CHAPTER D

The Carlitz Inversions and Rogers-Ramanujan Identities

According to the Jacobi triple product identity, we have

$$\big[q^4, \pm q, \pm q^3;\; q^4\big] \quad = \quad \sum_{k=-\infty}^{+\infty} (\mp 1)^k q^{2k^2+k}.$$

The sum of both triple products can be evaluated as a single triple product:

$$\begin{aligned} & \big[q^4,\, -q,\, -q^3;\; q^4\big]_\infty + \big[q^4,\, q,\, q^3;\; q^4\big]_\infty \\ = \;\; & 2\sum_{n=-\infty}^{+\infty} q^{8n^2+2n} \;=\; 2\sum_{n=-\infty}^{+\infty} q^{16\binom{n}{2}+10n} \\ = \;\; & 2\big[q^{16}, -q^6, -q^{10};\; q^{16}\big]_\infty. \end{aligned}$$

We can similarly treat their difference as follows:

$$\begin{aligned} & \big[q^4,\, -q,\, -q^3;\; q^4\big]_\infty - \big[q^4,\, q,\, q^3;\; q^4\big]_\infty \\ = \;\; & 2\sum_{n=-\infty}^{+\infty} q^{8n^2-6n+1} \;=\; 2\sum_{n=-\infty}^{+\infty} q^{16\binom{n}{2}+2n+1} \\ = \;\; & 2q\big[q^{16}, -q^2, -q^{14};\; q^{16}\big]_\infty. \end{aligned}$$

Dividing both equations by $(q^4; q^4)_\infty$ and noting the fact that the odd natural numbers are congruent to 1 or to 3 modulo 4, we get two q-difference equations:

$$(-q;\, q^2)_\infty \;+\; (q;\, q^2)_\infty = \frac{2}{(q^4;\, q^4)_\infty} \sum_n q^{8n^2+2n} \tag{D0.1a}$$

$$= 2\frac{\big[q^{16}, -q^6, -q^{10};\; q^{16}\big]}{(q^4;\, q^4)_\infty} \tag{D0.1b}$$

$$(-q;\, q^2)_\infty \;-\; (q;\, q^2)_\infty = \frac{2q}{(q^4;\, q^4)_\infty} \sum_n q^{8n^2-6n} \tag{D0.2a}$$

$$= 2q\frac{\big[q^{16}, -q^2, -q^{14};\; q^{16}\big]}{(q^4;\, q^4)_\infty}. \tag{D0.2b}$$

Further, if we specify with $x \mapsto \pm q^{1/2}$ in Euler's q-difference formula

$$(x;\, q)_\infty \quad = \quad \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(q;\, q)_m} q^{\binom{m}{2}}$$

then we find that

$$(\pm q^{1/2};\, q)_\infty \quad = \quad \sum_{m=0}^{\infty} \frac{(\mp 1)^m q^{m^2/2}}{(q;\, q)_m}$$

whose linear combinations lead us to two summation formulae as follows:

$$(-q^{1/2};\, q)_\infty + (q^{1/2};\, q)_\infty = 2 \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;\, q)_{2n}} \tag{D0.3a}$$

$$(-q^{1/2};\, q)_\infty - (q^{1/2};\, q)_\infty = 2q^{1/2} \sum_{n=0}^{\infty} \frac{q^{2n(2n+1)}}{(q;\, q)_{2n+1}}. \tag{D0.3b}$$

Replacing the base q by $q^{1/2}$ in (D0.1a-D0.1b) and (D0.2a-D0.2b), we can reformulate the left hand sides of both equations just displayed respectively as follows:

$$(-q^{1/2};\, q)_\infty + (q^{1/2};\, q)_\infty = 2 \frac{\left[q^8, -q^3, -q^5;\, q^8\right]}{(q^2;\, q^2)_\infty} \tag{D0.4a}$$

$$(-q^{1/2};\, q)_\infty - (q^{1/2};\, q)_\infty = 2q^{1/2} \frac{\left[q^8, -q, -q^7;\, q^8\right]}{(q^2;\, q^2)_\infty}. \tag{D0.4b}$$

Combining (D0.3a) and (D0.3b) respectively with (D0.4a) and (D0.4b), we establish two infinite series identities:

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;\, q)_{2n}} \quad = \quad \frac{\left[q^8, -q^3, -q^5;\, q^8\right]}{(q^2;\, q^2)_\infty} \tag{D0.5a}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;\, q)_{2n+1}} \quad = \quad \frac{\left[q^8, -q, -q^7;\, q^8\right]}{(q^2;\, q^2)_\infty}. \tag{D0.5b}$$

They are only very simple examples of classical partition identities of Roger-Ramanujan's type. By means of inverse series relations, we establish a finite series transformation, which leads us to an elementary derivation to the celebrated Rogers-Ramanujan identities and their finite forms.

D1. Combinatorial inversions and series transformations

D1.1. The Carlitz inversions. Let $\{a_i\}$ and $\{b_j\}$ be two complex sequences such that the polynomials defined by

$$\phi(x;\,0) = 1 \quad \text{and} \quad \phi(x;\,n) = \prod_{k=0}^{n-1}(a_k + xb_k), \quad \text{for } n = 1, 2, \cdots$$

differ from zero for $x = q^n$ with n being non-negative integers. Then we have the following inverse series relations due to Carlitz (1973)

$$\begin{cases} F(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \phi(q^k; n)\, G(k), & n = 0, 1, 2, \cdots \quad \text{(D1.1a)} \\ G(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \dfrac{a_k + q^k b_k}{\phi(q^n; k+1)}\, F(k), & n = 0, 1, 2, \cdots \quad \text{(D1.1b)} \end{cases}$$

which may be considered as q-analogue of Gould-Hsu Inversions (1973).

Proof. To prove the bilateral implications (D1.1a) $\rightleftharpoons$ (D1.1b), it is sufficient to verify one implication because one system of equations with $F(n)$ in terms of $G(k)$ can be considered as the (unique) solution of another system with $G(n)$ in terms of $F(k)$, and vice versa.

$\Longleftarrow$ We first reproduce the original proof due to Carlitz. Suppose that the relations of $G(n)$ in terms of $F(k)$ are valid. We have to verify the relations of $F(n)$ in terms of $G(k)$.

Substituting the relations of $G(n)$ in terms of $F(k)$ into the right hand sides of those of $F(n)$ in terms of $G(k)$ and observing that

$$\begin{bmatrix} n \\ k \end{bmatrix} \times \begin{bmatrix} k \\ i \end{bmatrix} = \begin{bmatrix} n \\ i \end{bmatrix} \times \begin{bmatrix} n-i \\ k-i \end{bmatrix}$$

we get the double sum

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n\\ k\end{bmatrix} \phi(q^k;n)\, G(k)\\
=\;& \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}} \phi(q^k;n) \sum_{i=0}^{k}(-1)^i \begin{bmatrix} k\\ i\end{bmatrix} \frac{a_i+q^i b_i}{\phi(q^k;i+1)} F(i)\\
=\;& \sum_{i=0}^{n}(a_i+q^i b_i) \begin{bmatrix} n\\ i\end{bmatrix} F(i) \sum_{k=i}^{n}(-1)^{k+i} \begin{bmatrix} n-i\\ k-i\end{bmatrix} \frac{\phi(q^k;n)}{\phi(q^k;i+1)} q^{\binom{n-k}{2}}\\
=\;& \sum_{i=0}^{n}(a_i+q^i b_i) \begin{bmatrix} n\\ i\end{bmatrix} F(i) \sum_{\ell=0}^{n-i}(-1)^{\ell} \begin{bmatrix} n-i\\ \ell\end{bmatrix} \frac{\phi(q^{i+\ell};n)}{\phi(q^{i+\ell};i+1)} q^{\binom{n-i-\ell}{2}}.
\end{aligned}$$

Let $S(i,n)$ stand for the inner sum with respect to ℓ:

$$S(i,n) := \sum_{\ell=0}^{n-i}(-1)^{\ell} \begin{bmatrix} n-i\\ \ell\end{bmatrix} q^{\binom{n-i-\ell}{2}} \frac{\phi(q^{i+\ell};n)}{\phi(q^{i+\ell};i+1)}.$$

It is trivial to see that

$$S(n,n) = \frac{\phi(q^n;n)}{\phi(q^n;n+1)} = \frac{1}{a_n+q^n b_n}$$

which implies that the double sum reduces to $F(n)$ when $i=n$.

In order to prove that the double sum is equal to $F(n)$, it suffices for us to verify that $S(i,n)=0$ for $0\le i<n$.

Noting that $\frac{\phi(q^{i+\ell};n)}{\phi(q^{i+\ell};i+1)}$ is a polynomial of degree $n-i-1$ in q^ℓ, we can write it formally as

$$\frac{\phi(q^{i+\ell};n)}{\phi(q^{i+\ell};i+1)} = \sum_{j=0}^{n-i-1} C_j\, q^{\ell(n-i-j-1)}$$

where $\{C_j\}$ are constants independent of ℓ. Therefore the sum $S(i,n)$ can be reformulated accordingly as follows:

$$\begin{aligned}
S(i,n) &= \sum_{\ell=0}^{n-i}(-1)^{\ell} \begin{bmatrix} n-i\\ \ell\end{bmatrix} q^{\binom{n-i-\ell}{2}} \sum_{j=0}^{n-i-1} C_j\, q^{\ell(n-i-j-1)}\\
&= \sum_{j=0}^{n-i-1} C_j\, q^{\binom{n-i}{2}} \sum_{\ell=0}^{n-i}(-1)^{\ell} q^{\binom{\ell}{2}} \begin{bmatrix} n-i\\ \ell\end{bmatrix} q^{-\ell j}
\end{aligned}$$

where we have applied the binomial relation

$$\binom{n-i-\ell}{2} = \binom{n-i}{2} + \binom{\ell}{2} - \ell(n-i-1).$$

Evaluating the sum with respect to ℓ by Euler's q-difference formula (B5.3)

$$\sum_{\ell=0}^{n-i}(-1)^\ell \begin{bmatrix} n-i \\ \ell \end{bmatrix} q^{\binom{\ell}{2}-\ell j} = (q^{-j};q)_{n-i}$$

which vanishes for $0 \leq j < n-i$.

This completes the proof of the Carlitz inversions stated in D1.1. □

$\Longrightarrow$ An alternative proof is worth to be included. Assuming that (D1.1a) is true for all $n \in \mathbb{N}_0$, we should verify the truth of (D1.1b).

In fact, substituting the first relation into the second, we reduce the question to the confirmation of the following orthogonal relation:

$$\sum_{k=i}^{n}(-1)^{k+i}\{a_k + q^k b_k\} \begin{bmatrix} n-i \\ k-i \end{bmatrix} \frac{\phi(q^i;k)}{\phi(q^n;k+1)} q^{\binom{k-i}{2}} = \begin{cases} 1, & i=n \\ 0, & i \neq n. \end{cases} \quad \text{(D1.2)}$$

It is obvious that the relation is valid for $i=n$. We therefore need to verify it only when $i<n$. For that purpose, we introduce the sequence

$$\tau_k := \begin{bmatrix} n-i-1 \\ k-i-1 \end{bmatrix} \frac{\phi(q^i;k)}{\phi(q^n;k)} q^{\binom{k-i}{2}}.$$

Then it is not hard to check that the summand in (D1.2) can be expressed as follows:

$$\tau_k + \tau_{k+1} = \{a_k + q^k b_k\} \begin{bmatrix} n-i \\ k-i \end{bmatrix} \frac{\phi(q^i;k)}{\phi(q^n;k+1)} q^{\binom{k-i}{2}}.$$

Separating the two extreme terms indexed with $k=i$ and $k=n$ from the sum displayed in (D1.2)

$$\tau_{i+1} = \frac{\phi(q^i;i+1)}{\phi(q^n;i+1)}$$

$$\tau_n = \frac{\phi(q^i;n)}{\phi(q^n;n)} q^{\binom{n-i}{2}}$$

and then appealing for the telescoping method, we find that

$$\begin{aligned} \text{LHS(D1.2)} &= \tau_{i+1} + (-1)^{n+i}\tau_n + \sum_{i<k<n} (-1)^{k+i}\{\tau_k + \tau_{k+1}\} \\ &= \{\tau_{i+1} + (-1)^{n+i}\tau_n\} - \{\tau_{i+1} + (-1)^{n+i}\tau_n\} = 0. \end{aligned}$$

This completes the proof of (D1.2). □

D1.2. Series transformation. For the polynomials $\phi(x;n) = (\lambda x;q)_n$ specified with $a_k = 1$ and $b_k = -q^k\lambda$, the inverse series relations displayed in D1.1 become the following:

$$f(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(q^k\lambda;q)_n\, g(k), \quad n = 0,1,2,\cdots \tag{D1.3a}$$

$$g(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(q^n\lambda;\ q)_{k+1}}\ f(k), \quad n = 0,1,2,\cdots. \tag{D1.3b}$$

By means of the finite version of Kummer's theorem and rearrangement of double sums, we may establish finite and infinite series transformations

$$\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} \frac{\lambda^n q^{n^2}}{(\lambda;q)_n}\, g(n) = \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(\lambda;q)_{m+k+1}} \lambda^k q^{k^2} f(k) \tag{D1.4a}$$

$$\sum_{n=0}^{\infty} \frac{\lambda^n q^{n^2}}{(q;q)_n(\lambda;q)_n} g(n) = \sum_{k=0}^{\infty}\ (-1)^k\ \frac{1-q^{2k}\lambda}{(\lambda;q)_\infty}\ \frac{\lambda^k q^{k^2}}{(q;q)_k}\ f(k). \tag{D1.4b}$$

Proof. By means of (D1.3b), we can express the left member of (D1.4a) as the following double sum

$$\begin{aligned}
\text{LHS(D1.4a)} &= \sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} \frac{\lambda^n q^{n^2}}{(\lambda;q)_n} \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(q^n\lambda;\ q)_{k+1}} f(k) \\
&= \sum_{k=0}^{m}(-1)^k(1-q^{2k}\lambda) \begin{bmatrix} m \\ k \end{bmatrix} f(k) \sum_{n=k}^{m} \begin{bmatrix} m-k \\ n-k \end{bmatrix} \frac{\lambda^n q^{n^2}}{(\lambda;q)_{n+k+1}} \\
&= \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(\lambda;q)_{2k+1}} \lambda^k q^{k^2} f(k) \sum_{j=0}^{m-k} \begin{bmatrix} m-k \\ j \end{bmatrix} \frac{\lambda^j q^{j(j+2k)}}{(q^{2k+1}\lambda;q)_j}
\end{aligned}$$

where we have applied relations on shifted factorials

$$(\lambda;q)_{n+k+1} = (\lambda;q)_n(q^n\lambda;q)_{k+1} = (\lambda;q)_{2k+1}(q^{2k+1}\lambda;q)_{n-k} \tag{D1.5}$$

and the substitution $j := n-k$ on summation indices.

In view of the finite version of Kummer's theorem stated in Corollary C1.2

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^k\, q^{k^2}}{(qx;\ q)_k} = \frac{1}{(qx;\ q)_n}$$

we can evaluate the inner sum as the following closed form:

$$\sum_{j=0}^{m-k} \begin{bmatrix} m-k \\ j \end{bmatrix} \frac{\lambda^j q^{j(j+2k)}}{(q^{2k+1}\lambda;q)_j} = \frac{1}{(q^{2k+1}\lambda;q)_{m-k}}.$$

Recalling (D1.5), we derive finally the following

$$\begin{aligned}\text{LHS(D1.4a)} &= \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(\lambda;q)_{2k+1}} \frac{\lambda^k q^{k^2}}{(q^{2k+1}\lambda;q)_{m-k}} f(k) \\ &= \sum_{k=0}^{m}(-1)^k \{1-q^{2k}\lambda\} \begin{bmatrix} m \\ k \end{bmatrix} \frac{\lambda^k q^{k^2}}{(\lambda;q)_{m+k+1}} f(k)\end{aligned}$$

which is the first identity (D1.4a).

The second identity (D1.4b) follows from the limit $m \to \infty$ of (D1.4a). □

D2. Finite q-differences and further transformation

On account of the inverse series relations

$$(x;q)_n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k \qquad \text{(D2.1a)}$$

$$q^{\binom{n}{2}} x^n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (x;q)_k \qquad \text{(D2.1b)}$$

we may determine, as an example of (D1.3a-D1.3b), two sequences as follows:

$$f(n) = \lambda^n q^{n^2+\binom{n}{2}} (\lambda;\ q)_n \quad \rightleftharpoons \quad g(n) = (\lambda;\ q)_n.$$

They may be used to reformulate the finite series transformation (D1.4a) explicitly

$$\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} \lambda^n q^{n^2} = \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(q^k\lambda;\ q)_{m+1}} \lambda^{2k} q^{2k^2+\binom{k}{2}}. \qquad \text{(D2.2)}$$

Proof. The first relation (D2.1a) is a restatement of Euler's q-finite difference formula (B5.3). Specifying the Carlitz inversions stated in D1.1 with

$$\phi(x;n) = 1, \quad f(n) = x^n q^{\binom{n}{2}}, \quad g(n) = (x;q)_n$$

we get the second relation (D2.1b) which is dual to the first one.

In order to verify that two sequences

$$f(n) = \lambda^n q^{n^2+\binom{n}{2}} (\lambda;\ q)_n \quad \rightleftharpoons \quad g(n) = (\lambda;\ q)_n$$

satisfy (D1.3a-D1.3b), it is sufficient to show that

$$\begin{aligned}\lambda^n q^{n^2+\binom{n}{2}}(\lambda;\ q)_n &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(\lambda;\ q)_k(q^k\lambda; q)_n \\ &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(\lambda; q)_{n+k}\end{aligned}$$

in view of the inverse series relations specified with $\phi(x;n) = (\lambda x; q)_n$.

Applying (D2.1b) with $x = q^n\lambda$, we confirm the last summation identity:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(\lambda; q)_{n+k} =&(\lambda; q)_n \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(q^n\lambda; q)_k \\ =&(\lambda; q)_n q^{n^2+\binom{n}{2}}\lambda^n.\end{aligned}$$

The transformation (D2.2) follows from (D1.4a) with the $\{f(k), g(n)\}$ sequences just displayed explicitly. □

D3. Rogers-Ramanujan identities and their finite forms

D3.1. Proposition. With the specifications $\lambda \mapsto 1$ and $\lambda \mapsto q$ in (D2.2), the finite forms of Rogers-Ramanujan identities can be derived as follows:

$$\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} q^{n^2} = \frac{(q;q)_m}{(q;q)_{2m}} \sum_{k=-m}^{m} (-1)^k \begin{bmatrix} 2m \\ m+k \end{bmatrix} q^{\binom{k}{2}+2k^2} \quad \text{(D3.1a)}$$

$$\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} q^{n^2+n} = \frac{(q;q)_m}{(q;q)_{2m+1}} \sum_{k=-m}^{m+1} (-1)^k \begin{bmatrix} 2m+1 \\ m+k \end{bmatrix} q^{\binom{k}{2}+2k^2-k}. \quad \text{(D3.1b)}$$

Proof. Separating the first term from (D2.2), we have

$$\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} \lambda^n q^{n^2} = \frac{1-\lambda}{(\lambda;q)_{m+1}} + \sum_{k=1}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k}\lambda}{(q^k\lambda;\ q)_{m+1}} \lambda^{2k} q^{2k^2+\binom{k}{2}}.$$

Its limiting case $\lambda \to 1$ may be manipulated as follows:

$$\begin{aligned}\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} q^{n^2} &= \frac{1}{(q;q)_m} + \sum_{k=1}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k}}{(q^k;\, q)_{m+1}} q^{2k^2+\binom{k}{2}} \\ &= \frac{1}{(q;q)_m} + \sum_{k=1}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1+q^k}{(q^{k+1};\, q)_m} q^{2k^2+\binom{k}{2}}.\end{aligned}$$

In view of the definition of q-Gauss binomial coefficient and the relation

$$(q;\, q)_{m+k} = (q;\, q)_k (q^{k+1};\, q)_m$$

we can further reformulate the sum as

$$\begin{aligned}\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} q^{n^2} &= \frac{1}{(q;q)_m} + \sum_{k=1}^{m} (-1)^k \frac{(q;\, q)_m}{(q;\, q)_{m-k}} \frac{1+q^k}{(q;\, q)_{m+k}} q^{2k^2+\binom{k}{2}} \\ &= \frac{1}{(q;q)_m} + \frac{(q;\, q)_m}{(q;\, q)_{2m}} \sum_{k=1}^{m} (-1)^k \begin{bmatrix} 2m \\ m+k \end{bmatrix} q^{2k^2+\binom{k}{2}} \\ &+ \frac{(q;\, q)_m}{(q;\, q)_{2m}} \sum_{k=1}^{m} (-1)^k \begin{bmatrix} 2m \\ m+k \end{bmatrix} q^{2k^2+\binom{k+1}{2}}.\end{aligned}$$

Performing the replacement $k \to -k$ in the last sum and noting that

$$\begin{bmatrix} 2m \\ m-k \end{bmatrix} = \begin{bmatrix} 2m \\ m+k \end{bmatrix}$$

we can combine the last three expressions as a single one:

$$\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} q^{n^2} = \sum_{k=-m}^{m} (-1)^k \begin{bmatrix} 2m \\ m+k \end{bmatrix} q^{2k^2+\binom{k}{2}}$$

which is the finite form of the first Rogers-Ramanujan identity (D3.1a).

Similarly, specifying (D2.2) with $\lambda \to q$, we have

$$\begin{aligned}\sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} q^{n+n^2} &= \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{1-q^{2k+1}}{(q^{k+1};\, q)_{m+1}} q^{2k^2+\binom{k}{2}+2k} \\ &= \frac{(q;q)_m}{(q;q)_{2m+1}} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} 2m+1 \\ m-k \end{bmatrix} (1-q^{2k+1})\, q^{2k^2+\binom{k}{2}+2k} \\ &= \frac{(q;q)_m}{(q;q)_{2m+1}} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} 2m+1 \\ m-k \end{bmatrix} q^{2k^2+\binom{k}{2}+2k} \\ &- \frac{(q;q)_m}{(q;q)_{2m+1}} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} 2m+1 \\ m-k \end{bmatrix} q^{2k^2+\binom{k}{2}+4k+1}.\end{aligned}$$

Replacing the summation index k by $-1-k$ in the second sum and then combining the result with the first one, we get the following simplified transformation

$$\sum_{n=0}^{m}\begin{bmatrix}m\\n\end{bmatrix}q^{n+n^2} \;=\; \frac{(q;q)_m}{(q;q)_{2m+1}}\sum_{k=-m-1}^{m}(-1)^k\begin{bmatrix}2m+1\\m-k\end{bmatrix}q^{2k^2+\binom{k}{2}+2k}$$

which is equivalent to the second finite form (D3.1b) of Rogers-Ramanujan identities under parameter replacement $k\to -k$. □

D3.2. Theorem. Their limiting cases give rise, with the help of the Jacobi-triple product identity, to the celebrated Rogers-Ramanujan identities:

$$\frac{1}{(q;\,q^5)_\infty\,(q^4;q^5)_\infty}=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;\,q)_n}\;=\prod_{k=0}^{\infty}\frac{1}{(1-q^{1+5k})(1-q^{4+5k})}\qquad\text{(D3.2a)}$$

$$\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;\,q)_n}\;=\prod_{k=0}^{\infty}\frac{1}{(1-q^{2+5k})(1-q^{3+5k})}.\qquad\text{(D3.2b)}$$

Proof. Letting $m\to\infty$, we can state (D3.1a) as

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}\;&=\;\frac{1}{(q,q)_\infty}\sum_{k=-\infty}^{+\infty}(-1)^k q^{\binom{k}{2}+2k^2}\\ &=\;\frac{1}{(q,q)_\infty}\sum_{k=-\infty}^{+\infty}(-1)^k q^{5\binom{k}{2}+2k}.\end{aligned}$$

The sum on the right hand side can be evaluated, by means of Jacobi triple product identity, as

$$\sum_{k=-\infty}^{+\infty}(-1)^k q^{5\binom{k}{2}+2k}\;=\;\left[q^5,q^2,q^3;q^5\right]_\infty.$$

Therefore the first identity (D3.2a) follows consequently:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}\;=\;\frac{\left[q^5,q^2,q^3;q^5\right]_\infty}{(q;\,q)_\infty}\;=\;\frac{1}{\left[q,q^4;q^5\right]_\infty}.$$

If we let $m\to\infty$ in (D3.1b), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}\;&=\;\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}(-1)^k\,q^{\binom{k}{2}+2k^2-k}\\ &=\;\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}(-1)^k\,q^{5\binom{k}{2}+k}.\end{aligned}$$

The sum on the right hand side reads as

$$\sum_{k=-\infty}^{+\infty} (-1)^k q^{5\binom{k}{2}+k} = \big[q^5, q, q^4; q^5\big]_\infty$$

in view of Jacobi triple product identity.

Hence we have established the following

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{\big[q^5, q, q^4; q^5\big]_\infty}{(q;\ q)_\infty} = \frac{1}{\big[q^2, q^3; q^5\big]_\infty}$$

which is the second identity (D3.2b). □

Up to now, about ten proofs have been provided for this beautiful pair of identities. The most recent ones are, respectively, due to Baxter (1982) based on the statistical mechanics and Lepowsky-Milne (1978) through the character formula on infinite dimensional Lie algebra (Kac-Moody algebra [**45**, 1985]).