1 Introduction

Most of the models of spatial discrimination with quantity competition exhibit a unique agglomerated equilibrium when the market space is linear and bounded. This result crucially depends on a restriction on the admissible levels of the unit transportation cost - restriction which is indeed imposed by many authors, in order to ensure that for any location pairs both firms deliver positive quantities over the whole market (Hamilton *et al* 1989, Anderson and Neven, 1991). However, agglomeration implies full symmetry of firms' behaviour at all market addresses, thus making the spatial dimension eventually irrelevant at equilibrium. Moreover, within the above framework, and under the same restriction on costs, Shimizu (2002) has shown that the agglomeration result is robust to the introduction of an element of product differentiation, and therefore that the degree of substitutability/complementarity is immaterial in the definition of the firms' optimal locations.

This paper discusses the role of product differentiation when the range of admissible values of the unit transport costs is extended to those consistent with full market coverage by both firms *at equilibrium*. By allowing for higher values of t, the existence of an additional dispersed solution with full coverage, originally suggested by Hamilton *et al*, is confirmed in the case of substitute goods for a range of the transportation costs, the width and bounds of which are shown to depend on the degree of substitutability. Moreover, the paper shows how the latter interacts with t in the definition of the optimal dispersed locations.

The paper is organized as follows. In the next section we modify the standard model of spatial discrimination with Cournot competition by introducing the Deneckere (1983) inverse demand function in order to capture product differentiation. The solution for the Subgame Perfect Nash Equilibria (SPNE) of the game is then followed by a brief discussion of their properties and of the role of product differentiation. Section 3 concludes.

2 The model

In a spatial market two firms (labeled 1 and 2) are assumed to decide their location along a segment of length l (normalized to 1 in the sequel) and then to engage in quantity competition at all sites. Consumers are uniformly distributed along the segment, a consumer's location being denoted by $x \in [0,1]$. Let a be the location of firm 1 and $1-b$ the location of firm 2 (i.e. b is the distance of firm 2 from the right endpoint of the segment), with $a+b \leq 1$. When firms 1 and 2 deliver their product to a location x, they bear a freight cost, linear in distance, respectively denoted by $t\,|a-x|$ and $t\,|1-b-x|$. We also assume that each firm incurs a constant and equal to zero marginal and average cost of production. The products of the two firms may be either substitutes or complements, so that in each address x market demand is given by $p_i\,(x) = 1-\gamma q_j\,(x) - q_i\,(x)$ (with $j \neq i$), where $\gamma \in [-1,1]$ (with $\gamma \neq 0$) denotes the degree of substitutability/complementarity between the two goods,[1] and $q_i(x)$ indicates the quantity shipped by firm i at x. As fairly standard in this literature, we rule out the possibility of arbitrage [2].

When solving the two stage game in quantities and locations, we remove the common assumption $t < 1/2$, and allow for higher values of t. In particular, our purpose is to extend the analysis of the quantity-location game with complement or substitute goods to all those values of t which support duopolistic competition at equilibrium over the entire space. In a perfect substitutability framework, i.e. when $\gamma = 1$, Chamorro Rivas (2000) has shown that the range of values of t for which an agglomerated equilibrium with the above properties exists, extends to $t = 1$, while a dispersed symmetric equilibrium arises for the range $t \in]2/3, 10/11[$. In what follows we investigate how different values of γ affect these ranges and the way in which γ may, under some circumstances, interact with t in determining the firms' optimal locations. It must be stressed that, since we want to concentrate on market configurations with full coverage by both firms, we do not analyze possible equilibria entailing monopolistic areas.

2.1 The two-stage game

The two-stage game is solved backwards. The structure of the problem is such that the decisions on quantities at a specific point x are independent of the decisions at other points.

Given the demand function $p_i = 1 - \gamma q_j - q_i$, by solving

$$\begin{aligned} \max_{\text{wrt } q_i} \pi_1^D(q_1, q_2, a, b, x) &= (1 - \gamma q_2 - q_1 - t\,|a - x|)\, q_1 \\ \max_{\text{wrt} q_2} \pi_2^D(q_1, q_2, a, b, x) &= (1 - \gamma q_1 - q_2 - t\,|1 - b - x|)\, q_2 \end{aligned}$$

we obtain the following Nash equilibrium in quantities at x:

$$q_1^*(a, b, \gamma, x) = \frac{2 - \gamma + \gamma t\,|1 - b - x| - 2t\,|a - x|}{(2 + \gamma)(2 - \gamma)} \quad (1)$$

$$q_2^*(a, b, \gamma) = \frac{2 - \gamma - 2t\,|1 - b - x| + \gamma t\,|a - x|}{(2 + \gamma)(2 - \gamma)} \quad (2)$$

If the entire market is served by the two firms at equilibrium, then at the location stage any equilibrium pair (a, b) maximizes

$$\begin{aligned} \Pi_1^D(a, b, \gamma) &= \int_0^1 \left(\pi_1^D(a, b, \gamma, x)\right) dx \\ \Pi_2^D(a, b, \gamma) &= \int_0^1 \left(\pi_2^D(a, b, \gamma, x)\right) dx \end{aligned}$$

[1] We recall that $\gamma < 0$ denotes complementarity, while $\gamma > 0$ denotes substitutability.

[2] See, e.g. Hamilton *et al.* (1989) for a discussion.

where

$$\pi_1^D(a,b,\gamma,x) = \left(\frac{(2-\gamma+\gamma t|1-b-x|-2t|a-x|)^2}{(4-\gamma^2)^2}\right) \quad (3)$$

$$\pi_2^D(a,b,\gamma,x) = \left(\frac{(2-\gamma-2t|1-b-x|+\gamma t|a-x|)^2}{(4-\gamma^2)^2}\right) \quad (4)$$

From the solution of the First Order Conditions, we obtain an agglomerate outcome

$$a^* = b^* = \frac{1}{2} \quad (5)$$

which is invariant with respect to γ, and a dispersed outcome

$$a^{'}(\gamma,t) = b^{'}(\gamma,t) = \frac{(\gamma-2)(t-2)}{4\gamma t} \quad (6)$$

where $a^{'}$ and $b^{'}$ depend negatively on γ and t. This latter configuration collapses to that found by Chamorro Rivas, $a = b = (2-t)/4t$ when $\gamma = 1$.

For (5) and (6) to be Nash solutions of the locations game, however, some restrictions on the parameters' values must be imposed. First of all, feasibility requires $a+b \leq 1$ which in the case of (6) implies $t \geq t_{\min}(\gamma) = 2(2-\gamma)/(\gamma+2)$. As far as the Second Order Conditions (SOCs) are concerned, they are satisfied at (5) for $0 < t < 2-\gamma$, at (6) for all non-negative values of γ. This enables us to rule out the existence of a dispersed equilibrium when the products are complements.

Finally, the above locations stem from profit maximization under the hypothesis of full market coverage by both firms. For them to be an equilibrium they must be consistent with this hypothesis and must be régime-change proof: i.e., we have to observe duopolistic interaction over the entire segment at equilibrium, and the firms must not perceive any incentive to deviate towards location that deliver a different coverage pattern. For this purpose, we establish the following proposition.

Proposition 1 *The agglomerated outcome is a SPNE with full market coverage by both firms when $t < 2$. The dispersed outcome is a SPNE with full market coverage by both firms when $\gamma > 0$ and $t \in [t_{\min}(\gamma), t^{cover}(\gamma)[$, where $t^{cover}(\gamma) = 2(3\gamma+2)(\gamma-2)/\left(\gamma^2-8\gamma-4\right)$.*

Proof. See the Appendix ■

Therefore, on the basis of the feasibility condition, the SOCs and Proposition 1, we can fully characterize the equilibria with global duopolistic interaction.

Proposition 2 *When the products of the two firms are complements ($\gamma < 0$) the agglomerated pattern $a^* = b^* = 1/2$ is the unique SPNE for all $t < 2$. When the products are substitutes ($\gamma > 0$), $a = b = 1/2$ is the unique SPNE for $t \in \left[0, t^{\min}(\gamma)\right]$, while it coexists with the dispersed solution $a^{'} = b^{'} = ((\gamma-2)(t-2))/4\gamma t$ for $t \in [t_{\min}(\gamma), t^{cover}(\gamma)[$.*

Proposition 2 states that a dispersed equilibrium with overall duopolistic interaction exists (and therefore multiplicity of equilibria arises) only when the goods are substitutes, provided that the transportation cost lies in a well-defined range. On the contrary, agglomeration is the unique equilibrium outcome consistent with complementarity. The intuition behind this result is related to the different forces driving the choice of locations: a cost-saving effect which induces firms to move towards the center in order to minimize transport costs, and a strategic effect leading firms to locate further apart from each other in order to soften competition, i.e. to exploit the advantages of serving at low costs areas where the rival firm delivers lower quantities. Only the former effect is relevant when the goods are complements. In this case, inspection of (1) and (2) reveals that the output of each firm is univocally decreasing in t. For this reason the two firms will only choose at equilibrium locations in which transportation costs are minimized. Conversely, it is the interplay of both effects which determines multiplicity of equilibria under the hypothesis of substitutability.

2.2 The properties of the equilibrium locations

Equilibria (5) and (6) exhibit some quite common properties. By evaluating at (6) the optimal quantities (1) and (2) over the entire market, it is easy to check that the so-called quantity median property is verified.[3] Moreover, when the conditions for multiplicity are met, the profits of the two firms turn out to be higher at the dispersed solution rather than at the agglomerated one (Chamorro Rivas, 2000).

The most relevant feature of the model, however, is that the dispersed equilibrium is actually influenced by the degree of substitutability. While low transportation costs leave no role for substitutability at the unique agglomerated equilibrium, at the dispersed equilibrium observed for $t \in [t_{\min}(\gamma), t^{cover}(\gamma)[$ the degree of substitutability matters. First of all, γ clearly influences the threshold values $t_{\min}(\gamma)$ and $t^{cover}(\gamma)$, and the width of the above interval. As γ decreases from 1 to 0 (in the limit), $t_{\min}(\gamma)$ increases from 2/3 to 2, and $t^{cover}(\gamma)$ increases from 10/11 to 2. Moreover, since $d\left(t^{cover}(\gamma) - t_{\min}(\gamma)\right)/d\gamma$ is always positive for $\gamma > 0$, the above interval progressively shrinks, clearly collapsing in the limit to $t = 2$. Therefore, for each value of γ we have a different range of values of t supporting the dispersed equilibrium: the lower is γ, the narrower is the interval $[t_{\min}(\gamma), t^{cover}(\gamma)[$, but the higher are both its threshold values: an appropriate degree of imperfect substitutability makes high levels of t (up to $t = 2$) consistent with duopolistic interaction over the entire market both at the agglomerated and at the dispersed equilibrium.

Inspection of (6) shows that for $t < 2$, the optimal distance from the endpoints is decreasing in γ for given t, and decreasing in t for given γ: the lower

[3] The quantity median property (each firm sells an identical total quantity at the left and at the right of its location) has been first proved by Hamilton *et al* (1989) in a framework of perfect substitutability and uniform distribution of consumers. It has been extended by Gupta *et al* (1997) to a variety of symmetric distributions. Gupta *et al* (2004) verify this property in a circular city context, while Pal and Sarkar (2002) prove it for multistore competition.

is product substitutability, the lower is the incentive of firms to separate from each other; the higher is the transportation costs, the wider is the distance separating the two firms at the dispersed equilibrium. Figure 1 synthesizes these results. It shows the optimal distance from the endpoints as a function of γ, for given values of t, the inner curves being associated to higher levels of the latter. The SPNE values of a $(= b)$, evaluated for each γ and t, are those comprised in the shaded area. For example, all points in the vertical segment AB represent firm i's optimal locations when $\gamma = \gamma_0$, as t varies from $t_{\min}(\gamma_0)$ to $t^{cover}(\gamma_0)$, while the interval $[\gamma_0, \gamma']$ is the range of values of γ which supports a dispersed equilibrium with full market coverage by both firms, when $t = 6/5$.

3 Final remarks

In this paper we have extended the analysis by Shimizu (2002) who argues that the degree of product substitutability doesn't alter the equilibrium solution in locations when firms compete on a linear city and the unit transportation cost is upper bounded at $t = 1/2$. This restriction ensures full market coverage by both firms from all pairs of locations; it is therefore a sufficient, but not a necessary condition for duopolistic interaction over the entire market *at the SPNE locations*. By deriving the less restrictive necessary conditions, we show that when the products are substitutes, the dispersed solution, coexisting with the agglomerated one, is indeed affected by the degree of substitutability. As goods become less substitutable, the distance between the firms at the dispersed equilibrium narrows, while the range of values of t consistent at equilibrium with full market coverage by both firms shrinks and shifts upwards. Imperfect substitutability softens competition: from the one side both firms may profitably reach distant locations even in the presence of high transportation costs; from the other side firms may interact from closer locations at the dispersed equilibrium. Shimizu's result turns out to be a special case; it applies when the agglomerated equilibrium is unique, i.e., for a subset of the admissible values of t when the goods are substitutes, for all admissible t when the goods are complements.