in \mathcal{T} \implies p_{\lambda i}p_{\mu i}^{-1}p_{\mu j}p_{\lambda j}^{-1} \in N.$$

For any admissible triple $(\mathcal{S},N,\mathcal{T})$, define the relation $\rho_{(\mathcal{S},N,\mathcal{T})}$ on S by

$$(i,a,\lambda)\ \rho_{(\mathcal{S},N,\mathcal{T})}\ (j,b,\mu) \iff (i,j) \in \mathcal{S},\ (\lambda,\mu) \in \mathcal{T} \text{ and } p_{\zeta i}ap_{\lambda x}p_{\mu x}^{-1}b^{-1}p_{\zeta j}^{-1} \in N \text{ for some (all) } x \in I,\ \zeta \in \Lambda.$$

THEOREM 2.1. For any admissible triple $(\mathcal{S},N,\mathcal{T})$, $\rho_{(\mathcal{S},N,\mathcal{T})}$ is a congruence on $S = \mathcal{M}(I,G,\Lambda;P)$ and all congruences on S are of this form.

Given the stucture theorems of Clifford (Theorem 1.4), Rees (Theorem 1.6) Petrich (Theorem 1.7), it would be natural to investigate the properties of congruences on a completely regular semigroups by considering their restrictions to the completely simple components and how they can be reconstituted from these components. This approach has been succesfully explored by Petrich [18]. However, here I wish to explore an approach to the study of congruences which is less direct but which has provided a rich harvest of insights into not only the behaviour of congruences but also the lattice of varieties of completely regular semigroups.

DEFINITION Let ρ be a congruence on a completely regular semigroup S. Then the kernel of ρ is

$$\ker \rho = \{ a \in S : a\ \rho\ a^0 \}$$

and the trace of ρ is

$$\operatorname{tr} \rho = \rho|_{E(S)}.$$

The key observation about the kernel and trace of a congruence is that in combination they completely determine the congruence.

LEMMA 2.2. (Pastijn and Petrich [14], Lemma 2.10) Let ρ be a congruence on a completely regular semigroup S. Then, for any elements $a,b \in S$,

$$a\ \rho\ b \iff a^0\ \operatorname{tr}\rho\ b^0 \quad \text{and} \quad ab^{-1} \in \ker\rho.$$

Proof. Let $a,b \in S$ and $a\ \rho\ b$. Then $a^0\ \rho\ b^0$ and $ab^{-1}\ \rho\ b^0$. Hence $a^0\ \operatorname{tr}\rho\ b^0$ and $ab^{-1} \in \ker \rho$. Conversely, suppose that $a^0\ \operatorname{tr}\rho\ b^0$ and $ab^{-1} \in \ker \rho$. Then

$$\begin{aligned} b &= b(b^{-1}b)b^{-1}b \\ &\ \rho\ b(a^{-1}a)b^{-1}b \\ &= ba^{-1}(ab^{-1})b \\ &\ \rho\ ba^{-1}(ab^{-1})(ab^{-1})b \\ &= b(a^{-1}a)b^{-1}a(b^{-1}b) \end{aligned}$$

$$\begin{aligned}
&\rho \quad b(b^{-1}b)b^{-1}a(a^{-1}a)\\
&= \quad bb^{-1}a\\
&\rho \quad aa^{-1}a\\
&= \quad a.
\end{aligned}$$

COROLLARY 2.3. (Feigenbaum [5], Theorem 4.1) Let λ, ρ be congruences on a completely regular semigroup S. Then

$$\lambda = \rho \quad \Longleftrightarrow \quad \ker \lambda = \ker \rho \quad \text{and} \quad \operatorname{tr} \lambda = \operatorname{tr} \rho.$$

This leads to natural questions concerning the nature of those subsets of a completely regular semigroup which are kernels for congruences and those equivalence relations on the set of idempotents which are the traces of congruences. The treatment presented here is essentially that of Pastijn and Petrich [14], specialized to completely regular semigroups as in (Petrich and Reilly [24]).

DEFINITION A subset K of a completely regular semigroup S is said to be a normal subset of S if it satisfies the following conditions:

(K1) $E(S) \subseteq K$,

(K2) $k \in K \Rightarrow k^{-1} \in K$,

(K3) $xy \in K \Rightarrow yx \in K$, $\quad (x,y \in S)$,

(K4) $x, x^0y \in K \Rightarrow xy \in K$ $\quad (x,y \in S)$.

For any subset K of a semigroup S, we denote by π_K the largest congruence on S for which K is a union of π_K-classes. Then

$$a \,\pi_K\, b \quad \Longleftrightarrow \quad [xay \in K \iff xby \in K \quad (x,y \in S^1)]$$

If γ is a relation on a semigroup S, then we denote by γ^* the congruence on S generated by γ, and if γ is an equivalence relation then we denote by γ^0 the largest congruence on S contained in γ.

THEOREM 2.4. (Pastijn and Petrich [14], Lemmas 2.4, 2.9 and Petrich and Reilly [24]) Let K be a subset of a completely regular semigroup S. Then the following statements are equivalent.

(1) K is a normal subset of S.

(2) K is the kernel of some congruence on S.

(3) K is the kernel of π_K.

When (1) - (3) hold, $\{(k,k^0): k \in K\}^*$ is the smallest congruence and π_K is the largest congruence on S with kernel K.

Next we consider the relations on the set of idempotents that arise from congruences.

<u>DEFINITION</u> Let S be a completely regular semigroup and τ be an equivalence relation on $E(S)$. Then τ is a <u>normal equivalence</u> if it satisfies the following condition:

$$e \ \tau \ f \ \Rightarrow \ (xey)^0 \ \tau \ (xfy)^0 \qquad (x,y \in S^1).$$

<u>THEOREM</u> 2.5. (Pastijn and Petrich [14], Lemma 1.3 and Petrich and Reilly [24]) Let S be a completely regular semigroup and τ be an equivalence relation on $E(S)$. Then the following conditions are equivalent.

(1) τ is a normal equivalence.

(2) τ is the trace of some congruence on S.

(3) $\tau = \operatorname{tr} \tau^*$.

When (1) - (3) hold, then τ^* is the smallest congruence and $(\mathcal{H}\tau\mathcal{H})^0$ is the largest congruence on S with trace τ.

Having successfully characterized those subsets of S that can be kernels and those equivalences on $E(S)$ that can be traces, it is natural to consider when a normal subset and a normal equivalence can be combined to be the kernel and trace of a single congruence.

<u>DEFINITION</u> Let S be a completely regular semigroup, K be a normal subset of S and τ be a normal equivalence relation on $E(S)$. Then (K,τ) is a <u>congruence pair</u> for S if K is a normal subset, τ is a normal equivalence and the following conditions are satisfied:

$$\text{(CP1)} \quad e \ \tau \ f \ \Longrightarrow \ [xey \in K \iff xfy \in K, \text{ for all } x,y \in S^1]$$

$$\text{(CP2)} \quad k \in K \ \Longrightarrow \ (xky)^0 \ \ \tau \ \ (xk^0y)^0, \text{ for } \ x,y \in S^1.$$

From the definition of π_K it follows that (CP1) could be replaced by the equivalent condition

$$\text{(CP1)}^* \quad e \ \tau \ f \ \Longrightarrow \ e \ \pi_K \ f, \quad (\text{equivalently, } \tau \subseteq \operatorname{tr} \pi_K)$$

or, alternatively, invoking (K3) we could replace (CP1) by

$$\text{(CP1)}^{**} \ e \ \tau \ f \ \Longrightarrow \ [ex \in K \iff fx \in K].$$

In the same spirit, (CP2) can be replaced by the equivalent condition

$(CP2)^*$ $K \subseteq \ker (\mathcal{H}\tau\mathcal{H})^0$.

For any congruence pair (K,τ) for S, define the relation $\rho_{(K,\tau)}$ on S by

$$a \,\rho_{(K,\tau)}\, b \iff a^0 \,\tau\, b^0 ,\ ab^{-1} \in K \qquad (a,b \in S).$$

THEOREM 2.6. (Pastijn and Petrich [14], Theorem 2.13 and Petrich and Reilly [24]) Let S be a completely regular semigroup, K be a normal subset of S and τ be a normal equivalence relation on $E(S)$. Then the following statements are equivalent.

(1) (K,τ) is a congruence pair for S.

(2) $\pi_K \cap (\mathcal{H}\tau\mathcal{H})^0$ has kernel K and trace τ.

(3) There exists a congruence ρ on S with kernel K and trace τ.

(4) There is a unique congruence ρ on S with kernel K and trace τ.

Whenever (1) - (4) hold, the unique congruence on S with kernel K and trace τ is

$$\rho_{(K,\tau)} = \pi_K \cap (\mathcal{H}\tau\mathcal{H})^0.$$

3. KERNEL AND TRACE RELATIONS

Throughout this section, let S denote a completely regular semigroup and $\mathcal{C}(S)$ its lattice of congruences. Let the kernel relation K and the trace relation T be defined on $\mathcal{C}(S)$ as follows.

$$\lambda \, K \, \rho \iff \ker \lambda = \ker \rho \qquad (\lambda,\rho \in \mathcal{C}(S)).$$

$$\lambda \, T \, \rho \iff \operatorname{tr} \lambda = \operatorname{tr} \rho \qquad (\lambda,\rho \in \mathcal{C}(S))$$

Clearly K and T are both equivalence relations. As an immediate consequence of Corollary 2.2, we have

LEMMA 3.1. $K \cap T = \epsilon$, the identical relation.

We consider K first. As a related characterization of the kernel relation we have the following interesting obvservation.

LEMMA 3.2. (Pastijn and Petrich [14], Lemma 3.9) Let $\lambda,\rho \in \mathcal{C}(S)$. Then

$$\lambda \, K \, \rho \iff \lambda \cap \mathcal{H} = \rho \cap \mathcal{H}.$$

Proof. First suppose that $\ker \lambda = \ker \rho$. Then

$$
\begin{aligned}
a\ \lambda \cap \mathcal{H}\ b &\iff a\ \mathcal{H}\ b,\ a\ \lambda\ b \\
&\iff a\ \mathcal{H}\ b,\ ab^{-1}\ \lambda\ b^0 \qquad (\text{since } a^0 = b^0) \\
&\iff a\ \mathcal{H}\ b,\ ab^{-1} \in \ker \lambda = \ker \rho \\
&\iff \ldots\ldots \\
&\iff a\ \rho\cap\mathcal{H}\ b.
\end{aligned}
$$

Thus $\lambda \cap \mathcal{H} = \rho \cap \mathcal{H}$. Conversely, let $\lambda \cap \mathcal{H} = \rho \cap \mathcal{H}$. Then

$$
\begin{aligned}
a \in \ker \lambda &\implies a\ \lambda \cap \mathcal{H}\ a^0 \\
&\implies a\ \rho \cap \mathcal{H}\ a^0 \\
&\implies a \in \ker \rho
\end{aligned}
$$

so that $\ker \lambda \subseteq \ker \rho$ and, by symmetry, equality follows.

NOTATION Let $\mathcal{K}(S)$ denote the set of normal subsets of S ordered by set theoretic inclusion.

For any family $\{K_i : i \in I\}$ of normal subsets of S, it is clear that $\bigcap_{i\in I} K_i$ is again a normal subset of S. From this it follows that $\mathcal{K}(S)$ is a complete lattice with respect to the operations

$$K_1 \wedge K_2 = K_1 \cap K_2 \qquad \text{and} \qquad K_1 \vee K_2 = \cap\{K \in \mathcal{K}(S) : K_1 \cup K_2 \subseteq K\}.$$

THEOREM 3.3. (Pastijn and Petrich [14], Lemma 2.9 and Petrich and Reilly [24]) The mapping

$$\ker : \rho \longrightarrow \ker \rho \qquad\qquad (\rho \in \mathcal{C}(S))$$

is a complete $\cap$ - homomorphism of $\mathcal{C}(S)$ onto $\mathcal{K}(S)$ which induces the relation K on $\mathcal{C}(S)$. For all $\rho \in \mathcal{C}(S)$ the K - class of ρ is an interval $[\rho_K, \rho^K]$ where

$$\rho_K = (\rho \cap \mathcal{H})^* \qquad \text{and} \qquad \rho^K = \pi_{\ker \rho}.$$

Unfortunately, K is not always a congruence. Let G be any non-trivial group, $Y = \{0,1\}$ be the two element semilattice and $S = G \times Y$. Let ϵ denote the identical relation, ω the universal relation, σ the minimum group congruence and ρ the Rees congruence determined by the ideal $G \times \{0\}$. Then $\epsilon\ K\ \sigma$ but

$$\epsilon \vee \rho = \rho \qquad \text{and} \qquad \sigma \vee \rho = \omega$$

where ρ and ω do not have the same kernels.

However, there are circumstances under which K is a congruence. The method of proof used by Pastijn to establish the fact (Theorem 4.4 below) that K is a congruence on the lattice of fully invariant congruences on the free completely regular semigroup suggests the following discussion. We begin with completely simple semigroups. Let $(\mathcal{S},N,\mathcal{T})$, $(\mathcal{S}',N',\mathcal{T}')$ and $(\mathcal{P},M,Q)$ be admissible triples for $S = \mathcal{M}(I,G,\Lambda;P)$ and let

$$\rho = \rho_{(\mathcal{S},N,\mathcal{T})}, \quad \rho' = \rho_{(\mathcal{S}',N',\mathcal{T}')} \quad \text{and} \quad \sigma = \sigma_{(\mathcal{P},M,Q)}.$$

A straightforward calculation will show that

$$\ker \rho = \{(i,a,\lambda) : ap_{\lambda i} \in N\}$$

with similar expressions for $\ker \rho'$ and $\ker \sigma$. Consequently,

$$\ker \rho = \ker \rho' \iff N = N'.$$

Now it is also the case that

$$\rho \vee \sigma = \rho_{(\mathcal{S}\vee\mathcal{P},MN,\mathcal{T}\vee Q)}$$

so that

$$\ker \rho \vee \sigma = \{(i,a,\lambda) : ap_{\lambda i} \in MN\}$$

with a similar expression for $\ker \rho' \vee \sigma$. Therefore, it is clear that

$$\ker \rho = \ker \rho' \implies \ker \rho \vee \sigma = \ker \rho' \vee \sigma$$

whence K is a congruence on $\mathcal{C}(S)$ and the mapping ker is a homomorphism on $\mathcal{C}(S)$ for any completely simple semigroup S.

This observation has consequences for any completely regular semigroup. To see this, let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a completely regular semigroup with completely simple components S_α and let $\rho, \rho', \sigma \in (\mathcal{D}]$, the sublattice of $\mathcal{C}(S)$ consisting of those congruences contained in $\mathcal{D}$, be such that $\ker \rho = \ker \rho'$. Let

$$\rho_\alpha = \rho_{|S_\alpha}, \quad \rho'_\alpha = \rho'_{|S_\alpha} \quad \text{and} \quad \sigma_\alpha = \sigma_{|S_\alpha} \qquad (\alpha \in Y).$$

Then $\ker \rho_\alpha = \ker \rho'_\alpha$. Also

$\rho \vee \sigma = \cup\, \rho\circ\sigma\circ\rho\circ\ldots\circ\rho$ where the union runs over compositions of arbitrary length

$= \cup \bigcup_{\alpha\in Y} \rho_\alpha\circ\sigma_\alpha\circ\ldots\circ\rho_\alpha$ since $\rho, \sigma \in (\mathcal{D}]$

$= \bigcup_{\alpha\in Y} \cup\rho_\alpha\circ\sigma_\alpha\circ\ldots\circ\rho_\alpha$

$= \bigcup_{\alpha\in Y} \rho_\alpha\vee\sigma_\alpha.$

Hence

$$(\rho \vee \sigma)_\alpha = \rho_\alpha \vee \sigma_\alpha$$

and

$\ker \rho \vee \sigma = \cup \ker(\rho \vee \sigma)_\alpha$

$= \cup \ker (\rho_\alpha \vee \sigma_\alpha)$

$= \cup (\ker \rho_\alpha \vee \ker \sigma_\alpha)$ since ker is a homomorphism when applied to the lattice of congruences on a completely simple semigroup

$$= \cup (\ker \rho'_\alpha \vee \ker \sigma_\alpha)$$
$$= \ldots\ldots$$
$$= \ker \rho' \vee \sigma.$$

Thus we have established the following theorem:

THEOREM 3.4. For any completely regular semigroup, the mapping ker is a homomorphism on $(\mathcal{D}]$.

Parallelling Lemma 3.1, we have the following result characterizing the trace relation.

LEMMA 3.5. (Pastijn and Petrich [14], Lemma 6.5) Let $\lambda, \rho \in \mathcal{C}(S)$. Then

$$\lambda \; T \; \rho \iff \lambda \vee \mathcal{H} = \rho \vee \mathcal{H}.$$

Combining Lemmas 3.1 and 3.5, we obtain a rather curious test for the equality of congruences.

LEMMA 3.6. Let $\lambda, \rho \in \mathcal{C}(S)$. Then

$$\lambda = \rho \iff \lambda \cap \mathcal{H} = \rho \cap \mathcal{H} \quad \text{and} \quad \lambda \vee \mathcal{H} = \rho \vee \mathcal{H}.$$

In dealing with expressions of the form $\rho \vee \mathcal{H}$, it is sometimes useful to know the following simpler descriptions.

LEMMA 3.7. For any $\rho \in \mathcal{C}(S)$,

$$\rho \vee \mathcal{H} = \rho\mathcal{H}\rho = \mathcal{H}\rho\mathcal{H}.$$

NOTATION Let $\mathcal{T}(S)$ denote the set of all normal equivalence relations on $E(S)$.

Clearly the intersection of any family of normal equivalences is again a normal equivalence. From this it follows that the set $\mathcal{T}(S)$ is a complete lattice with respect to the operations

$$\sigma \wedge \tau = \sigma \cap \tau \quad \text{and} \quad \sigma \vee \tau = \cap \{\rho \in \mathcal{T}(S) : \sigma \cup \tau \subseteq \rho\}.$$

THEOREM 3.8. (Pastijn and Petrich [14], Theorem 4.20) The mapping

$$\mathrm{tr}: \rho \longrightarrow \mathrm{tr}\, \rho \qquad (\rho \in \mathcal{C}(S))$$

is a complete homomorphism of $\mathcal{C}(S)$ onto $\mathcal{T}(S)$ inducing the relation T on $\mathcal{C}(S)$. Moreover, for each $\rho \in \mathcal{C}(S)$, the T-class of ρ is an interval $[\rho_T, \rho^T]$ where

$$\rho_T = (\text{tr}\ \rho)^* \qquad \text{and} \qquad \rho^T = (\rho \vee \mathcal{H})^0 .$$

In contrast to the fact that K need not always be a congruence on $\mathcal{C}(S)$, we have the following immediate consequence of Theorem 3.8.

<u>COROLLARY</u> 3.9. T is a complete congruence on $\mathcal{C}(S)$.

From Theorems 3.3 and 3.8, we see that the equivalence relations K and T are such that every class is an interval in the lattice $\mathcal{C}(S)$. These facts, together with Lemma 3.1 enable us to give a purely lattice theoretic proof of the next observation.

<u>PROPOSITION</u> 3.10. (Pastijn and Petrich [14], Theorem 3.5) Let $\rho \in \mathcal{C}(S)$. Then

$$\rho_K \vee \rho_T = \rho = \rho^K \wedge \rho^T .$$

<u>Proof</u>. We have

$$\rho_K \leq \rho_K \vee \rho_T \leq \rho$$

and, by the convexity of the class ρK, it follows that $\rho_K \vee \rho_T \ K \ \rho$. Similarly, $\rho_K \vee \rho_T \ T \ \rho$ which, by Lemma 3.1, implies that $\rho_K \vee \rho_T = \rho$. The second equality in the statement of the proposition follows by duality.

There are two additional relations on $\mathcal{C}(S)$ that are closely related to T. In order to recognize that these relations are natural relatives of K and T, it is helpful to consider slightly different characterizations of K and T.

Let $\rho \in \mathcal{C}(S)$. Then

ρ is <u>idempotent pure</u> if $\ker \rho = E(S)$,

ρ is <u>idempotent separating</u> if $\text{tr}\ \rho = \epsilon$ or, equivalently, $\rho \subseteq \mathcal{H}$.

Clearly,

$$\lambda \ K \ \rho \iff \ker \lambda = \ker \rho = \ker \lambda\cap\rho$$
$$\iff \ker \lambda/(\lambda\cap\rho) = E(S/(\lambda\cap\rho)) = \ker \rho/(\lambda\cap\rho)$$
$$\iff \lambda/(\lambda\cap\rho) \text{ and } \rho/(\lambda\cap\rho) \text{ are both idempotent pure.}$$

Similarly,

$$\lambda \ T \ \rho \iff \text{tr}\ \lambda = \text{tr}\ \rho = \text{tr}\ \lambda\cap\rho$$
$$\iff \text{tr}\ \lambda/(\lambda\cap\rho) = \epsilon = \text{tr}\ \rho/(\lambda\cap\rho)$$

$\iff \lambda/(\lambda\cap\rho)$ and $\rho/(\lambda\cap\rho)$ are both idempotent separating

$\iff \lambda/(\lambda\cap\rho),\ \rho/(\lambda\cap\rho) \subseteq \mathcal{H}$.

It is this very last characterization of T that leads to two additional relations on $\mathcal{C}(S)$: for $\lambda,\rho \in \mathcal{C}(S)$,

$$\lambda\ T_\ell\ \rho \iff \lambda/(\lambda\cap\rho),\ \rho/(\lambda\cap\rho) \subseteq \mathcal{L}$$

$$\lambda\ T_r\ \rho \iff \lambda/(\lambda\cap\rho),\ \rho/(\lambda\cap\rho) \subseteq \mathcal{R}.$$

We refer to T_ℓ as the <u>left trace relation</u> and to T_r as the <u>right trace relation</u> on $\mathcal{C}(S)$.

For any congruence $\rho \in \mathcal{C}(S)$, the <u>left trace</u> and <u>right trace</u> of ρ are defined to be

$$\text{ltr}\ \rho = (\rho \vee \mathcal{L})^0 \qquad \text{and} \qquad \text{rtr}\ \rho = (\rho \vee \mathcal{R})^0.$$

Then an equivalent characterization of the relations T_ℓ and T_r is given by the following: for $\lambda,\rho \in \mathcal{C}(S)$,

$$\lambda\ T_\ell\ \rho \iff \text{ltr}\ \lambda = \text{ltr}\ \rho \qquad \text{and} \qquad \lambda\ T_r\ \rho \iff \text{rtr}\ \lambda = \text{rtr}\ \rho.$$

The parallelism between the relations T, T_ℓ and T_r is brought out strongly in the next result.

<u>THEOREM</u> 3.11. (Pastijn and Petrich [14], Lemma 6.5) The mappings

$$\rho \longrightarrow \rho \vee \mathcal{H}, \qquad \rho \longrightarrow \rho \vee \mathcal{L}, \qquad \rho \longrightarrow \rho \vee \mathcal{R}$$

are complete homomorphisms of the lattice $\mathcal{C}(S)$ into the lattice $\mathcal{E}(S)$ of equivalence relations on S inducing the relations T, T_ℓ and T_r, respectively. Consequently, the relations T, T_ℓ and T_r are complete congruences on $\mathcal{C}(S)$.

As an immediate consequence, to match Lemma 3.5, we have

<u>COROLLARY</u> 3.12. (i) $\lambda\ T_\ell\ \rho \iff \lambda \vee \mathcal{L} = \rho \vee \mathcal{L}$.

(ii) $\lambda\ T_r\ \rho \iff \lambda \vee \mathcal{R} = \rho \vee \mathcal{R}$.

Since T_ℓ and T_r are complete congruences, it follows that all the T_ℓ-classes and T_r-classes are intervals. For any $\rho \in \mathcal{C}(S)$, we define ρ_{T_ℓ}, ρ_{T_r}, ρ^{T_ℓ} and ρ^{T_r} by setting

$$\rho T_\ell = [\rho_{T_\ell}, \rho^{T_\ell}] \qquad \text{and} \qquad \rho T_r = [\rho_{T_r}, \rho^{T_r}].$$

The next result sets out some important basic connections between the relations T, T_ℓ and T_r.

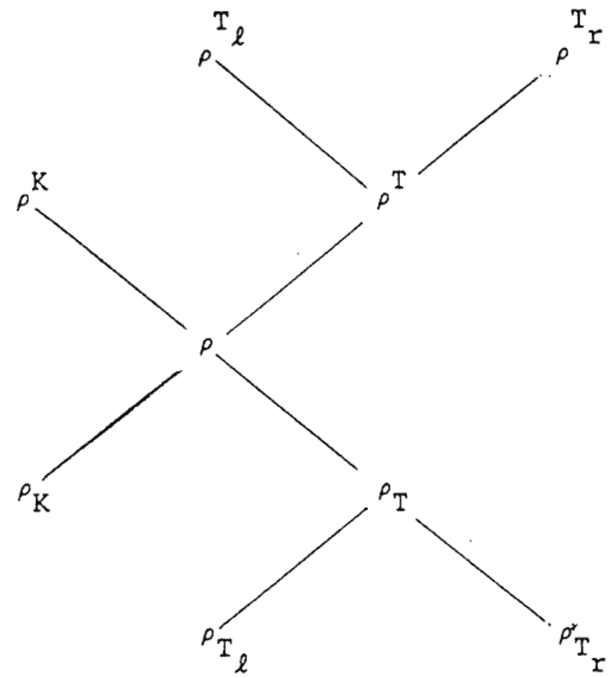
<u>THEOREM</u> 3.13. (Pastijn and Petrich [14], Corollary 4.8 and Theorem 4.14)

(i) $T_\ell \cap T_r = T$.

(ii) For any $\rho \in \mathcal{C}(S)$,

$$\rho_{T_\ell} \vee \rho_{T_r} = \rho_T \qquad \text{and} \qquad \rho^{T_\ell} \wedge \rho^{T_r} = \rho^T.$$

This leads to the following diagram from [14].



In order to give more explicit descriptions of the endpoints of T_ℓ- and T_r-classes, it is convenient to introduce the following relations. Define

$$e \leq_\ell f \quad \Leftrightarrow \quad e = ef \qquad (e, f \in E(S))$$

and define the relation $\leq_r$ dually.

<u>PROPOSITION</u> 3.14. (Pastijn and Petrich [14], Theorem 4.12)

Let $\rho \in \mathcal{C}(S)$.

(i) $\rho_{T_r} = (\rho \cap \leq_\ell)^*$ and $\rho^{T_r} = (\rho \vee \mathcal{R})^0$.

(ii) $\qquad \rho_{T_\ell} = (\rho \cap \leq_r)^* \quad$ and $\quad \rho^{T_\ell} = (\rho \vee \mathcal{L})^0.$

4. THE LATTICE OF VARIETIES

We shall require some notation. For any subvariety $\mathcal{V}$ of $\mathcal{CR}$, we shall write

$\mathcal{L}(\mathcal{V})$ = the lattice of subvarieties of $\mathcal{V}$

$F\mathcal{V}$ = the relatively free completely regular semigroup in $\mathcal{V}$ on a countably infinite set X

Γ = the lattice of fully invariant congruences on $F\mathcal{CR}$.

Fundamental to the discussion of varieties is the standard correspondence between varieties and fully invariant congruences.

For $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, let $\rho_{\mathcal{V}}$ be defined on $F\mathcal{CR}$ by

$$\rho_{\mathcal{V}} = \{(u,v) \in F\mathcal{CR}\times F\mathcal{CR}:\ u\theta = v\theta, \text{ for all homomorphisms } \theta: F\mathcal{CR} \longrightarrow S \in \mathcal{V}\}.$$

Then the mapping

$$\pi:\ \mathcal{V} \longrightarrow \rho_{\mathcal{V}} \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR}))$$

is an antiisomorphism of $\mathcal{L}(\mathcal{CR})$ onto Γ.

The study of $\mathcal{L}(\mathcal{CR})$ involves many special varieties as reference points.

$\mathcal{T}$	- trivial semigroups	$[x = y]$
$\mathcal{LZ}$	- left zero semigroups	$[xy = x]$
$\mathcal{RZ}$	- right zero semigroups	$[xy = y]$
$\mathcal{RB}$	- rectangular bands	$[xyz = xz]$
$\mathcal{R}e\mathcal{G}$	- rectangular groups	$[x^0y^0z^0 = (xz)^0]$
$\mathcal{S}$	- semilattices	$[x^2 = x,\ xy = yx]$
$\mathcal{B}$	- bands	$[x^2 = x]$
$\mathcal{NB}$	- normal bands	$[x^2 = x,\ axya = ayxa]$
$\mathcal{G}$	- groups	$[x^0 = y^0]$
$\mathcal{AG}$	- abelian groups	$[x^0 = y^0,\ xy = yx]$
$\mathcal{A}_n$	- abelian groups of exponent n	$[x^0 = y^0,\ xy = yx,\ x^n = x^0]$
$\mathcal{LG}$	- left groups	$[x^0y^0 = x^0]$
$\mathcal{RG}$	- right groups	$[x^0y^0 = y^0]$
$\mathcal{SG}$	- semilattices of groups	$[x^0y^0 = y^0x^0]$
$\mathcal{CS}$	- completely simple semigroups	$[(xyz)^0 = (xz)^0]$
$\mathcal{OG}$	- orthogroups	$[x^0y^0 = (x^0y^0)^0]$
$\mathcal{CG}$	- cryptogroups	$[(x^0y^0)^0 = (xy)^0]$

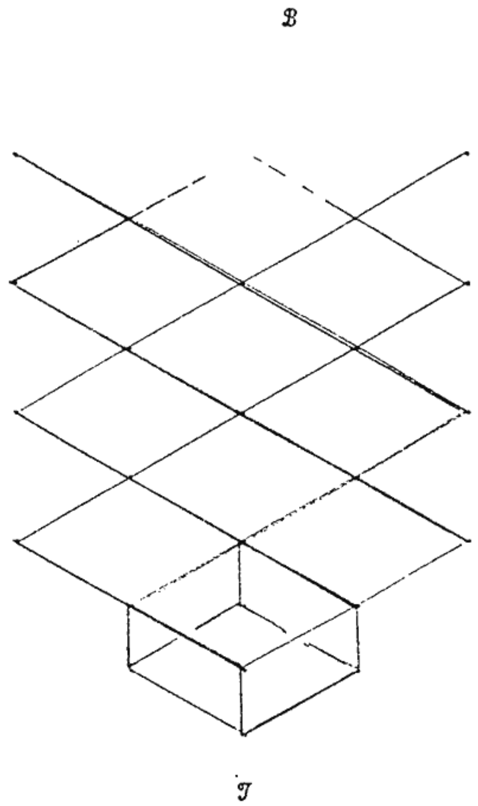
$\mathcal{OCG}$ - orthocryptogroups $[x^0y^0 = (x^0y^0)^0,\ (x^0y^0)^0 = (xy)^0]$

$\mathcal{NCG}$ - normal cryptogroups (completely regular semigroups for which $\mathcal{H}$ is a congruence and $S/\mathcal{H}$ is a normal band).

$L\mathcal{OCG}$ - locally orthodox cryptogroups (that is, all $S \in \mathcal{CR}$ such that $eSe \in \mathcal{OCG}$ for all $e \in E(S)$).

$CL\mathcal{OCG}$ - completely regular semigroups for which the core (that is, the subsemigroup generated by the idempotents) lies in $L\mathcal{OCG}$

The first part of $\mathcal{L}(\mathcal{CR})$ to be studied in any depth was the lattice $\mathcal{L}(\mathcal{B})$ of subvarieties of the variety $\mathcal{B}$ of bands. Here is the familiar diagram for $[\mathcal{S},\mathcal{B}]$ due to Birjukov [1], Fennemore [6] and Gerhard [8]:



The next part of the lattice $\mathcal{L}(\mathcal{CR})$ to be studied in depth (excluding the lattice of varieties of groups, which has been studied for many years, of course) was the lattice $\mathcal{L}(\mathcal{CS})$ of subvarieties of the variety of completely simple semigroups. Most of the work on $\mathcal{L}(\mathcal{CS})$ to date has taken advantage of the description of the free completely simple semigroup described by Clifford and Rasin (independently), in 1979.

THEOREM 4.1. (Clifford [3] Theorem 7.4, Rasin [28] Theorem 1) Let X be a non-empty set and fix $z \in X$. Let $Y = \{p_{x,y} : x,y \in X\backslash\{z\}\}$ be a set indexed by pairs of elements from X different from z and let G be the free group on $Z = X \cup Y$. Let $p_{z,x} = p_{x,z} = 1$, the identity of G, for all $x \in X$, and let $P = (p_{x,y})$ be the $X \times X$ matrix with $(x,y)^{th}$ entry equal to $p_{x,y}$. Then $F\mathcal{CS}(X) = (\mathcal{M}(X,G,X;P),\theta)$ where $x\theta = (x,x,x)$, for all $x \in X$.

NOTATION Let $\mathcal{E}$ denote the set of endomorphisms ω of G for which there exist mappings φ and ψ of X into itself such that, for all $x,y \in X$,

$$p_{x,y}\omega = p_{z\psi,z\varphi}(p_{x\psi,z\varphi})^{-1}p_{x\psi,y\varphi}(p_{z\psi,y\varphi})^{-1}.$$

Let $\mathcal{N}$ denote the set of normal subgroups of G which are invariant under all elements of $\mathcal{E}$. It is easily verified that $\mathcal{N}$ is a sublattice of the lattice of normal subgroups of G.

THEOREM 4.2. (Rasin [28], Theorem 3) The interval $[\mathcal{RB},\mathcal{CS}]$ is anti-isomorphic to the lattice $\mathcal{N}$.

Because of this result, most of the advances to date in the study of $\mathcal{L}(\mathcal{CS})$ have involved the study of the structure of G and $\mathcal{N}$.

NOTATION Let $\mathcal{C}$ denote the variety of all completely simple semigroups with the property that the product of any two idempotents lies in the centre of the $\mathcal{H}$-class containing it. This variety is defined by the identity

$$ax^0a^0ya = aya^0x^0a.$$

For any $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$, let $\mathcal{I}(\mathcal{V})$ denote the class of all idempotent generated members of $\mathcal{V}$ and let $\langle\mathcal{I}(\mathcal{V})\rangle$ denote the variety of completely simple semigroups generated by $\mathcal{I}(\mathcal{V})$.

The largest ideal of $\mathcal{L}(\mathcal{CS})$ to have been given a fairly precise characterization is $\mathcal{L}(\mathcal{C})$.

THEOREM 4.3. (Petrich and Reilly [20], Theorem 3.11) The mapping

$$\zeta:\mathcal{A} \longrightarrow (\mathcal{A} \cap \mathcal{RB}, \langle\mathcal{IS}\rangle \cap \mathcal{AG}, \mathcal{A} \cap \mathcal{G}) \qquad (\mathcal{A} \in \mathcal{L}(\mathcal{C}))$$

is an isomorphism of $\mathcal{L}(\mathcal{C})$ onto the subdirect product

$\{(\mathcal{U},\mathcal{V},\mathcal{W}) \in \mathcal{L}(\mathcal{RB}) \times \mathcal{L}(\mathcal{AG}) \times \mathcal{L}(\mathcal{G}): \mathcal{V} \subseteq \mathcal{W},\ \mathcal{U} \neq \mathcal{RB} \implies \mathcal{V} = \mathcal{T}\}.$

Despite the "simple" characterízation of completely simple semigroups provided by the Rees Theorem, the structure of $\mathcal{L}(\mathcal{CS})$ outside of the ideal $\mathcal{L}(\mathcal{C})$, remains a mystery.

In order to probe deeper into the structure of $\mathcal{L}(\mathcal{CR})$, we take advantage of the recent techniques for investigating congruences that were described in earlier sections.

In Theorem 3.10, we saw that the relations T, T_ℓ and T_r are complete congruences on $\mathcal{C}(S)$, for any completely regular semigroup S, but that K need not be. We now have:

<u>THEOREM</u> 4.4.(Polák [25] Theorem 1, Pastijn [12] Theorem 11) K is a complete congruence on Γ.

Thus K, T, T_ℓ and T_r are all complete congruences on Γ. Under the antiisomorphism π^{-1}, these carry over to complete congruences on $\mathcal{L}(\mathcal{CR})$:

$$\mathcal{U}\ K\ \mathcal{V} \iff \rho_{\mathcal{U}}\ K\ \rho_{\mathcal{V}}, \quad \mathcal{U}\ T\ \mathcal{V} \iff \rho_{\mathcal{U}}\ T\ \rho_{\mathcal{V}}$$
$$\mathcal{U}\ T_\ell\ \mathcal{V} \iff \rho_{\mathcal{U}}\ T_\ell\ \rho_{\mathcal{V}}, \quad \mathcal{U}\ T_r\ \mathcal{V} \iff \rho_{\mathcal{U}}\ T_r\ \rho_{\mathcal{V}}$$

The classes of any complete congruence are intervals and so it is convenient to denote the intervals for these four congruences as follows:

$$\mathcal{V}K = [\mathcal{V}_K, \mathcal{V}^K], \qquad \mathcal{V}T = [\mathcal{V}_T, \mathcal{V}^T]$$
$$\mathcal{V}T_\ell = [\mathcal{V}_{T_\ell}, \mathcal{V}^{T_\ell}], \quad \mathcal{V}T_r = [\mathcal{V}_{T_r}, \mathcal{V}^{T_r}]$$

<u>THEOREM</u> 4.5.(Polák [25] Theorem 1 and [26] Theorem 1.6, Pastijn [12] Theorem 8) The mappings

$$\mathcal{V} \longrightarrow \mathcal{V}^K, \quad \mathcal{V} \longrightarrow \mathcal{V}_{T_\ell}, \quad \mathcal{V} \longrightarrow \mathcal{V}_{T_r} \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR}))$$

are complete endomorphisms of $\mathcal{L}(\mathcal{CR})$ inducing the congruences K, T_ℓ and T_r.

Somewhat surprisingly, the mapping

$$\mathcal{V} \longrightarrow \mathcal{V}_T \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR}))$$

is not an endomorphism of $\mathcal{L}(\mathcal{CR})$ (see Petrich and Reilly [22], Proposition 7.6). In addition, the mappings associated with the other ends of the intervals of K, T, T_ℓ and T_r are not endomorphisms. An interesting and useful fact is that the upper ends of the intervals of K, T_ℓ and T_r can be described in terms of Mal'cev products (Pastijn [12] Lemma 3, Theorem 13):

$$\mathcal{V}^K = \mathcal{RB} \circ (\mathcal{V} \vee \mathcal{S}), \quad \mathcal{V}^{T_\ell} = \mathcal{LG} \circ \mathcal{V}, \quad \mathcal{V}^{T_r} = \mathcal{RG} \circ \mathcal{V}.$$

An alternative expression for $\mathcal{V}^K$ is $\mathcal{V}^K = \mathcal{B} \circ \mathcal{V}$.

One approach used in the study of $\mathcal{L}(\mathcal{CR})$ has been to describe certain intervals of the form $[\mathcal{U} \wedge \mathcal{V}, \mathcal{U} \vee \mathcal{V}]$, for suitable $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, as particular subdirect products of the intervals $[\mathcal{U} \wedge \mathcal{V}, \mathcal{U}]$ and $[\mathcal{U} \wedge \mathcal{V}, \mathcal{V}]$.

We begin by studying the circumstances under which an interval of the form $[a \wedge b, a \vee b]$ in a lattice may be isomorphic to the product $[a \wedge b, a] \times [a \wedge b, b]$ with a view to applying this to the lattice $\mathcal{L}(\mathcal{CR})$.

For any complete congruence λ on a complete lattice L and any $a \in L$, the class $a\lambda$ is an interval. We define a_λ and a^λ by $a\lambda = [a_\lambda, a^\lambda]$. The following discussion is taken from (Petrich and Reilly [23]).

<u>LEMMA</u> 4.6. Let κ and τ be congruences on a lattice L and $a,b \in L$. The following statements are equivalent.

(i) $a \;\kappa\; a \wedge b \;\tau\; b$. (ii) $a \;\tau\; a \vee b \;\kappa\; b$.

<u>Proof</u>. If (i) holds then

$$a = a \vee (a \wedge b) \;\;\tau\;\; a \vee b \quad \text{and} \quad b = (a \wedge b) \vee b \;\;\kappa\;\; a \vee b$$

which gives (i). The proof that (ii) implies (i) is similar.

<u>DEFINITION</u> If L, a, b, κ and τ satisfy (i) and (ii) in Lemma 6.1, then we will say that a and b are $\kappa\tau$-<u>neighbours</u>. Congruences κ and τ on a lattice L are said to be <u>disjoint</u> if $\kappa \cap \tau = \epsilon$.

<u>LEMMA</u> 4.7. Let κ and τ be disjoint complete congruences on a complete lattice L and let $a \in L$. Then

$$a = a_\kappa \vee a_\tau = a^\kappa \wedge a^\tau .$$

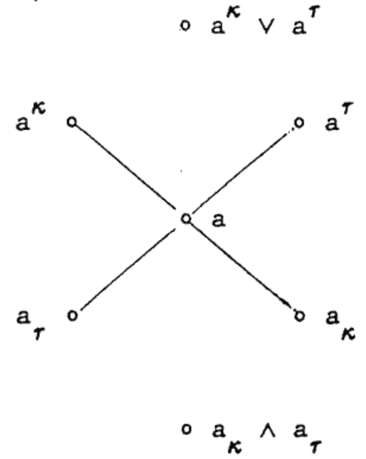
<u>Proof.</u> Since κ and τ are congruences, we have

$$a_\kappa \vee a_\tau \;\;\kappa\;\; a \vee a_\tau = a \quad \text{and} \quad a_\kappa \vee a_\tau \;\;\tau\;\; a_\kappa \vee a = a$$

so that $a_\kappa \vee a_\tau \;(\kappa \cap \tau)\; a$. But κ and τ are disjoint. Therefore $a = a_\kappa \vee a_\tau$. The second equality follows by duality.

<u>COROLLARY</u> 4.8. Let κ and τ be disjoint complete congruences on a complete lattice L and let $a \in L$. Then a^κ, a^τ are

$\kappa\tau$-neighbours and a_κ, a_τ are $\tau\kappa$-neighbours.



<u>Proof</u>. By Lemma 4.7, we have

$$a^\kappa \wedge a^\tau = a \;\kappa\; a^\kappa \quad \text{and} \quad a^\kappa \wedge a^\tau = a \;\tau\; a^\tau$$

from which we deduce the first claim. The second claim follows similarly using Lemmas 4.6 and 4.7.

We are now ready for the main lattice theoretic observation. One of the striking features of this result is the fact that neither modularity nor neutrality appear in the hypotheses.

<u>THEOREM</u> 4.9. Let κ,τ be disjoint congruences on a lattice L and $a,b \in L$ be $\kappa\tau$-neighbours. Then the mappings

$$\varphi\colon z \longrightarrow (z \wedge a, z \wedge b), \qquad \psi\colon (x,y) \longrightarrow x \vee y$$

are mutually inverse isomorphisms between $[a \wedge b, a \vee b]$ and $[a \wedge b, a] \times [a \wedge b, b]$.

Applying these lattice theoretic considerations to congruences, we obtain:

<u>THEOREM</u> 4.10. (Pastijn and Trotter [15], Theorems 5.1 and 5.2) Let $\rho \in \Gamma$.

(i) The mappings

$$\theta \longrightarrow (\theta \cap \rho^K, \theta \cap \rho^T), \qquad (\xi,\eta) \longrightarrow \xi \vee \eta$$

are mutually inverse isomorphisms between $[\rho, \rho^K \vee \rho^T]$ and $[\rho,\rho^K] \times [\rho,\rho^T]$.

(ii) The mappings

$$\theta \longrightarrow (\theta \vee \rho_K, \theta \vee \rho_T), \qquad (\xi,\eta) \longrightarrow \xi \cap \eta$$

are mutually inverse isomorphisms between $[\rho_K \cap \rho_T, \rho]$ and $[\rho_K, \rho] \times [\rho_T, \rho]$.

Proof. (i) From Lemma 3.1, we know that K and T are disjoint complete congruences on Γ. It follows from Corollary 4.8 that ρ^K and ρ^T are KT-neighbours and the claim follows by Theorem 4.9.

(ii) This follows from (i) by duality.

In order to provide some specific illustrations of the preceding discussions in terms of varieties rather than fully invariant congruences, we need to know some specific values for the upper end points of some of the K- and T-classes.

LEMMA 4.11. (i) $\mathcal{T}^K = \mathcal{B}$, $\mathcal{G}^K = \mathcal{R}e\mathcal{G}^K = \mathcal{O}\mathcal{G}$.
(ii) $\mathcal{T}^T = \mathcal{G}$, $\mathcal{RB}^T = \mathcal{R}e\mathcal{G}^T = \mathcal{CS}$.

Proof. (i) See (Polák [25], Theorem 2).
(ii) See (Petrich and Reilly [21], Section 9).

We can now give some examples of applications in $\mathcal{L}(\mathcal{CR})$.

LEMMA 4.12. (i) (Petrich [16], Theorem) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{B}, \mathcal{V} \cap \mathcal{G}), \qquad (\mathcal{U},\mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{OCG})$ and $\mathcal{L}(\mathcal{B}) \times \mathcal{L}(\mathcal{G})$.

(ii) (Hall and Jones [9], Corollary 5.5 and Rașin [30], Proposition 1) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{B}, \mathcal{V} \cap \mathcal{CS}), \qquad (\mathcal{U},\mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $[\mathcal{RB}, \mathcal{LOCG}]$ and $[\mathcal{RB},\mathcal{B}] \times [\mathcal{RB},\mathcal{CS}]$.

(iii) (Reilly [32], Theorem 4.9) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{OG}, \mathcal{V} \cap \mathcal{CS}) \qquad (\mathcal{U},\mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $[\mathcal{R}e\mathcal{G}, \mathcal{CLOCG}]$ and $[\mathcal{R}e\mathcal{G},\mathcal{OG}] \times [\mathcal{R}e\mathcal{G},\mathcal{CS}]$.

5. POLAK'S THEOREM

The following subset has a special role to play in the study of $\mathcal{L}(\mathcal{CR})$:

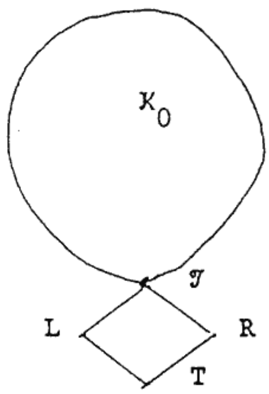
$$\mathcal{K}_0 = \{\mathcal{V}_K : \mathcal{V} \in \mathcal{L}(\mathcal{CR})\}$$

Examples of members of $\mathcal{K}_0$ are plentiful and include all group varieties and all non-orthodox varieties of completely simple semigroups.

Since K is a complete congruence on $\mathcal{L}(\mathcal{CR})$ and $\mathcal{K}_0$ contains exactly one representative from each K-class, we may consider $\mathcal{K}_0$ as being a lattice with the lattice structure inherited from $\mathcal{L}(\mathcal{CR})/K$. Thus, for $\mathcal{U}$, $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, $\mathcal{U} \leq \mathcal{V}$ if and only if $\mathcal{U}K \leq \mathcal{V}K$.

We now adjoin three elements to the bottom of $\mathcal{K}_0$ (below the trivial variety $\mathcal{T}$) and extend the order on $\mathcal{K}_0$ to $\mathcal{K} = \mathcal{K}_0 \cup \{L,T,R\}$ as indicated in the diagram below so that $\mathcal{K}$ becomes a lattice with $\mathcal{K}_0$ as a sublattice.

$\mathcal{K}$ =



Before proceeding, we require some additional notation:

$\mathcal{LNB}$ = the variety of left normal bands = $[x^2 = x,\ xyz = xzy]$

$\mathcal{RNB}$ = the variety of right normal bands = $[x^2 = x,\ xyz = yxz]$.

For $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, let the mapping

$$\mathcal{V} \longrightarrow \mathcal{V}_K{}^* \qquad (\mathcal{V} \in [\mathcal{S},\mathcal{CR}])$$

be defined by the following:

$$\mathcal{V}_K{}^* = \begin{cases} \mathcal{V}_K & \text{if } \mathcal{V} \neq \mathcal{S},\mathcal{LNB},\mathcal{RNB} \\ L & \text{if } \mathcal{V} = \mathcal{LNB} \\ T & \text{if } \mathcal{V} = \mathcal{S} \\ R & \text{if } \mathcal{V} = \mathcal{RNB}. \end{cases}$$

We wish to combine the above mapping with mappings associated with T_ℓ and T_r. Towards this end we introduce "products" of T_ℓ and T_r. Let

$$\Theta = \langle T_\ell, T_r \mid T_\ell^2 = T_\ell,\ T_r^2 = T_r \rangle \quad \text{and} \quad \Theta^1 = \Theta \cup \{1\}$$

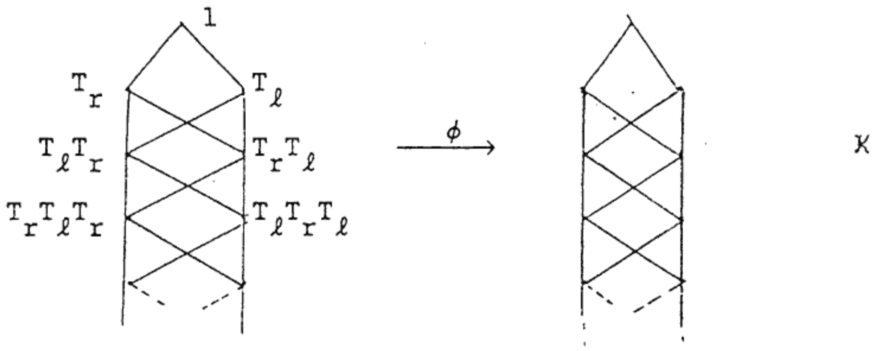
be the monoid with generators T_ℓ and T_r subject to the relations $T_\ell^2 = T_\ell$, $T_r^2 = T_r$. It is easy to see that every element of Θ can be written uniquely in canonical form as

$$\tau = T_1 T_2 \ldots T_n \quad \text{where } T_i \in \{T_\ell, T_r\},\ T_i \neq T_{i+1}$$

For such an element τ, let $|\tau| = n$, $h(\tau) = T_1$ and $t(\tau) = T_n$. Define a relation $\leq$ on Θ^1 by

$$\sigma \leq \tau \iff |\sigma| > |\tau| \text{ or } \sigma = \tau \text{ or } \tau = 1.$$

Then $(\Theta^1, \leq)$ is the partially ordered set depicted on the left of the diagram:



We also extend the definitions of V_{T_ℓ} and V_{T_r} to cover V_τ for any $\tau \in \Theta^1$ by defining $V_1 = V$ and otherwise defining V_τ inductively as follows: for $\tau = T_1 T_2 \ldots T_n \in \Theta$ and $V \in \mathcal{L}(\mathcal{CR})$ let

$$V_\tau = (V_{T_1 \ldots T_{n-1}})_{T_n}.$$

Our main interest is in certain mappings of Θ^1 into K.

Let Φ denote the set of all $\phi \in K^{\Theta^1}$ satisfying the following conditions:

D(i) $1\phi \in K_0$,

D(ii) ϕ is order preserving,

D(iii) $\tau \in \Theta$, $\tau\phi = L \implies t(\tau) = T_r$,

D(iv) $\tau \in \Theta$, $\tau\phi = R \implies t(\tau) = T_\ell$, D(v) $\sigma \in \Theta^1, \tau \in \Theta,\ \sigma\phi \in K_0$ and either $\sigma = \emptyset$

$$\text{or } t(\sigma) \neq h(\tau) \implies (\sigma\tau)\phi \supseteq (\sigma\phi)_{\tau K^*}.$$

<u>THEOREM</u> 5.1.(Polák [26], Theorem 3.6) Φ is a complete lattice (with respect to the component-wise order).

Polák's main theorem concerns those subvarieties of $\mathcal{CR}$ that contain the variety of semilattices.

<u>THEOREM</u> 5.2.(Polák [26], Theorem 3.6) For any $\mathcal{V} \in [\mathcal{S},\mathcal{CR}]$, let $\chi_{\mathcal{V}} \in K^{\Theta^1}$ be defined by:

$$\tau\chi_{\mathcal{V}} = \begin{cases} \mathcal{V}_K & \text{if } \tau = \emptyset \\ \mathcal{V}_{\tau K^*} & \text{otherwise} \end{cases}$$

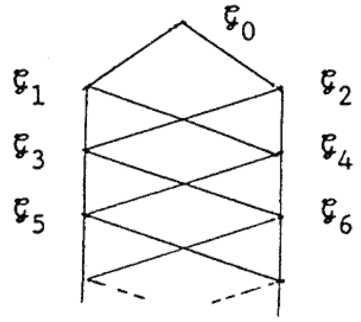
Then the mapping

$$\chi\colon \mathcal{V} \longrightarrow \chi_{\mathcal{V}} \qquad (\mathcal{V} \in [\mathcal{S},\mathcal{CR}])$$

is an isomorphism of $[\mathcal{S},\mathcal{CR}]$ onto Φ.

Many interesting subsidiary facts and applications of this theorem can be found in Polák's three papers [25], [26] and [27].

A case to which Polák's Theorem can be quickly applied to give new information, is the lattice $\mathcal{L}(\mathcal{OG})$ of subvarieties of the variety $\mathcal{OG}$ of orthodox completely regular semigroups It is not hard to show that $\mathcal{OG}_K = \mathcal{G}$, where $\mathcal{G}$ denotes the variety of groups. Therefore, for any $\mathcal{V} \in \mathcal{L}(\mathcal{OG})$, the partially ordered set of values of $\chi_{\mathcal{V}}$ may be depicted as follows:



where $\mathcal{G}_0 \in \mathcal{L}(\mathcal{G})$, the lattice of varieties of groups and, for each $n \geq 1$, $\mathcal{G}_n \in \mathcal{L}(\mathcal{G}) \cup \{ L , T , R \}$. From this it is easy to deduce the following result.

THEOREM 5.3. (Polák [26], Theorem 4.2) $\mathcal{L}(\mathcal{OG})$ is a subdirect product of countably many copies of $\mathcal{L}(\mathcal{G})$ and a single copy of $\mathcal{L}(\mathcal{B})$.

One question about $\mathcal{L}(\mathcal{CR})$ that remained unanswered for a considerable time was whether or not it is a modular lattice. Various results had been obtained concerning various sublattices of $\mathcal{L}(\mathcal{CR})$ (see, for example, Rasin [29] for the lattice of varieties of completely simple semigroups and Hall and Jones [9] for the lattice of varieties of completely regular semigroups for which $\mathcal{H}$ is a congruence). The question was finally answered in full generality with the aid of Polák's Theorem by Pastijn:

THEOREM 5.4. (Pastijn [12], Theorem 18) $\mathcal{L}(\mathcal{CR})$ is modular.

Verifications of the modularity of $\mathcal{L}(\mathcal{CR})$ that are not dependent on Polák's Theorem have been obtained by Pastijn [13] and Petrich and Reilly [23].

Since the lattice of group varieties is a sublattice of $\mathcal{L}(\mathcal{CR})$ it follows that $\mathcal{L}(\mathcal{CR})$ is not distributive. However, even in a non-distributive lattice, there may be elements which have properties that are normally associated with distributivity. More exactly, an element a in a lattice L is neutral if the mapping

$$x \longrightarrow (x \wedge a,\ x \vee a)$$

is a monomorphism of L onto a subdirect product of (a] and [a) (where (a] and [a) denote the ideal and filter of L, respectively, generated by a).

The usefulness of a neutral element a in a lattice L is that it makes it possible to convert certain types of problems on the whole lattice L to (hopefully simpler) problems on the (hopefully simpler) sublattices (a] and [a). One nice feature of modular lattices is that, by virtue of the lemma below, in order to establish that an element is neutral it is not necessary to verify all the conditions in the definition each time.

LEMMA 5.5. ([7]) For any element a in a modular lattice L, the following statements are equivalent:

(i) a is neutral in L;

(ii) the mapping

$$\mu_a : x \longrightarrow x \wedge a \qquad (x \in L)$$

is an endomorphism of L;

(iii) the mapping

$$\nu_a : x \longrightarrow x \vee a \qquad (x \in L)$$

is an endomorphism of L.

Prior to Polák's Theorem, a few simple examples of neutral elements in $\mathcal{L}(\mathcal{CR})$ were known. For example, Hall and Jones [9] had shown that the variety $\mathcal{S}$ of semilattices is neutral and Jones [11] extended the list to include all subvarieties of the variety $\mathcal{NB}$ of normal bands.

Also Jones [11] had shown that $\mu_{\mathcal{G}}$ and $\mu_{\mathcal{CS}}$ are homomorphisms so that, by the preceding theorem and lemma, we may conclude immediately that $\mathcal{G}$ and $\mathcal{CS}$ are both neutral in $\mathcal{L}(\mathcal{CR})$. But now, with the techniques available on account of Polák's Theorem it is possible to determine many more neutral elements and to approach the search for neutral elements in a much more systematic way.

The following is a partial listing of the varieties that are now known to be neutral in $\mathcal{L}(\mathcal{CR})$:(for details, see Hall and Jones [9], Jones [11] and Reilly [33])

$\mathcal{G}$, $\mathcal{CS}$, $\mathcal{AG}$, $\mathcal{B}$, $\mathcal{OG}$, $\mathcal{LOCG}$,

$\mathcal{CS}(\mathcal{AG})$ - the variety of completely simple semigroups with abelian subgroups.

$\mathcal{OG}(\mathcal{AG})$ - the variety of orthodox completely regular semigroups with abelian subgroups.

$\mathcal{LOCG}(\mathcal{AG})$ - the variety of locally orthodox cryptogroups with abelian subgroups.

$\mathcal{L}(\mathcal{B})$ - all subvarieties of $\mathcal{B}$

$\mathcal{L}(\mathcal{OG}(\mathcal{AG}))$ - all subvarieties of $\mathcal{OG}(\mathcal{AG})$.

$\mathcal{L}(\mathcal{LOCG}(\mathcal{AG}))$ - all subvarieties of $\mathcal{LOCG}(\mathcal{AG})$.

Some partial results can also be obtained, such as the following.

<u>COROLLARY</u> 5.6. (Reilly [33], Corollary 5.8) The variety $\mathcal{CG}$ is neutral in $\mathcal{L}(\mathcal{CG}^K)$.

Since $\mathcal{G} \subseteq \mathcal{CG}$, we must also have $\mathcal{OG} - \mathcal{G}^K \subseteq \mathcal{CG}^K$ and therefore also

$\mathcal{CG} \vee \mathcal{OG} \subseteq \mathcal{CG}^K$. From this it can be shown that $\mathcal{CG}$ is neutral in $\mathcal{L}(\mathcal{CG} \vee \mathcal{OG})$.

An important feature of the next theorem is the fact that certain varieties are expressible as joins of well known varieties.

<u>LEMMA</u> 5.7. (i) $\mathcal{B} \vee \mathcal{G} = \mathcal{OCG}$. (ii) $\mathcal{B} \vee \mathcal{CS} = \mathcal{LOCG}$.

(iii) $\mathcal{OG} \vee \mathcal{CS} = \mathcal{CLOCG}$.

<u>Proof</u>. (i) See (Petrich [16], Lemma 1).

(ii) See (Hall and Jones [9], Corollary 5.4).

(iii) See (Hall and Jones [9], Theorem 5.3 and Reilly [32], Proposition 5.3).

<u>COROLLARY</u> 5.8. (i) (Petrich [16], Theorem) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{B},\ \mathcal{V} \cap \mathcal{G}), \qquad (\mathcal{U},\mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{OCG})$ and $\mathcal{L}(\mathcal{B}) \times \mathcal{L}(\mathcal{G})$.

(ii) (Hall and Jones [9], Corollary 5.5, Rasin [30], Proposition 1) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{B},\ \mathcal{V} \cap \mathcal{CS}), \qquad (\mathcal{U},\mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{LOCG})$ and the subdirect product of $\mathcal{L}(\mathcal{B})$ and $\mathcal{L}(\mathcal{CS})$ consisting of all those pairs $(\mathcal{U}, \mathcal{W})$ with $\mathcal{U} \cap \mathcal{B} = \mathcal{W} \cap \mathcal{CS}$.

(iii) (Reilly [33], Theorem 5.9) The mappings

$$\mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{CG},\ \mathcal{V} \cap \mathcal{OG}), \qquad (\mathcal{U}, \mathcal{W}) \longrightarrow \mathcal{U} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{CG} \vee \mathcal{OG})$ and the subdirect product of $\mathcal{L}(\mathcal{CG})$ and $\mathcal{L}(\mathcal{OG})$ consisting of all those pairs $(\mathcal{U}, \mathcal{W})$ with $\mathcal{U} \cap \mathcal{OCG} = \mathcal{W} \cap \mathcal{OCG}$.

<u>REFERENCES</u>

[1] A.P.Birjukov, Varieties of idempotent semigroups, Algebra i Logika, 9(1970), 255-273 (Russian), AMS Translation.

[2] A.H.Clifford, Semigroups admitting relative inverses, Annals of Math. 42(1941), 1037-1049.

[3] A.H. Clifford, The free completely regular semigroup on a

set, J. Algebra 59 (1979), 434-451.
[4] A.H.Clifford and G.B.Preston, The algebraic theory of semigroups, Amer. Math. Surveys No. 7, 1961, Providence.
[5] R.Feigenbaum, Regular semigroup congruences, Semigroup Forum 17 (1979), 373-377.
[6] C.F.Fennemore, All varieties of bands, Math. Nachr. 48 (1971), I:237-252, II: 253-262.
[7] G.Grätzer, General lattice theory, Academic Press, New York, 1978.
[8] J.A. Gerhard, The lattice of equational classes of idempotent semigroups, J.Algebra 15(1970), 195-224.
[9] T.E.Hall and P.R. Jones, On the lattice of varieties of bands of groups, Pacific J. Math. 91(1980), 327-337.
[10] J.M.Howie, An Introduction to Semigroup Theory, Academic Press, London, 1976.
[11] P.R.Jones, On the lattice of varieties of completely regular semigroups, J. Austral. Math. Soc. (Series A) 35 (1983), 227-235.
[12] F.Pastijn, The lattice of completely regular semigroup varieties, preprint.
[13] F.Pastijn, Commuting fully invariant congruences on free completely regular semigroups, (manuscript).
[14] F.Pastijn and M. Petrich, Congruences on regular semigroups, Trans. Amer. Math. Soc. 295(1986), 607-633.
[15] F. Pastijn and P.G. Trotter, Lattices of completely regular semigroup varieties, Pacific J. Math. 119 (1985), 191-214.
[16] M.Petrich, Varieties of orthodox bands of groups, Pacific J. Math., 58(1975), 209-217.
[17] M.Petrich, A structure theorem for completely regular semigroups, Proc. Amer. Math. Soc. 99(1987), 617-622.
[18] M.Petrich, Congruences on completely regular semigroups, preprint.
[19] M.Petrich and N.R.Reilly, Bands of groups with universal properties, Monatshefte für Mathematik 94(1982), 45-67.
[20] M.Petrich and N.R.Reilly, All varieties of central completely simple semigroups, Trans. Amer. Math. Soc. 280(1983), 623-636.
[21] M. Petrich and N.R. Reilly, Semigroups generated by certain operators on varieties of completely regular semigroups, Pacific J. Math. 132 (1988), 151-175.
[22] M.Petrich and N.R.Reilly, Operators related to E-disjunctive and fundamental completely regular semigroups (to appear).

[23] M.Petrich and N.R.Reilly, The modularity of the lattice of varieties of completely regular semigroups and related representations (to appear).
[24] M.Petrich and N.R.Reilly, Completely regular semigroups (manuscript).
[25] L.Polák, On varieties of completely regular semigroups I, Semigroup Forum 32(1985), 97-123.
[26] L.Polák, On varieties of completely regular semigroups II, Semigroup Forum 36(1987), 253-284.
[27] L.Polák, On varieties of completely regular semigroups III, Semigroup Forum 37(1988), 1-30.
[28] V.V. Rasin, Free completely simple semigroups, Ural. Gos. Univ. Mat. Zap. 11 (1979), 140-151 (Russian).
[29] V.V.Rasin, On the lattice of varieties of completely simple semigroups, Semigroup Forum 17(1979), 113-122.
[30] V.V.Rasin, On the varieties of Cliffordian semigroups, Semigroup Forum 23(1981), 201-220.
[31] D.Rees, On semi-groups, Proc. Cambridge Phil. Soc. 36(1940), 387-400.
[32] N.R.Reilly, Varieties of completely regular semigroups, J. Austral. Math. Soc. (Series A) 38 (1985), 372-393.
[33] N.R.Reilly, Completely regular semigroups, Proc. of the Inter. Conf. on Algebra, Lisbon (1988).

Dept. of Math. and Stat.,
Simon Fraser University,
B.C., Canada V5A 1S6.