CHAPTER H

Congruence Properties of Partition Function

Congruence properties of $p(n)$, the number of partitions of n, were first discovered by Ramanujan on examining the table of the first 200 values of $p(n)$ constructed by MacMahon (1915):

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5 \\ p(7n+5) &\equiv 0 \pmod 7 \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

In general, if β is a prime, then the congruence with some fixed $\gamma \in \mathbb{N}$

$$p(\beta n+\gamma) \equiv_\beta 0 \quad \text{for all} \quad n \in \mathbb{N}$$

is called the Ramanujan congruence modulo β. Ahlgren and Boylan (2003) confirmed recently that the three congruences displayed above are the only Ramanujan ones.

There exist other congruences of partition function, but non Ramanujan one's. For example, the simplest congruence modulo 13 recorded by Atkin and O'Brien (1967) can be reproduced as follows:

$$p(1331 \times 13n + 237) \equiv 0 \pmod{13}.$$

In order to facilitate the demonstration of the three Ramanujan congruences, we will first show the following general congruence relation about the partition function.

Congruence Lemma on Partition Function: *Let β be a prime and γ an integer. Define the $\wp(m)$-sequence by*

$$\wp(m) = \left[q^m\right] \left\{ q^{\beta-\gamma}(q;q)_\infty^{\beta-1} \right\}.$$

If all the coefficients $\wp(\beta m)$ for $m \in \mathbb{N}$ are multiples of β, then there holds the corresponding Ramanujan congruence, i.e., $p(\beta n+\gamma)$ are divisible by β for all $n \in \mathbb{N}$.

PROOF. Writing $x \equiv_\beta y$ for congruence relation $x \equiv y \pmod{\beta}$, then we have the binomial congruence

$$\binom{\beta-1+k}{\beta-1} \equiv_\beta \begin{cases} 1, & k \equiv_\beta 0 \\ 0, & k \not\equiv_\beta 0. \end{cases}$$

By means of binomial expansion, we can derive congruence relation

$$\begin{aligned} \frac{1-q^\beta}{(1-q)^\beta} &= (1-q^\beta)\sum_{k=0}^{\infty}\binom{\beta-1+k}{\beta-1}q^k \\ &\equiv_\beta (1-q^\beta)\sum_{k=0}^{\infty} q^{\beta k} = 1. \end{aligned}$$

Therefore we have accordingly the formal power series congruence

$$\begin{aligned} q^{\beta-\gamma}\frac{(q^\beta;\,q^\beta)_\infty}{(q;\,q)_\infty} &= q^{\beta-\gamma}(q;\,q)_\infty^{\beta-1}\frac{(q^\beta;\,q^\beta)_\infty}{(q;\,q)_\infty^\beta} \\ &\equiv q^{\beta-\gamma}\prod_{k=1}^{\infty}(1-q^k)^{\beta-1} \pmod{\beta} \end{aligned}$$

which implies consequently the following congruence relation

$$\wp(\beta m) \equiv \left[q^{\beta m}\right]\left\{q^{\beta-\gamma}\frac{(q^\beta;\,q^\beta)_\infty}{(q;\,q)_\infty}\right\} \pmod{\beta} \quad \text{for} \quad m \in \mathbb{N}.$$

According to the generating function of partitions and the Cauchy product of formal power series, we get the following relation

$$\begin{aligned} p(\beta n+\gamma) &= \left[q^{\beta(1+n)}\right]\left\{q^{\beta-\gamma}\Big/\prod_{k=1}^{\infty}(1-q^k)\right\} \\ &= \left[q^{\beta(1+n)}\right]\left\{q^{\beta-\gamma}\frac{(q^\beta;\,q^\beta)_\infty}{(q;\,q)_\infty}\Big/(q^\beta;\,q^\beta)_\infty\right\} \\ &= \sum_{m=0}^{1+n} p(1+n-m)\left[q^{\beta m}\right]\left\{q^{\beta-\gamma}\frac{(q^\beta;\,q^\beta)_\infty}{(q;\,q)_\infty}\right\} \\ &\equiv \sum_{m=0}^{1+n} p(1+n-m)\,\wp(\beta m) \pmod{\beta}. \end{aligned}$$

Hence $p(\beta n+\gamma)$ is divisible by β as long as all the coefficients $\wp(\beta m)$ for $m \in \mathbb{N}$ are multiples of β. This completes the proof of the congruence lemma. □

By means of this lemma, we will present Ramanujan's original proof for the first two congruences and the proof for the third one due to Winquist (1969). In addition, the corresponding generating functions will be determined for the first two cases.

H1. Proof of $p(5n+4) \equiv 0 \pmod 5$

There holds the Ramanujan congruence modulo 5:

$$p(5n+4) \;\equiv\; 0 \pmod{5}. \tag{H1.1}$$

In view of the congruence lemma on partition function, we should show that the coefficients of q^{5m} in the formal power series expansion of $q(q;\,q)_\infty^4$ are divisible by 5 for all $m \in \mathbb{N}$.

By means of Euler's pentagon number theorem and the Jacobi triple product identity, consider the formal power series expansion

$$\begin{aligned} q(q;\,q)_\infty^4 &= q\prod_{m=1}^{\infty}(1-q^m)\prod_{n=1}^{\infty}(1-q^n)^3 \\ &= \sum_{i=0}^{+\infty}\sum_{j=-\infty}^{+\infty}(-1)^{i+j}\,(1+2i)\,q^{1+j^2+\binom{1+i}{2}+\binom{1+j}{2}}. \end{aligned}$$

In accordance with congruences

$$\binom{k+1}{2} \equiv_5 \begin{cases} 0, & k \equiv_5 0 \\ 1, & k \equiv_5 1 \\ 3, & k \equiv_5 2 \\ 1, & k \equiv_5 3 \\ 0, & k \equiv_5 4 \end{cases}$$

it is not hard to check that the residues of q-exponent in the formal power series just-displayed

$$1+j^2+\binom{1+i}{2}+\binom{1+j}{2}$$

modulo 5 are given by the following table:

$j\backslash i$	0	1	2	3	4
0	1	2	4	2	1
1	3	4	1	4	3
2	3	4	1	4	3
3	1	2	4	2	1
4	2	3	0	3	2

From this table, we see that if the q-exponent $1+j^2+\binom{1+i}{2}+\binom{1+j}{2}$ is a multiple of 5, so is the coefficient $1+2i$, which corresponds to the only case $i \equiv_5 2$ and $j \equiv_5 4$.

We can also verify this fact by reformulating the congruence relation on the q-exponent as

$$\begin{aligned} 0 &\equiv_5 1+j^2+\binom{1+i}{2}+\binom{1+j}{2} \\ &\equiv_5 8\Big\{1+j^2+\binom{1+i}{2}+\binom{1+j}{2}\Big\} \\ &\equiv_5 (1+2i)^2+2(1+j)^2. \end{aligned}$$

This congruence can be reached only when

$$\begin{aligned} (1+2i)^2 \equiv_5 0 &\implies i \equiv_5 2 \\ 2(1+j)^2 \equiv_5 0 &\implies j \equiv_5 4 \end{aligned}$$

because the corresponding residues modulo 5 read respectively as

$$(1+2i)^2 \equiv_5 0,1,4 \quad \text{and} \quad 2(1+j)^2 \equiv_5 0,2,3.$$

Therefore the coefficients of q^{5m} in the formal power series expansion of $q(q;q)_\infty^4$ are divisible by 5. This completes the proof of the Ramanujan congruence (H1.1).

H2. Generating function for $p(5n+4)$

Furthermore, Ramanujan computed explicitly the generating function:

$$\sum_{n=0}^{\infty} p(5n+4)\, q^n = 5\frac{(q^5;\, q^5)_\infty^5}{(q;\, q)_\infty^6}. \tag{H2.1}$$

About this identity, Hardy wrote that if he were to select one formula from Ramanujan's work for supreme beauty, he would agree with MacMahon in selecting this one.

The proof presented here is essentially due to Ramanujan.

H2.1. Let $p := q^{1/5}$ and ω be a 5-th primitive root of unit $\omega = e^{\frac{2\pi}{5}\sqrt{-1}}$.

Recall the generating function of partitions

$$\frac{q}{(q;\, q)_\infty} = q\sum_{n=0}^{\infty} p(n)q^n.$$

Replacing q by $q\omega^k$ and then summing the equations with k over $0 \leq k \leq 4$, we have

$$\sum_{k=0}^{4} \frac{q\omega^k}{(q\omega^k; q\omega^k)_\infty} = \sum_{n=0}^{\infty} p(n)q^{n+1}\sum_{k=0}^{4}\omega^{k(n+1)}.$$

It is not hard to verify that

$$\sum_{k=0}^{4}\omega^{k(n+1)} = \begin{cases} 5, & n+1 \equiv_5 0 \\ 0, & n+1 \not\equiv_5 0 \end{cases} \tag{H2.2}$$

where the last line is justified by the finite geometric series

$$\sum_{k=0}^{4}\omega^{k(n+1)} = \frac{1-\omega^{5(n+1)}}{1-\omega^{n+1}} \quad \text{provided that } n+1 \not\equiv_5 0.$$

Specifying $n+1$ with $5m+5$, we have

$$\sum_{m=0}^{\infty} p(5m+4)q^{5m+5} = \frac{1}{5}\sum_{k=0}^{4}\frac{q\omega^k}{(q\omega^k; q\omega^k)_\infty}.$$

Replacing q by $p := q^{1/5}$, we can reformulate the last equation as

$$\sum_{m=0}^{\infty} p(5m+4)q^{m} = \frac{1}{5q}\sum_{k=0}^{4}\frac{p\omega^k}{(p\omega^k; p\omega^k)_\infty}. \tag{H2.3}$$

H2.2. In order to evaluate the sum displayed in (H2.3), we first show that:

$$\frac{p(q^5; q^5)_\infty}{(p; p)_\infty} = \frac{1}{\lambda/p - 1 - p/\lambda} \tag{H2.4}$$

where λ is an infinite factorial fraction defined by

$$\lambda := \frac{\left[q^2,\, q^3;\, q^5\right]_\infty}{\left[q,\, q^4;\, q^5\right]_\infty}. \tag{H2.5}$$

Recall the Euler pentagon number theorem:

$$(p;p)_\infty = \sum_{j=-\infty}^{+\infty}(-1)^j p^{\frac{j(3j+1)}{2}}.$$

It is easy to verify that the pentagon numbers admit only three residue classes modulo 5:

$$\frac{j(3j+1)}{2} \equiv_5 \begin{cases} 0, & j = 0, -2 \pmod 5 \\ 1, & j = -1 \pmod 5 \\ 2, & j = 1,\ 2 \pmod 5. \end{cases}$$

We can accordingly write

$$(p;p)_\infty \; = \; A - pB - p^2C. \tag{H2.6}$$

The coefficients A, B and C can be individually determined by means of Jacobi's triple and the quintuple product identities.

A-Coefficient: Specifying the summation index j with $5j$ and $-2-5j$, we can compute A, by means of the quintuple product identity as follows:

$$\begin{aligned} A \;&=\; \sum_{j=-\infty}^{+\infty} (-1)^j \Big\{ p^{\frac{5j(15j+1)}{2}} + p^{\frac{(5j+2)(15j+5)}{2}} \Big\} \\ &=\; \sum_{j=-\infty}^{+\infty} (-1)^j \big\{1 + p^{5+25j}\big\} p^{75\binom{j}{2}+40j} \\ &=\; \big[p^{25}, -p^5, -p^{20}; p^{25}\big]_\infty \big[p^{35}, p^{15}; p^{50}\big]_\infty \\ &=\; \big[q^5, -q, -q^4; q^5\big]_\infty \big[q^7, q^3; q^{10}\big]_\infty. \end{aligned}$$

B-Coefficient: It can be evaluated through the Jacobi triple product identity as follows:

$$\begin{aligned} B \;&=\; p^{-1} \sum_{j=-\infty}^{+\infty} (-1)^j p^{\frac{(5j-1)(15j-2)}{2}} \;=\; \sum_{j=-\infty}^{+\infty} (-1)^j p^{75\binom{j}{2}+25j} \\ &=\; \big[p^{75}, p^{25}, p^{50}; p^{75}\big]_\infty \;=\; \big[q^{15}, q^5, q^{10}; q^{15}\big] \;=\; \big[q^5; q^5\big]_\infty. \end{aligned}$$

C-Coefficient: Similar to the computation of A, we can compute C, by specifying the summation index j with $5j+1$ and $5j+2$, as follows:

$$\begin{aligned} C \;&=\; p^{-2} \sum_{j=-\infty}^{+\infty} (-1)^j \Big\{ p^{\frac{(5j+1)(15j+4)}{2}} - p^{\frac{(5j+2)(15j+7)}{2}} \Big\} \\ &=\; \sum_{j=-\infty}^{+\infty} (-1)^j p^{75\binom{j}{2}+55j} \big\{1 - p^{5+15j}\big\} \\ &=\; \big[p^{25}, -p^{10}, -p^{15}; p^{25}\big]_\infty \big[p^{45}, p^5; p^{50}\big]_\infty \\ &=\; \big[q^5, -q^2, -q^3; q^5\big]_\infty \big[q^9, q; q^{10}\big]_\infty. \end{aligned}$$

In accordance with (H2.6), we find the following relation

$$\frac{(p;p)_\infty}{(q^5;q^5)_\infty} = \frac{A}{B} - p - p^2\frac{C}{B}$$

where the coefficient-fractions can be simplified as follows:

$$\begin{aligned}\frac{A}{B} &= \big[-q,-q^4;q^5\big]_\infty \times \big[q^7,q^3;q^{10}\big]_\infty\\ &= \frac{\big[q^2,q^3,q^7,q^8;q^{10}\big]_\infty}{\big[q,\ q^4;q^5\big]_\infty} = \frac{\big[q^2,q^3;q^5\big]_\infty}{\big[q,\ q^4;q^5\big]_\infty}\\ \frac{C}{B} &= \big[-q^2,-q^3;q^5\big]_\infty \times \big[q,q^9;q^{10}\big]_\infty\\ &= \frac{\big[q,q^4,q^6,q^9;q^{10}\big]_\infty}{\big[q^2,q^3;q^5\big]_\infty} = \frac{\big[q,\ q^4;q^5\big]_\infty}{\big[q^2,q^3;q^5\big]_\infty}.\end{aligned}$$

Observing further that

$$\lambda := \frac{A}{B} = \frac{B}{C} = \frac{\big[q^2,q^3;q^5\big]_\infty}{\big[q,\ q^4;q^5\big]_\infty}$$

we can reformulate (H2.6) as the following reduced expression

$$\frac{(p;p)_\infty}{(q^5;q^5)_\infty} = \lambda - p - \frac{p^2}{\lambda}$$

which is equivalent to

$$\frac{p(q^5;q^5)_\infty}{(p;p)_\infty} = \frac{1}{\lambda/p - 1 - p/\lambda}.$$

H2.3. For the sum displayed in (H2.3), we then compute the common denominator:

$$\prod_{k=0}^{4}(p\omega^k;p\omega^k)_\infty = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty}. \tag{H2.7}$$

In fact, the general term of the product with index n reads as

$$\prod_{k=0}^{4}\big\{1-(p\omega^k)^n\big\} = \prod_{k=0}^{4}(1-p^n\omega^{kn}) = \begin{cases}(1-p^n)^5, & n\equiv_5 0\\ (1-p^{5n}), & n\not\equiv_5 0.\end{cases}$$

Therefore we have the following simplified product

$$\begin{aligned}\prod_{k=0}^{4}(p\omega^k;p\omega^k)_\infty &= \prod_{n=1}^{\infty}\prod_{k=0}^{4}(1-p^n\omega^{kn}) = \prod_{\substack{n=1\\ n\not\equiv_5 0}}^{\infty}(1-p^{5n})\prod_{n=1}^{\infty}(1-p^{5n})^5\\ &= \prod_{\substack{n=1\\ n\not\equiv_5 0}}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1-q^n)^5 = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty}, \quad p := q^{1/5}.\end{aligned}$$

This can be stated equivalently as the product of λ-polynomials:

$$\prod_{k=0}^{4}\frac{1}{\lambda/p\omega^k-1-p\omega^k/\lambda} = (q^5;q^5)_\infty^5\prod_{k=0}^{4}\frac{p\omega^k}{(p\omega^k;p\omega^k)_\infty} = q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6}.$$

H2.4. Performing replacement $p\to p\omega^\ell$ in (H2.4) and then summing both sides over $0\le\ell\le4$, we have

$$\sum_{\ell=0}^{4}\frac{p\omega^\ell}{(p\omega^\ell;p\omega^\ell)_\infty} = \frac{1}{(q^5;q^5)_\infty}\sum_{\ell=0}^{4}\frac{1}{\lambda/(p\omega^\ell)-1-p\omega^\ell/\lambda}. \tag{H2.8}$$

For the sum on the right hand side, there holds the following closed form:

$$\sum_{\ell=0}^{4}\frac{1}{\lambda/p\omega^\ell-1-p\omega^\ell/\lambda} = \frac{25}{\prod_{k=0}^{4}\Big\{\lambda/p\omega^k-1-p\omega^k/\lambda\Big\}}. \tag{H2.9}$$

Then the generating function (H2.1) can be derived from (H2.3) consequently as follows:

$$\begin{aligned}\sum_{m=0}^{\infty}p(5m+4)q^m &= \frac{1}{5q}\sum_{k=0}^{4}\frac{p\omega^k}{(p\omega^k;p\omega^k)_\infty}\\
&= \frac{1}{5q(q^5;q^5)_\infty}\sum_{\ell=0}^{4}\frac{1}{\lambda/(p\omega^\ell)-1-p\omega^\ell/\lambda}\\
&= \frac{5}{q(q^5;q^5)_\infty}\prod_{k=0}^{4}\frac{1}{\lambda/p\omega^k-1-p\omega^k/\lambda}\\
&= \frac{5}{q(q^5;q^5)_\infty}\times\frac{q(q^5;q^5)_\infty^6}{(q;q)_\infty^6} = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}.\end{aligned}$$

H2.5. In order to show the algebraic identity (H2.9), we first reformulate it as follows:

$$\begin{aligned}\sum_{\ell=0}^{4}\frac{1}{\lambda/p\omega^\ell-1-p\omega^\ell/\lambda} &= \prod_{k=0}^{4}\frac{1}{\lambda/p\omega^k-1-p\omega^k/\lambda}\\
&\times\sum_{\ell=0}^{4}\prod_{\substack{k=0\\k\neq\ell}}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\}.\end{aligned}$$

If we can show that the last sum on the right hand side equals 25, then the generating function (H2.1) will be confirmed.

Surprisingly enough, it is true that the sum just mentioned is indeed equal to 25:

$$\sum_{\ell=0}^{4}\prod_{\substack{k=0\\k\neq\ell}}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\} \;=\; 25. \tag{H2.10}$$

As the Laurent polynomial in p, we can expand the product

$$\prod_{k=1}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\} \;=\; \sum_{\kappa=-4}^{4} W(\kappa)\,p^\kappa$$

where $\{W(\kappa)\}$ are constants independent of p. Keeping in mind that for each ℓ with $0\leq\ell\leq4$, the residues of $\{k+\ell\}_{k=1}^4$ modulo 5 are, in effect, $\{0\leq k\leq4\}_{k\neq\ell}$, we can accordingly simplify the sum displayed in (H2.10) as follows:

$$\begin{aligned}
&\sum_{\ell=0}^{4}\prod_{\substack{k=0\\k\neq\ell}}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\}\\
=\;&\sum_{\ell=0}^{4}\prod_{k=1}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\}\Big|_{p\to p\omega^\ell}\\
=\;&\sum_{\ell=0}^{4}\sum_{\kappa=-4}^{4}W(\kappa)p^\kappa\omega^{\kappa\ell} \;=\; \sum_{\kappa=-4}^{4}W(\kappa)\,p^\kappa\sum_{\ell=0}^{4}\omega^{\kappa\ell}\\
=\;&5W(0) \;=\; 5[p^0]\prod_{k=1}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\}.
\end{aligned}$$

Recalling two simple facts about ω with $\omega=e^{\frac{2\pi}{5}\sqrt{-1}}$

$$\prod_{k=1}^{4}\omega^k \;=\; +1 \quad\text{and}\quad \sum_{k=1}^{4}\omega^k \;=\; -1$$

we can compute $W(0)$, by matching the powers of p, as follows:

$$\begin{aligned}
W(0) &= [p^0]\prod_{k=1}^{4}\Big\{\lambda/(p\omega^k)-1-p\omega^k/\lambda\Big\}\\
&= (-1)^4-(-1)^2\sum_{i=1}^{4}\omega^{-i}\sum_{j\neq i}\omega^j+\sum_{1\le i<j\le 4}\omega^{-2i-2j}\\
&= 1-\sum_{i=1}^{4}\omega^{-i}\{-1-\omega^i\}+\frac{1}{2}\sum_{i\neq j}\omega^{-2i-2j}\\
&= 1+\sum_{i=1}^{4}\{1+\omega^{-i}\}+\frac{1}{2}\Big\{\Big(\sum_{k=1}^{4}\omega^{-2k}\Big)^2-\sum_{k=1}^{4}\omega^{-4k}\Big\}\\
&= 1+(4-1)+\frac{1}{2}\big\{(-1)^2+1\big\} \;=\; 5.
\end{aligned}$$

H2.6. Partial fraction method for (H2.9)**.** The algebraic identity (H2.9) can also be demonstrated by means of partial fraction method.

Define the quadratic polynomial by

$$\vartheta(\lambda,p) \;=\; \lambda^2-\lambda p-p^2 \;=\; \big\{\lambda-\lambda_1(p)\big\}\times\big\{\lambda-\lambda_2(p)\big\}$$

where two zeros are given explicitly by

$$\lambda_1(p)=\frac{p}{2}(1+\sqrt{5})\quad\text{and}\quad\lambda_2(p)=\frac{p}{2}(1-\sqrt{5}).$$

Then we have the partial fraction decomposition

$$25q\prod_{k=0}^{4}\frac{\lambda}{\vartheta(\lambda,p\omega^k)} \;=\; \frac{25\lambda^5 q}{\big\{\lambda^5-\lambda_1^5(p)\big\}\times\big\{\lambda^5-\lambda_2^5(p)\big\}} \tag{H2.11a}$$

$$= \frac{25}{\prod_{k=0}^{4}\Big\{\lambda/p\omega^k-1-p\omega^k/\lambda\Big\}} \tag{H2.11b}$$

$$= \sum_{\ell=0}^{4}\Big\{\frac{u_\ell}{\lambda-\lambda_1(p\omega^\ell)}+\frac{v_\ell}{\lambda-\lambda_2(p\omega^\ell)}\Big\} \tag{H2.11c}$$

where the coefficients u_ℓ and v_ℓ remain to be determined.

By means of the L'Hôspital rule, we can compute the u_ℓ-coefficient:

$$\begin{aligned} u_\ell &= \lim_{\lambda\to\lambda_1(p\omega^\ell)} \frac{25q\lambda^5\big\{\lambda-\lambda_1(p\omega^\ell)\big\}}{\big\{\lambda^5-\lambda_1^5(p)\big\}\times\big\{\lambda^5-\lambda_2^5(p)\big\}} \\ &= \frac{25q\lambda_1^5(p\omega^\ell)}{\lambda_1^5(p)-\lambda_2^5(p)} \lim_{\lambda\to\lambda_1(p\omega^\ell)} \frac{\lambda-\lambda_1(p\omega^\ell)}{\lambda^5-\lambda_1^5(p)} \\ &= \frac{5q\lambda_1(p\omega^\ell)}{\lambda_1^5(p)-\lambda_2^5(p)} = \frac{\lambda_1(p\omega^\ell)}{\sqrt{5}} \end{aligned}$$

where we have simplified the difference

$$\lambda_1^5(p)-\lambda_2^5(p) = 5q\sqrt{5}.$$

Similarly, we can also determine the v_ℓ-coefficient:

$$\begin{aligned} v_\ell &= \lim_{\lambda\to\lambda_2(p\omega^\ell)} \frac{25q\lambda^5\big\{\lambda-\lambda_2(p\omega^\ell)\big\}}{\big\{\lambda^5-\lambda_1^5(p)\big\}\times\big\{\lambda^5-\lambda_2^5(p)\big\}} \\ &= \frac{25q\lambda_2^5(p\omega^\ell)}{\lambda_2^5(p)-\lambda_1^5(p)} \lim_{\lambda\to\lambda_2(p\omega^\ell)} \frac{\lambda-\lambda_2(p\omega^\ell)}{\lambda^5-\lambda_2^5(p)} \\ &= \frac{5q\lambda_2(p\omega^\ell)}{\lambda_2^5(p)-\lambda_1^5(p)} = -\frac{\lambda_2(p\omega^\ell)}{\sqrt{5}}. \end{aligned}$$

Combining two summand terms in (H2.11c) into a single one

$$\begin{aligned} \frac{u_\ell}{\lambda-\lambda_1(p\omega^\ell)} + \frac{v_\ell}{\lambda-\lambda_2(p\omega^\ell)} &= \frac{\lambda}{\sqrt{5}}\frac{\lambda_1(p\omega^\ell)-\lambda_2(p\omega^\ell)}{\vartheta(\lambda,p\omega^\ell)} \\ &= \frac{\lambda p\omega^\ell}{\vartheta(\lambda,p\omega^\ell)} = \frac{1}{\lambda/p\omega^\ell-1-p\omega^\ell/\lambda} \end{aligned}$$

we establish the algebraic identity

$$\sum_{\ell=0}^{4}\frac{1}{\lambda/p\omega^\ell-1-p\omega^\ell/\lambda} = \frac{25}{\prod_{k=0}^{4}\Big\{\lambda/p\omega^k-1-p\omega^k/\lambda\Big\}}$$

which is exactly (H2.9) as desired.

H2.7. There exists a polynomial expression of the common denominator in terms of λ:

$$\prod_{k=0}^{4}(\lambda/(p\omega^k)-1-p\omega^k/\lambda) = \lambda^5/q-11-q/\lambda^5.$$

In fact, replacing λ/p by y, we can restate the product as

$$\prod_{k=0}^{4}(y/\omega^k-1-\omega^k/y).$$

Noticing that the zeros of the first factor $y-1-1/y$ are solutions of equation $y-1/y=1$ and so solutions of equation

$$(y-1/y)^3 = (y^3-1/y^3) - 3(y-1/y) = 1$$

which is equivalent to

$$y^3 - 1/y^3 = 4.$$

Furthermore, these zeros of $y-1-1/y$ are also solutions of equation

$$(y-1/y)^5 = y^5 - 1/y^5 - 5(y^3-1/y^3) + 10(y-1/y) = 1.$$

The last equation reads in fact as the following simplified form

$$y^5 - 1/y^5 = 11$$

which implies therefore that $y^5 - 11 - 1/y^5$ is a multiple of $y-1-1/y$.

Noting that $y^5 - 11 - 1/y^5$ is invariant under $y \to y/\omega^k$ for $k = 0, 1, 2, 3, 4$, we deduce that it is also a multiple of the product $\prod_{k=0}^4 (y/\omega^k - 1 - \omega^k/y)$. Hence we have established the following equation

$$\prod_{k=0}^{4} (y/\omega^k - 1 - \omega^k/y) = y^5 - 11 - 1/y^5$$

thanks for the fact that both sides are *monic polynomials* of the same degree.

H3. Proof of $p(7n+5) \equiv 0 \pmod 7$

There holds the Ramanujan congruence modulo 7:

$$p(7n+5) \equiv 0 \pmod{7}. \tag{H3.1}$$

According to the congruence lemma on partition function, we should show that the coefficients of q^{7m} in the formal power series expansion of $q^2(q;\, q)_\infty^6$ are divisible by 7 for all $m \in \mathbb{N}$.

By means of the limiting version of the Jacobi triple product identity, consider the formal power series expansion

$$\begin{aligned} q^2(q;\, q)_\infty^6 &= q^2 \prod_{m=1}^{\infty} (1-q^m)^3 \prod_{n=1}^{\infty} (1-q^n)^3 \\ &= \sum_{i,j=0}^{+\infty} (-1)^{i+j}\, (1+2i)(1+2j)\, q^{2+\binom{1+i}{2}+\binom{1+j}{2}}. \end{aligned}$$

Observe that the congruence relation on the q-exponent

$$\begin{aligned} 0 &\equiv_7 2+\binom{1+i}{2}+\binom{1+j}{2} \\ &\equiv_7 8\Big\{2+\binom{1+i}{2}+\binom{1+j}{2}\Big\} \\ &\equiv_7 (1+2i)^2+(1+2j)^2 \end{aligned}$$

can be reached only when

$$\begin{aligned} (1+2i)^2 \equiv_7 0 &\implies i \equiv_7 3 \\ (1+2j)^2 \equiv_7 0 &\implies j \equiv_7 3 \end{aligned}$$

because the corresponding residues modulo 7 read as

$$(1+2k)^2 \equiv_7 0, 1, 2, 4.$$

The coefficients of q^{7m} in the formal power series expansion of $q^2(q;q)_\infty^6$ are therefore divisible by 7. This completes the proof of congruence (H3.1).

H4. Generating function for $p(7n+6)$

Ramanujan discovered also explicitly the generating function.

$$\sum_{n=0}^{\infty} p(7n+6)\, q^n = 7\frac{(q^7;\, q^7)_\infty^3}{(q;\, q)_\infty^4} + 49q\frac{(q^7;\, q^7)_\infty^7}{(q;\, q)_\infty^8}. \tag{H4.1}$$

Following the same line to the proof of (H2.1), we present a derivation of this generating function, which is much more difficult.

H4.1. Let $\rho := q^{1/7}$ and ϖ be a 7-th primitive root of unit $\varpi = e^{\frac{2\pi}{7}\sqrt{-1}}$.

Recall the generating function of partitions

$$\frac{q^2}{(q;\, q)_\infty} = q^2 \sum_{n=0}^{\infty} p(n) q^n.$$

Replacing q by $q\varpi^k$ and then summing the equations with k over $0 \le k \le 6$, we have

$$\sum_{k=0}^{6} \frac{q^2\varpi^{2k}}{(q\varpi^k; q\varpi^k)_\infty} = \sum_{n=0}^{\infty} p(n) q^{n+2} \sum_{k=0}^{6} \varpi^{k(n+2)}.$$

It is not hard to verify that

$$\sum_{k=0}^{6} \varpi^{k(n+2)} = \begin{cases} 7, & n+2 \equiv_7 0 \\ 0, & n+2 \not\equiv_7 0 \end{cases} \tag{H4.2}$$

where the last line is justified by the finite geometric series

$$\sum_{k=0}^{6} \varpi^{k(n+2)} = \frac{1-\varpi^{7(n+2)}}{1-\varpi^{n+2}} \quad \text{provided that } n+2 \not\equiv_7 0.$$

Specifying $n+2$ with $7m+7$, we have

$$\sum_{m=0}^{\infty} p(7m+5) q^{7m+7} = \frac{1}{7} \sum_{k=0}^{6} \frac{q^2 \varpi^{2k}}{(q\varpi^k; q\varpi^k)_\infty}.$$

Replacing q by $\rho := q^{1/7}$, we can reformulate the last equation as

$$\sum_{m=0}^{\infty} p(7m+5) q^{m} = \frac{1}{7q} \sum_{k=0}^{6} \frac{\rho^2 \varpi^{2k}}{(\rho\varpi^k; \rho\varpi^k)_\infty}. \tag{H4.3}$$

H4.2. In order to simplify the sum displayed in (H4.3), we show that:

$$\frac{\rho^2 (q^7; q^7)_\infty}{(\rho;\rho)_\infty} = \frac{1}{A/\rho^2 - B/\rho - 1 + \rho^3/AB} \tag{H4.4}$$

where A and B are two infinite factorial fractions defined by

$$A := \left[\begin{matrix} q^2,\ q^5 \\ q,\ q^6 \end{matrix} \,\middle|\, q^7\right]_\infty \quad \text{and} \quad B := \left[\begin{matrix} q^3,\ q^4 \\ q^2,\ q^5 \end{matrix} \,\middle|\, q^7\right]_\infty. \tag{H4.5}$$

Recall again the Euler pentagon number theorem:

$$(\rho;\rho)_\infty = \sum_{j=-\infty}^{+\infty} (-1)^j \rho^{\frac{j(3j+1)}{2}}.$$

It is easy to verify that the pentagon numbers admit only four residue classes modulo 7:

$$\frac{j(3j+1)}{2} \equiv_7 \begin{cases} 0, & j = 0, 2 \pmod 7 \\ 1, & j = 3, 6 \pmod 7 \\ 2, & j = 1 \pmod 7 \\ 5, & j = 4, 5 \pmod 7). \end{cases}$$

We can accordingly write

$$(\rho;\rho)_\infty = C_0 - \rho C_1 - \rho^2 C_2 + \rho^5 C_5. \tag{H4.6}$$

The coefficients C_0, C_1, C_2 and C_5 can be individually determined by means of Jacobi's triple and the quintuple product identities.

C_0-Coefficient: Specifying the summation index j with $7n$ and $7n+2$, we can compute C_0, by means of the quintuple product identity as follows:

$$\begin{aligned} C_0 &= \sum_{n=-\infty}^{+\infty} (-1)^n \{1+\rho^{7(1+6n)}\}\rho^{147\binom{n}{2}+77n} \\ &= [\rho^{49}, -\rho^{7}, -\rho^{42}; \rho^{49}]_\infty [\rho^{63}, \rho^{35}; \rho^{98}]_\infty \\ &= (q^7;q^7)_\infty \left[\begin{matrix} q^2,\ q^5 \\ q,\ q^6 \end{matrix} \,\Big|\, q^7\right]_\infty = A \times (q^7;q^7)_\infty. \end{aligned}$$

C_1-Coefficient: Similar to the computation of C_0, we can compute C_1, by specifying the summation index j with $7n-1$ and $7n+3$, as follows:

$$\begin{aligned} C_1 &= \sum_{n=-\infty}^{+\infty} (-1)^n \{1+\rho^{14(1+6n)}\}\rho^{147\binom{n}{2}+56n} \\ &= [\rho^{49}, -\rho^{14}, -\rho^{35}; \rho^{49}]_\infty [\rho^{77}, \rho^{21}; \rho^{98}]_\infty \\ &= (q^7;q^7)_\infty \left[\begin{matrix} q^3,\ q^4 \\ q^2,\ q^5 \end{matrix} \,\Big|\, q^7\right]_\infty = B \times (q^7;q^7)_\infty. \end{aligned}$$

C_2-Coefficient: It can be evaluated through the Jacobi triple product identity with $j = 1+7n$ as follows:

$$\begin{aligned} C_2 &= \sum_{n=-\infty}^{+\infty} (-1)^n \rho^{147\binom{n}{2}+98n} \\ &= [\rho^{147}, \rho^{49}, \rho^{98}; \rho^{147}]_\infty \\ &= [q^{21}, q^{7}, q^{14}; q^{21}]_\infty = (q^7;q^7)_\infty. \end{aligned}$$

C_5-Coefficient: Similar to the computation of C_0 and C_1, we can evaluate C_5, by specifying the summation index j with $-7n-2$ and $-7n-3$, as follows:

$$\begin{aligned} C_5 &= \sum_{n=-\infty}^{+\infty} (-1)^n \{1-\rho^{7(1+3n)}\}\rho^{147\binom{n}{2}+112n} \\ &= [\rho^{49}, -\rho^{21}, -\rho^{28}; \rho^{49}]_\infty [\rho^{91}, \rho^{7}; \rho^{98}]_\infty \\ &= (q^7;q^7)_\infty \left[\begin{matrix} q,\ q^6 \\ q^3,\ q^4 \end{matrix} \,\Big|\, q^7\right]_\infty = \frac{(q^7;q^7)_\infty}{AB}. \end{aligned}$$

In accordance with (H4.6), we find the following relation

$$\frac{(\rho;\rho)_\infty}{\rho^2 (q^7;q^7)_\infty} = A/\rho^2 - B/\rho - 1 + \rho^3/AB$$

which is equivalent to (H4.4):

$$\frac{\rho^2(q^7;q^7)_\infty}{(\rho;\rho)_\infty} = \frac{1}{A/\rho^2-B/\rho-1+\rho^3/AB}.$$

H4.3. Replacing $\rho\to\rho\varpi^\ell$ in (H4.4) and then summing both sides over $0\le\ell\le6$, we can express the generating function defined by (H4.3) as

$$\sum_{m=0}^{\infty}p(7m+5)q^m = \frac{1}{7q}\sum_{\ell=0}^{6}\frac{\rho^2\varpi^{2\ell}}{(\rho\varpi^\ell;\rho\varpi^\ell)_\infty}$$

$$= \frac{1}{7q(q^7;q^7)_\infty}\sum_{\ell=0}^{6}\frac{1}{A/\rho^2\varpi^{2\ell}-B/\rho\varpi^\ell-1+\rho^3\varpi^{3\ell}/AB}.$$

Observing that for each ℓ with $0\le\ell\le6$, the residues of $\{k+\ell\}_{k=1}^{6}$ modulo 7 are $\{0\le k\le6\}_{k\neq\ell}$, we can accordingly reformulate the sum as follows:

$$\sum_{\ell=0}^{6}\frac{1}{A/\rho^2\varpi^{2\ell}-B/\rho\varpi^\ell-1+\rho^3\varpi^{3\ell}/AB} \tag{H4.7a}$$

$$= \sum_{\ell=0}^{6}\prod_{\substack{k=0\\k\neq\ell}}^{6}\Big\{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB\Big\} \tag{H4.7b}$$

$$\div \prod_{k=0}^{6}\Big\{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB\Big\} \tag{H4.7c}$$

$$= \sum_{\ell=0}^{6}\prod_{k=1}^{6}\Big\{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB\Big\}\Big|_{\rho\to\rho\varpi^\ell} \tag{H4.7d}$$

$$\div \prod_{k=0}^{6}\Big\{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB\Big\}. \tag{H4.7e}$$

Let "nn" and "dd" stand for the sum and the product displayed in (H4.7d) and (H4.7e) respectively. We shall reduce these algebraic expressions and find a functional equation between them.

H4.4. As the Laurent polynomial in ρ, we can expand the product displayed in (H4.7d) as follows:

$$\prod_{k=1}^{6}\Big\{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB\Big\} = \sum_{\kappa=-12}^{18}U(\kappa)\,\rho^\kappa$$

where $\{U(\kappa)\}$ are constants independent of ρ. The sum displayed in (H4.7d) can be accordingly reduced to

$$\begin{aligned}
\text{nn} \quad &:= \quad \sum_{\ell=0}^{6}\prod_{k=1}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^{k} - 1 + \rho^3\varpi^{3k}/AB\Big\}\Big|_{\rho\to\rho\varpi^{\ell}} \\
&= \quad \sum_{\ell=0}^{6}\sum_{\kappa=-12}^{18} U(\kappa)\,\rho^{\kappa}\,\varpi^{\kappa\ell} \;=\; \sum_{\kappa=-12}^{18} U(\kappa)\,\rho^{\kappa}\sum_{\ell=0}^{6}\varpi^{\kappa\ell} \\
&= \quad 7\Big\{U(0) + qU(7) + q^{-1}U(-7) + q^2U(14)\Big\}.
\end{aligned}$$

Similarly, we expand the denominator "dd" as a Laurent polynomial in ρ:

$$\prod_{k=0}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^{k} - 1 + \rho^3\varpi^{3k}/AB\Big\} \;=\; \sum_{\kappa=-14}^{21} V(\kappa)\,\rho^{\kappa}.$$

Noting that the product is invariant under replacement $\rho \to \rho\varpi^{\ell}$ with $\ell \in \mathbb{Z}$, we can reduce the expression to following:

$$\begin{aligned}
\text{dd} := &\prod_{k=0}^{6}\Big\{A/\rho^2\varpi^{2k} - B/\rho\varpi^{k} - 1 + \rho^3\varpi^{3k}/AB\Big\} \\
= \;& V(0) + qV(7) + q^{-1}V(-7) + q^2V(14) + q^{-2}V(-14) + q^3V(21).
\end{aligned}$$

Analogously to the reasoning on the determination of the $W(0)$-coefficient in the proof of the generating function (H2.1), one can respectively compute (manually or by computer algebra) the coefficients for numerator

$$U(0) \quad = \quad 8 + 3\frac{B^2}{A} - 4\frac{A}{B^2} \tag{H4.8a}$$

$$U(7) \quad = \quad -\frac{3}{A^2B^3} - \frac{4}{A^3B} \tag{H4.8b}$$

$$U(-7) \quad = \quad AB^5 - \frac{A^4}{B} + 3A^3B + 4A^2B^3 \tag{H4.8c}$$

$$U(14) \quad = \quad \frac{1}{A^5B^4} \tag{H4.8d}$$

and the coefficients for denominator

$$\begin{aligned} V(0) &= -8+14\frac{A}{B^2} && \text{(H4.9a)}\\ V(7) &= \frac{14}{A^3B} && \text{(H4.9b)}\\ V(-7) &= 7\frac{A^4}{B}-14A^2B^3-7AB^5-B^7 && \text{(H4.9c)}\\ V(14) &= -\frac{7}{A^5B^4} && \text{(H4.9d)}\\ V(-14) &= A^7 && \text{(H4.9e)}\\ V(21) &= \frac{1}{A^7B^7}. && \text{(H4.9f)} \end{aligned}$$

They lead us to the polynomial expression for numerator

$$\begin{aligned} \text{nn} &= \sum_{\ell=0}^{6}\prod_{k=1}^{6}\Big\{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB\Big\}\Big|_{\rho\to\rho\varpi^\ell} && \text{(H4.10a)}\\ &= 7\Big\{8+3\frac{B^2}{A}-4\frac{A}{B^2}-3\frac{q}{A^2B^3}-4\frac{q}{A^3B} && \text{(H4.10b)}\\ &\quad +\frac{AB^5}{q}-\frac{A^4}{Bq}+3\frac{A^3B}{q}+4\frac{A^2B^3}{q}+\frac{q^2}{A^5B^4}\Big\} && \text{(H4.10c)} \end{aligned}$$

and the polynomial expression for denominator

$$\begin{aligned} \text{dd} &= \prod_{k=0}^{6}\Big\{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB\Big\} && \text{(H4.11a)}\\ &= -8+14\frac{A}{B^2}+14\frac{q}{A^3B}+7\frac{A^4}{Bq}-14\frac{A^2B^3}{q} && \text{(H4.11b)}\\ &\quad -7\frac{AB^5}{q}-\frac{B^7}{q}-7\frac{q^2}{A^5B^4}+\frac{A^7}{q^2}+\frac{q^3}{A^7B^7}. && \text{(H4.11c)} \end{aligned}$$

H4.5. In order to simplify the polynomial expressions for numerator "nn" and denominator "dd", we prove the following astonishing algebraic equation:

$$A^3B-A^2B^3 = q. \qquad \text{(H4.12)}$$

Recalling the definition of A and B in (H4.5), we can restate the equation as

$$\begin{aligned} &\langle q^2;q^7\rangle_\infty\,\langle q^2;q^7\rangle_\infty\,\langle q^2;q^7\rangle_\infty\,\langle q^4;q^7\rangle_\infty\\ -\;&\langle q;q^7\rangle_\infty\,\langle q^3;q^7\rangle_\infty\,\langle q^3;q^7\rangle_\infty\,\langle q^3;q^7\rangle_\infty\\ =\;&q\,\langle q^5;q^7\rangle_\infty\,\langle q;q^7\rangle_\infty\,\langle q;q^7\rangle_\infty\,\langle q;q^7\rangle_\infty \end{aligned}$$

which follows immediately from the q-difference equation stated in Theorem G5.2 under parameter specification $Q = q^7$, $b = q$, $c = d = e = q^3$ and $A = q^5$.

With the help of algebraic equation $q = A^3B - A^2B^3$, we can simplify further "nn" and "dd" as the following polynomial expressions

$$\text{nn} = \frac{7^2A}{Bq}\Big\{A^2B^2 - \frac{q}{B} + q\frac{B^3}{A^2}\Big\} \tag{H4.13a}$$

$$= 7^3 + \frac{7^2A}{Bq}\Big\{8AB^4 - 5A^2B^2 - A^3 - B^6\Big\} \tag{H4.13b}$$

$$\text{dd} = \frac{A^2}{B^2q^2}\Big\{8AB^4 - 5A^2B^2 - A^3 - B^6\Big\}^2. \tag{H4.13c}$$

H4.6. In order to determine generating function explicitly, we need an alternative expression for denominator "dd" in terms of infinite shifted factorial fraction.

Observing that the general term of the product with index n reads as

$$\prod_{k=0}^{6}\{1-(\rho\varpi^k)^n\} = \prod_{k=0}^{6}(1-\rho^n\varpi^{kn}) = \begin{cases}(1-\rho^n)^7, & n \equiv_7 0\\(1-\rho^{7n}), & n \not\equiv_7 0\end{cases}$$

we have therefore the following simplified product

$$\begin{aligned}\prod_{k=0}^{6}(\rho\varpi^k;\rho\varpi^k)_\infty &= \prod_{n=1}^{\infty}\prod_{k=0}^{6}\Big\{1-(\rho\varpi^k)^n\Big\}\\ &= \prod_{\substack{n=1\\ n\not\equiv_7 0}}^{\infty}(1-\rho^{7n})\prod_{n=1}^{\infty}(1-\rho^{7n})^7\\ &= \prod_{\substack{n=1\\ n\not\equiv_7 0}}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1-q^n)^7\end{aligned}$$

which can restated as the following identity:

$$\prod_{k=0}^{6}(\rho\varpi^k;\rho\varpi^k)_\infty = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty}. \tag{H4.14}$$

In view of (H4.4), this gives also another expression for the common denominator:

$$\begin{aligned}\frac{1}{\text{dd}} &= \prod_{k=0}^{6}\frac{1}{A/\rho^2\varpi^{2k}-B/\rho\varpi^k-1+\rho^3\varpi^{3k}/AB}\\ &= (q^7;q^7)_\infty^7\prod_{k=0}^{6}\frac{\rho^2\varpi^{2k}}{(\rho\varpi^k;\rho\varpi^k)_\infty} = q^2\frac{(q^7;q^7)_\infty^8}{(q;q)_\infty^8}.\end{aligned}$$

H4.7. By comparing (H4.13b) with (H4.13c), we find that

$$\sum_{\ell=0}^{6}\frac{1}{A/\rho^2\varpi^{2\ell}-B/\rho\varpi^\ell-1+\rho^3\varpi^{3\ell}/AB} \tag{H4.15a}$$

$$= \frac{\text{nn}}{\text{dd}} = \frac{7^3}{\text{dd}}+\frac{7^2}{\sqrt{\text{dd}}}. \tag{H4.15b}$$

Substituting the factorial expression for "dd" in the last fraction, we can finally determine the generating function

$$\begin{aligned}\sum_{m=0}^{\infty}p(7m+5)q^m &= \frac{1}{7q}\sum_{k=0}^{6}\frac{\rho^2\varpi^{2k}}{(\rho\varpi^k;\rho\varpi^k)_\infty} = \frac{1}{7q(q^7;q^7)_\infty}\frac{\text{nn}}{\text{dd}}\\ &= \frac{1}{7q(q^7;q^7)_\infty}\left\{\frac{7^3}{\text{dd}}+\frac{7^2}{\sqrt{\text{dd}}}\right\}\\ &= 7\frac{(q^7;\,q^7)_\infty^3}{(q;\,q)_\infty^4}+49q\frac{(q^7;\,q^7)_\infty^7}{(q;\,q)_\infty^8}.\end{aligned}$$

If we combine (H4.13a) with (H4.13c), we would get another expression of the generating function

$$\begin{aligned}\sum_{m=0}^{\infty}p(7m+5)q^m &= \frac{1}{7q}\sum_{k=0}^{6}\frac{\rho^2\varpi^{2k}}{(\rho\varpi^k;\rho\varpi^k)_\infty} = \frac{1}{7q(q^7;q^7)_\infty}\frac{\text{nn}}{\text{dd}}\\ &= 7\frac{(q^7;\,q^7)_\infty^7}{(q;\,q)_\infty^8}\left\{A^3B-\frac{qA}{B^2}+q\frac{B^2}{A}\right\}\end{aligned}$$

where A and B are shifted factorial fractions given by (H4.5).

Naturally, this form is less elegant that stated in (H4.1). However, it confirms again the Ramanujan congruence modulo 7.

H5. Proof of $p(11n+6) \equiv 0 \pmod{11}$

There holds the Ramanujan congruence modulo 11:

$$p(11n+6) \equiv 0 \pmod{11}. \tag{H5.1}$$

Recalling the congruence lemma on partition function, we should show that the coefficients of q^{11m} in the formal power series expansion of $q^5(q;\,q)_\infty^{10}$ are divisible by 11 for all $m \in \mathbb{N}$. The simplest proof of this congruence is due to Winquist (1969), which is based on the following formal power series expansion formula:

$$6q^5(q;q)_\infty^{10} = \sum_{i,j}(-1)^{i+j}(3i-3j-1)(3i+3j-2)^3 q^{3\binom{i}{2}+3\binom{j}{2}+j+5}. \tag{H5.2}$$

H5.1. If the q-exponent in the double sum is a multiple of 11, then we have the following congruence relation

$$\begin{aligned} 0 &\equiv_{11} 5+j+3\binom{i}{2}+3\binom{j}{2} \\ &\equiv_{11} 8\Big\{5+j+3\binom{i}{2}+3\binom{j}{2}\Big\} \\ &\equiv_{11} (i-6)^2+(j-2)^2. \end{aligned}$$

This can be reached only when

$$\begin{aligned} (i-6)^2 \equiv_{11} 0 &\implies i \equiv_{11} 6 \\ (j-2)^2 \equiv_{11} 0 &\implies j \equiv_{11} 2 \end{aligned}$$

in view of the following table on the quadratic residues modulo 11:

$k \pmod{11}$	0	±1	±2	±3	±4	±5
$k^2 \pmod{11}$	0	1	4	9	5	3

The coefficients corresponding to $i \equiv_{11} 6$ and $j \equiv_{11} 2$ are divisible by 11^4 because they contain two factors displayed in (H5.2):

$$\begin{aligned} 3i-3j-1 &\equiv_{11} 18-6-1 \equiv_{11} 0 \\ 3i+3j-2 &\equiv_{11} 18+6-2 \equiv_{11} 0. \end{aligned}$$

Therefore the coefficients of q^{11m} in the formal power series expansion of $q^5(q;q)_\infty^{10}$ are divisible by 11.

In order to complete the proof of congruence (H5.1), it remains to show the infinite series identity (H5.2).

H5.2. Define the bivariate function $F(x,y)$ by the following product of ten infinite shifted factorials:

$$F(x,y) := (q;q)_\infty^2 \langle x;q\rangle_\infty \langle y;q\rangle_\infty \langle xy;q\rangle_\infty \langle x/y;q\rangle_\infty. \tag{H5.3}$$

We can expand it formally as a Laurent series in x

$$F(x,y) = \sum_{k=-\infty}^{+\infty} \gamma_k(y)\, x^k.$$

It is trivial to check the functional equation

$$F(x,y) = -x^3\, F(qx,y)$$

which corresponds to the recurrence relation

$$\gamma_{k+3}(y) = -q^k\, \gamma_k(y).$$

Iterating this relation for k-times, we find that there exist three formal power series $A(y)$, $B(y)$ and $C(y)$ such that there hold

$$\begin{aligned}
\gamma_{3k}(y) &= -q^{3k-3}\,\gamma_{3k-3}(y) = (-1)^k q^{3\binom{k}{2}}\, A(y)\\
\gamma_{3k+1}(y) &= -q^{3k-2}\,\gamma_{3k-2}(y) = (-1)^k q^{3\binom{k}{2}+k}\, B(y)\\
\gamma_{3k+2}(y) &= -q^{3k-1}\,\gamma_{3k-1}(y) = (-1)^k q^{3\binom{k}{2}+2k}\, C(y).
\end{aligned}$$

Therefore $F(x,y)$ can be written as

$$\begin{aligned}
F(x,y) = {} & A(y)\sum_{k=-\infty}^{+\infty}(-1)^k q^{3\binom{k}{2}}\, x^{3k} && \text{(H5.4a)}\\
&+ B(y)\sum_{k=-\infty}^{+\infty}(-1)^k q^{3\binom{k}{2}+k}\, x^{3k+1} && \text{(H5.4b)}\\
&+ C(y)\sum_{k=-\infty}^{+\infty}(-1)^k q^{3\binom{k}{2}+2k}\, x^{3k+2}. && \text{(H5.4c)}
\end{aligned}$$

Again from the definition of $F(x,y)$, it is easy to verify another functional equation

$$F(x,y) = -x^3\, F(1/x,y)$$

which can be translated into the following

$$\begin{aligned} F(x,y) &= A(y)\sum_{k=-\infty}^{+\infty}(-1)^{k+1}q^{3\binom{k}{2}}\,x^{3-3k} \\ &+ B(y)\sum_{k=-\infty}^{+\infty}(-1)^{k+1}q^{3\binom{k}{2}+k}\,x^{2-3k} \\ &+ C(y)\sum_{k=-\infty}^{+\infty}(-1)^{k+1}q^{3\binom{k}{2}+2k}\,x^{1-3k}. \end{aligned}$$

The reversal of the bilateral series just displayed reads as

$$\begin{aligned} F(x,y) &= A(y)\sum_{k=-\infty}^{+\infty}(-1)^{k}q^{3\binom{k}{2}}\,x^{3k} && \text{(H5.5a)} \\ &+ B(y)\sum_{k=-\infty}^{+\infty}(-1)^{k+1}q^{3\binom{k}{2}+2k}\,x^{2+3k} && \text{(H5.5b)} \\ &+ C(y)\sum_{k=-\infty}^{+\infty}(-1)^{k+1}q^{3\binom{k}{2}+k}\,x^{1+3k}. && \text{(H5.5c)} \end{aligned}$$

Comparing both expansions (H5.4) and (H5.5) of $F(x,y)$, we find that $B(y) = -C(y)$. This allows us to restate $F(x,y)$ as follows:

$$\begin{aligned} F(x,y) &= A(y)\sum_{k=-\infty}^{+\infty}(-1)^{k}q^{3\binom{k}{2}}\,x^{3k} && \text{(H5.6a)} \\ &+ B(y)\sum_{k=-\infty}^{+\infty}(-1)^{k}\Big\{x^{1+3k}-x^{2-3k}\Big\}q^{3\binom{k}{2}+k} && \text{(H5.6b)} \end{aligned}$$

where the formal power series $A(y)$ and $B(y)$ remain to be determined.

H5.3. By means of the Jacobi triple product identity, the last expansion for $F(x,y)$ can be reformulated as

$$\begin{aligned} F(x,y) &= A(y)\,\big[q^3,\,x^3,\,q^3/x^3;q^3\big]_\infty \\ &+ x\,B(y)\,\big[q^3,\,qx^3,\,q^2/x^3;q^3\big]_\infty \\ &- x^2\,B(y)\,\big[q^3,\,q^2x^3,\,q/x^3;q^3\big]_\infty. \end{aligned}$$

Putting $x = q^{1/3}$ in the last equation and then recalling the definition of $F(x, y)$, we find that

$$\begin{aligned} A(y) + q^{1/3}\, B(y) &= \frac{F(q^{1/3}, y)}{(q;q)_\infty} = \Big[q^{1/3},\ y,\ q^{1/3}/y; q^{1/3}\Big]_\infty \\ &= \sum_{k=-\infty}^{+\infty} (-1)^k\, q^{\frac{1}{3}\binom{k}{2}}\, y^k. \end{aligned}$$

Based on the binomial congruences

$$\binom{k}{2} \equiv_3 \begin{cases} 0, & k \equiv_3 \ \ 0 \\ 0, & k \equiv_3 +1 \\ 1, & k \equiv_3 -1 \end{cases}$$

we can determine $A(y)$ and $B(y)$ respectively as follows:

$$\begin{aligned} A(y) &= \sum_{k=-\infty}^{+\infty} (-1)^k \Big\{ q^{\frac{1}{3}\binom{3k}{2}} y^{3k} - q^{\frac{1}{3}\binom{3k+1}{2}} y^{3k+1} \Big\} \\ &= \sum_{k=-\infty}^{+\infty} (-1)^k \Big\{ y^{3k} - y^{1-3k} \Big\} q^{3\binom{k}{2}+k} \\ B(y) &= -\sum_{k=-\infty}^{+\infty} (-1)^k q^{\frac{1}{3}\binom{3k-1}{2} - \frac{1}{3}} y^{3k-1} \\ &= -\sum_{k=-\infty}^{+\infty} (-1)^k q^{3\binom{k}{2}} y^{3k-1}. \end{aligned}$$

We therefore have the following bivariate formal power series expression

$$F(x, y) = \sum_{i=-\infty}^{+\infty} (-1)^i q^{3\binom{i}{2}} x^{3i} \sum_{j=-\infty}^{+\infty} (-1)^j \Big\{ y^{3j} - y^{1-3j} \Big\} q^{3\binom{j}{2}+j} \tag{H5.7a}$$

$$- \frac{x}{y} \sum_{i=-\infty}^{+\infty} (-1)^i q^{3\binom{i}{2}} y^{3i} \sum_{j=-\infty}^{+\infty} (-1)^j \Big\{ x^{3j} - x^{1-3j} \Big\} q^{3\binom{j}{2}+j}. \tag{H5.7b}$$

H5.4. Define further the bivariate function by formal power series

$$G(x, y) = \sum_{i=-\infty}^{+\infty} (-1)^i q^{3\binom{i}{2}}\, x^{3i} \sum_{j=-\infty}^{+\infty} (-1)^j \Big\{ y^{3j} - y^{1-3j} \Big\} q^{3\binom{j}{2}+j}. \tag{H5.8}$$

Then $F(x, y)$ can be expressed as a skew-symmetric function of x and y:

$$y\, F(x, y) = y\, G(x, y) - x\, G(y, x). \tag{H5.9}$$

Recalling the definition of $F(x, y)$, we have

$$\lim_{y \to x} \frac{yF(x, y)}{y - x} = (q;q)_\infty^4 \langle x; q\rangle_\infty^2 \langle x^2; q\rangle_\infty.$$

In view of the symmetric property, we write

$$yG(x,y) - xG(y,x) \;=\; \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j}\begin{Bmatrix} x^{3i}(y^{1+3j}-y^{2-3j}) \\ -y^{3i}(x^{1+3j}-x^{2-3j}) \end{Bmatrix}$$

which permits us to compute the corresponding limit:

$$\begin{aligned}\lim_{y\to x}\frac{yF(x,y)}{y-x} &= \lim_{y\to x}\frac{yG(x,y)-xG(y,x)}{y-x} \\ &= \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j}\lim_{y\to x}\frac{\begin{Bmatrix} x^{3i}(y^{1+3j}-y^{2-3j}) \\ -y^{3i}(x^{1+3j}-x^{2-3j}) \end{Bmatrix}}{y-x} \\ &= \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j}\begin{Bmatrix} (1+3j-3i)\;x^{3i+3j} \\ +(3i+3j-2)x^{1+3i-3j} \end{Bmatrix} \\ &= \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j}(1+3j-3i)\Big\{x^{3i+3j}-x^{4-3i-3j}\Big\}\end{aligned}$$

where the last line follows from the index involution $i\to 1-i$ on double sums.

Therefore we have established the following expansion formula:

$$\begin{aligned}(q;q)_\infty^4\langle x;q\rangle_\infty^2\langle x^2;q\rangle_\infty \;&=\; \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j} &\text{(H5.10a)} \\ &\times\;\; (1+3j-3i)\Big\{x^{3i+3j}-x^{4-3i-3j}\Big\}. &\text{(H5.10b)}\end{aligned}$$

Multiplying across by x^{-2}, we can rewrite (H5.10) as

$$\begin{aligned}\frac{(q;q)_\infty^4\langle x;q\rangle_\infty^2\langle x^2;q\rangle_\infty}{x^2} \;&=\; \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j} \\ &\times\;\; (1+3j-3i)\Big\{x^{3i+3j-2}-x^{2-3i-3j}\Big\}.\end{aligned}$$

Applying the derivative operator $\frac{x\partial}{\partial x}$ for three times at $x=1$, we find that

$$6(q;\,q)_\infty^{10} \;=\; \sum_{i,j}(-1)^{i+j}(3i-3j-1)(3i+3j-2)^3\,q^{3\binom{i}{2}+3\binom{j}{2}+j}.$$

This is exactly the formal power series expansion (H5.2), which has played the key role in the proof of congruence (H5.1).

H5.5. The crucial identity (H5.10) due to Winquist (1969) can alternatively be proved by means of the quintuple product identity.

$$\begin{aligned}(q;q)_\infty^4\langle x;q\rangle_\infty^2\langle x^2;q\rangle_\infty &= \sum_{i,j}(-1)^{i+j}q^{3\binom{i}{2}+3\binom{j}{2}+j}\\ &\times\ (1+3j-3i)\Big\{x^{3i+3j}-x^{4-3i-3j}\Big\}.\end{aligned}$$

The strategy is to simplify the double sum on the right hand side and then reduce it to the product form on the left hand side.

Performing the replacement on summation indices:

$$\left.\begin{aligned}j+i&=m\\ j-i&=n\end{aligned}\right\}\quad\rightleftharpoons\quad\left.\begin{aligned}i&=\frac{m-n}{2}\\ j&=\frac{m+n}{2}\end{aligned}\right\}\quad m\equiv_2 n$$

then we can reformulate the double sum as

$$\sum_{m\equiv_2 n}(-1)^m q^{3\binom{(m+n)/2}{2}+3\binom{(m-n)/2}{2}+\frac{m+n}{2}}(1+3n)\Big\{x^{3m}-x^{4-3m}\Big\}\qquad\text{(H5.11)}$$

where the double sum runs over $-\infty<m,\ n<+\infty$ with m and n having the same parity.

H5.6. Recall the quintuple product identities

$$\begin{aligned}[q,\,z,\,q/z;q]_\infty\big[qz^2,\,q/z^2;q^2\big]_\infty &= \sum_{k=-\infty}^{+\infty}\Big\{1-z^{1+6k}\Big\}\,q^{3\binom{k}{2}}\,\big(q^2/z^3\big)^k\\ &= \sum_{k=-\infty}^{+\infty}\Big\{1-(q/z^2)^{1+3k}\Big\}q^{3\binom{k}{2}}\big(qz^3\big)^k\end{aligned}$$

and their limiting forms:

$$\begin{aligned}\sum_{k=-\infty}^{+\infty}(1+6k)\,q^{3\binom{k}{2}+2k} &= [q,\,q,\,q;q]_\infty\,\big[q,\,q;q^2\big]_\infty\\ \sum_{k=-\infty}^{+\infty}(1+3k)q^{3\binom{k}{2}+\frac{5}{2}k} &= \Big[q,\,q^{1/2},q^{1/2};q\Big]_\infty\,\big[q^2,q^2;q^2\big]_\infty\,.\end{aligned}$$

We can evaluate the double sum (H5.11) with both m and n being even as

$$\begin{aligned}
&\sum_{m,\,n} q^{3\binom{m+n}{2}+3\binom{m-n}{2}+m+n}\left(1+6n\right)\left\{x^{6m}-x^{4-6m}\right\}\\
=\;& \sum_{m} q^{6\binom{m}{2}+m}\left\{x^{6m}-x^{4-6m}\right\}\sum_{n}\left(1+6n\right)q^{6\binom{n}{2}+4n}\\
=\;& (q^2;q^2)^3_\infty(q^2;q^4)^2_\infty\sum_{m} q^{6\binom{m}{2}}\left\{1-x^{4(1+3m)}\right\}\left(q^5/x^6\right)^m\\
=\;& (q^2;q^2)^3_\infty(q^2;q^4)^2_\infty\left[q^2,qx^2,q/x^2;q^2\right]_\infty\left[q^4/x^4,x^4;q^4\right]_\infty.
\end{aligned}$$

The double sum (H5.11) with both m and n being odd can be reduced similarly to the product:

$$\begin{aligned}
&\sum_{m,\,n} q^{3\binom{m+n+1}{2}+3\binom{m-n}{2}+m+n+1}\left(4+6n\right)\left\{x^{1-6m}-x^{3+6m}\right\}\\
=\;& \sum_{m} q^{6\binom{m}{2}+4m}\left\{x^{3+6m}-x^{1-6m}\right\}\sum_{n}\left(2+6n\right)q^{6\binom{n}{2}+5n}\\
=\;& -2x(q;q^2)^2_\infty(q^2;q^2)_\infty(q^4;q^4)^2_\infty\sum_{m} q^{6\binom{m}{2}}\left\{1-x^{2(1+6m)}\right\}\left(q^4/x^6\right)^m\\
=\;& -2x(q;q^2)^2_\infty(q^2;q^2)_\infty(q^4;q^4)^2_\infty\left[q^2,x^2,q^2/x^2;q^2\right]_\infty\left[q^2x^4,q^2/x^4;q^4\right]_\infty.
\end{aligned}$$

Their sum leads the identity (H5.10) equivalently to the following equation:

$$\begin{aligned}
&(q;q)^4_\infty\langle x;q\rangle^2_\infty\langle x^2;q\rangle_\infty\\
=\;& (q^2;q^2)^3_\infty(q^2;q^4)^2_\infty\left[q^2,qx^2,q/x^2;q^2\right]_\infty\left[q^4/x^4,x^4;q^4\right]_\infty\\
-\;& 2x(q;q^2)^2_\infty(q^2;q^2)_\infty(q^4;q^4)^2_\infty\left[q^2,x^2,q^2/x^2;q^2\right]_\infty\left[q^2x^4,q^2/x^4;q^4\right]_\infty.
\end{aligned}$$

We can reduce it by canceling the common factors to the following equivalent q-difference equation:

$$[q,\ x,\ q/x;q]^2_\infty=(q^2;q^2)^2_\infty\left\{\begin{matrix}\left[-q,\ -q,\ -x^2,\ -q^2/x^2;\ q^2\right]_\infty\\ -2x\left[-q^2,-q^2,-qx^2,-q/x^2;q^2\right]_\infty\end{matrix}\right\}\tag{H5.12}$$

whose terms can be reorganized, for convenience, as follows:

$$\begin{aligned}
&\left[-q,-q,-x^2,-q^2/x^2;q^2\right]_\infty-(q;q^2)^2_\infty\left[x,\ q/x;q\right]^2_\infty\\
=\;& x\left[-1,-q^2,-qx^2,-q/x^2;q^2\right]_\infty.
\end{aligned}$$

Rewriting the last identity as

$$\begin{aligned}
&\langle x\sqrt{-1};q\rangle_\infty\ \langle -x\sqrt{-1};q\rangle_\infty\ \langle q^{1/2}\sqrt{-1};q\rangle_\infty\ \langle q^{1/2}\sqrt{-1};q\rangle_\infty && \text{(H5.13a)}\\
-\;& \langle x;q\rangle_\infty\ \langle x;q\rangle_\infty\ \langle q^{1/2};q\rangle_\infty\ \langle -q^{1/2};q\rangle_\infty && \text{(H5.13b)}\\
=\;& x\ \langle\sqrt{-1};q\rangle_\infty\ \langle -\sqrt{-1};q\rangle_\infty\ \langle q^{1/2}x\sqrt{-1};q\rangle_\infty\ \langle q^{1/2}\sqrt{-1}/x;q\rangle_\infty && \text{(H5.13c)}
\end{aligned}$$

we can see without difficulty that it is the special case $b=c=x$, $d=q^{1/2}$, $e=-q^{1/2}$ and $A=q^{1/2}x\sqrt{-1}$ of the identity stated in Theorem G5.2.

This completes the proof of (H5.10). □

H5.7. The identity (H5.12) can also be proved directly.

In fact, by means of the Jacobi triple product identity, its right hand side can be expanded as

$$\text{RHS(H5.12)} \;=\; \sum_{i,\,j} q^{i^2+j^2}\Big\{q^{-j}x^{2j} - q^{-i}x^{1+2j}\Big\}.$$

Interchanging two summation indices i and j for the first part and then letting $k := j - i$, we can manipulate the last double sum as follows:

$$\begin{aligned}
\text{RHS(H5.12)} \;&=\; \sum_{i,\,j} q^{i^2+j^2-i}\Big\{x^{2i} - x^{1+2j}\Big\}\\
&=\; \sum_{k} q^{k^2}\Big\{1 - x^{1+2k}\Big\}\sum_{i} q^{4\binom{i}{2}+(1+2k)i}x^{2i}\\
&=\; \sum_{k} q^{k^2}\Big\{1 - x^{1+2k}\Big\}\left[q^4, -q^{1+2k}x^2, -q^{3-2k}/x^2; q^4\right]_\infty .
\end{aligned}$$

The last triple product can be restated as

$$\begin{aligned}
&\left[q^4, -q^{1+2k}x^2, -q^{3-2k}/x^2; q^4\right]_\infty\\
=\;&\begin{cases} x^{-2\ell}q^{-4\binom{\ell}{2}-\ell}\left[q^4, -qx^2, -q^3/x^2; q^4\right]_\infty, & k = 2\ell;\\ x^{-2\ell}q^{-4\binom{\ell}{2}-3\ell}\left[q^4, -q^3x^2, -q/x^2; q^4\right]_\infty, & k = 2\ell+1.\end{cases}
\end{aligned}$$

Now reformulating the k-sum according to the parity of k, we can express it as a combination of two infinite series:

$$\begin{aligned}
\text{RHS(H5.12)} \;&=\; \left[q^4, -qx^2, -q^3/x^2; q^4\right]_\infty \sum_{\ell} q^{4\binom{\ell}{2}+3\ell}x^{-2\ell}\Big\{1 - x^{1+4\ell}\Big\}\\
&+\; q\left[q^4, -q^3x^2, -q/x^2; q^4\right]_\infty \sum_{\ell} q^{4\binom{\ell}{2}+5\ell}x^{-2\ell}\Big\{1 - x^{3+4\ell}\Big\}.
\end{aligned}$$

By feeding back the parity of k, we can evaluate the first ℓ-sum as follows:

$$\begin{aligned}
&\sum_{\ell} q^{4\binom{\ell}{2}+3\ell}x^{-2\ell}\Big\{1 - x^{1+4\ell}\Big\}\\
=\;& \sum_{\ell}\Big\{q^{2\ell^2-\ell}x^{2\ell} - q^{2\ell^2+\ell}x^{1+2\ell}\Big\}\\
=\;& \sum_{k}(-1)^k\, q^{\binom{k}{2}}\, x^k \;=\; [q, x, q/x; q]_\infty .
\end{aligned}$$

The second ℓ-sum can be reduced similarly as follows:

$$\begin{aligned}
&\sum_{\ell} q^{4\binom{\ell}{2}+5\ell} x^{-2\ell}\Big\{1 - x^{3+4\ell}\Big\} \\
=\;& \sum_{\ell} \Big\{q^{2\ell^2-3\ell} x^{2\ell} - q^{2\ell^2+3\ell} x^{3+2\ell}\Big\} \\
=\;& \sum_{k} (-1)^k\, q^{\binom{k}{2}} \,\big(x/q\big)^k \;=\; -x/q\,[q, x, q/x; q]_\infty\,.
\end{aligned}$$

Combining these expressions, we arrive at the final assault

$$\begin{aligned}
\text{RHS(H5.12)} &= [q, x, q/x; q]_\infty \begin{Bmatrix} \big[q^4, -qx^2, -q^3/x^2; q^4\big]_\infty \\ -x\,\big[q^4, -q^3x^2, -q/x^2; q^4\big]_\infty \end{Bmatrix} \\
&= [q, x, q/x; q]_\infty \sum_{\ell} \Big\{q^{2\ell^2-\ell} x^{2\ell} - q^{2\ell^2+\ell} x^{1+2\ell}\Big\} \\
&= [q, x, q/x; q]_\infty \sum_{k} (-1)^k\, q^{\binom{k}{2}}\, x^k \\
&= [q, x, q/x; q]^2_\infty \;=\; \text{LHS(H5.12)}.
\end{aligned}$$

This completes the proof of (H5.12).