Chapter 2

Equations of Fluid Mechanics

2.1 Equations of balance of continuum mechanics

Let $Ox_1x_2x_3$ be a Cartesian frame of reference with fundamental unit vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$, $\mathbf{e}_3$ pointed vertically upward, and let V be a volume whose surface ∂V moves with velocity $\mathbf{v} = v_j\mathbf{e}_j$ of a body. Therefore at time t the rate of change of a generic quantity

$$\Psi = \int_V \rho\psi \mathrm{d}V$$

inside V is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\psi \mathrm{d}V = \int_V \frac{\partial(\rho\psi)}{\partial t}\mathrm{d}V + \int_{\partial V} \rho\psi v_j n_j \mathrm{d}A \tag{2.1}$$

where $\rho\psi$ is the density of the quantity Ψ, ψ being the specific value of Ψ, and $\mathbf{n} = n_j\mathbf{e}_j$ is the outer unit normal. Equation (2.1) is known as *Reynolds' transport theorem* [25].

The quantity Ψ may change in time due to a flux of Ψ through the surface ∂V, due to a production of Ψ and due to a supply from outside. For V the rate of change of Ψ may be expressed by the generic equation of balance

$$\int_V \frac{\partial(\rho\psi)}{\partial t}\mathrm{d}V = -\int_{\partial V} (\rho\psi v_j + \Phi_j)n_j \mathrm{d}A + \int_V \pi \mathrm{d}V + \int_V \rho\sigma \mathrm{d}V, \tag{2.2}$$

where $\boldsymbol{\Phi}$ is the non-convective flux density vector of Ψ, π is the production density and σ is the specific supply from outside. Given the appropriate smoothness properties, the surface integral in (2.2) may be converted into a volume integral by use of the Gauss Theorem and then (2.2) may be written as

$$\int_V \left[\frac{\partial(\rho\psi)}{\partial t} + \frac{\partial(\rho\psi v_j + \Phi_j)}{\partial x_j} - \pi - \rho\sigma\right]\mathrm{d}V = 0.$$

Since this equation must hold for all volumes, even infinitesimally small ones, the integrand itself must vanish. Thus we obtain the generic local equation of balance

$$\frac{\partial(\rho\psi)}{\partial t} + \frac{\partial(\rho\psi v_j + \Phi_j)}{\partial x_j} - \pi - \rho\sigma = 0. \tag{2.3}$$

The prototype of equation (2.3) is the mass balance which results from setting $\psi = 1$ and $\Phi_j = \pi = \sigma = 0$ so that we obtain

$$\frac{\partial\rho}{\partial t} + \frac{(\rho v_j)}{\partial x_j} = 0,$$

which is known as the continuity equation and may be used to simplify the generic local balance equation (2.3) to read

$$\rho\dot{\psi} + \mathrm{div}\,\mathbf{\Phi} = \pi + \rho\sigma,$$

where

$$\dot{\psi} = \frac{\partial\psi}{\partial t} + v_j\frac{\partial\psi}{\partial x_j} \tag{2.4}$$

is the *material derivative* of ψ.

The most commonly appearing balance equations of continuum mechanics are those of mass, linear momentum and internal energy. In those cases the generic quantities ψ, Φ_j, π and σ have concrete physical significance and are all denoted by canonical letters. Table 2.1 gives a list.

Ψ	ψ	Φ_i	π	σ
mass	1	0	0	0
linear momentum	v_i	$-t_{ij}$	0	b_i
internal energy	e	q_j	$t_{ij}d_{ij}$	r

Table 2.1: Canonical notation for specific values of mass, linear momentum and internal energy and their fluxes and source contributions.

$\mathbf{T} = t_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ in the flux density of linear momentum is the *Cauchy stress tensor* and the flux density $\mathbf{q}$ of internal energy is called the *heat flux vector*. In absence of body couples, the balance of the angular momentum requires that the stress tensor $\mathbf{T}$ is symmetric, i.e. $t_{ij} = t_{ji}\,\forall i = 1, 2, 3$. The external supply $\mathbf{b}$ of linear momentum is the *specific external body force field* and the supply r of internal energy is the *specific radiant heating*. Finally the second order tensor

$$\mathbf{D} = d_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

in the production density of internal energy is the symmetric part of velocity gradient $\mathbf{L} = \nabla \mathbf{v}$. Then, according to (2.3) and (2.4), the equations of balance of mass, linear momentum and internal energy are:

$$\dot{\rho} + \rho \text{div}\, \mathbf{v} = 0, \tag{2.5}$$

$$\rho \dot{\mathbf{v}} = \text{div}\, \mathbf{T} + \rho \mathbf{b}, \tag{2.6}$$

$$\rho \dot{e} + \text{div}\, \mathbf{q} = \mathbf{T} \cdot \mathbf{D} + \rho r. \tag{2.7}$$

2.2 Constitutive assumptions for fluid behavior

The equations of balance (2.5)-(2.7) are common to most bodies in Nature. These laws, however, are insufficient to fully characterize the behavior of bodies because they do not distinguish between different types of materials. We therefore introduce additional hypothesis, called *constitutive assumptions*, which serve to distinguish different types of material behavior.

Here we shall consider three types of constitutive assumptions in order to describe the fluid behavior.

(*i*) Constraint on the possible deformations the fluid may undergo.

(*ii*) Assumptions on the form of the stress tensor.

(*iii*) Constitutive equations relating the material parameters of the fluid to the motion.

From now on we shall be interested in isotropic linearly viscous fluids that can only undergo isochoric motions in isothermal processes, but can sustain motions that are not necessarily isochoric in processes that are not isothermal. Such fluids are said to be, roughly speaking, mechanically incompressible but thermally compressible. Experience tells us the possibility that a fluid be mechanically incompressible but thermally compressible seems a reasonable description of observations. The restriction that the fluid can undergo only isochoric motions in isothermal processes implies that the determinant of the deformation gradient is a function of temperature θ,

$$\det \mathbf{F} = f(\theta). \tag{2.8}$$

If $\mathbf{F}$ is differentiable with respect to time, (2.8) can be expressed as

$$\text{div}\, \mathbf{v} = \text{tr}\mathbf{D} = \alpha(\theta)\dot{\theta} \tag{2.9}$$

where

$$\alpha(\theta) = \frac{1}{f(\theta)} \frac{\mathrm{d}f}{\mathrm{d}\theta}(\theta)$$

is the *coefficient of volumetric thermal expansion*.

Constitutive expressions for the stress within the context of classical continuum mechanics such as those for the linearized response of solids due to Hooke and Navier, and for the linear response of fluids due to Newton, Navier, Poisson, St. Venant and Stokes provide explicit relationships for the stress in terms of appropriate kinematical quantities and the density. For instance, in the case of the classical incompressible Navier-Stokes fluid the Cauchy stress tensor takes the explicit form

$$\mathbf{T} = -p\mathbf{1} + 2\mu(\theta)\mathbf{D}, \tag{2.10}$$

where $-p\mathbf{1}$ is the indeterminate part of the stress due to the constraint of incompressibility (i.e. the constraint stress), p being the *pressure*, and μ is the *viscosity* of the fluid. In contrast, many constitutive relations for inelastic and viscoelastic fluids are implicit relations. Here, following Rajagopal [65], we shall discuss a generalization of the classical incompressible Navier-Stokes fluid, as envisioned by Stokes [85], that leads to implicit constitutive relations.

In his celebrated paper on the response of fluids Stokes [85] recognized that the viscosity of a fluid could depend upon the pressure. However, based on the experiments of Du Buat on the flow of water in canals and pipes under normal operating conditions, Stokes suggested that the viscosity could be considered a constant for such flows. Stokes was however very careful to delineate the class of flows wherein viscosity might be considered a constant and he also remarked that such an assumption would be invalid under other flow conditions. As early as Barus [5] proposed an empirical relationship between the viscosity and the pressure, namely

$$\mu(p) = \mu_0 \exp[\beta(p - p_0)], \tag{2.11}$$

where μ_0 is the viscosity at the pressure p_0 and β is a piezoviscous coefficient that varies with temperature. Later, Andrade [2] suggested the following expression for the viscosity

$$\mu(p, \rho, \theta) = A\rho^{1/2} \exp\left[(p + \rho r^2)\frac{s}{T}\right],$$

based on experiments. In the above expression ρ denotes the density, T the temperature, p the pressure, and r, s and A are constants. More recently, Laun [36] has modelled the viscosity of polymer melts through

$$\mu(p, T) = \mu_0 \exp[\beta(p - p_0) - \gamma(T - T_0)],$$

where μ_0 is the viscosity at pressure p_0 and temperature T_0, and β and γ are non-negative constants. There have been numerous other experiments by Bair and co-workers that shows that the dependence of the viscosity on the pressure is exponential (see recent experiments of Bair and Kottke [4]). Mention must be made of the works of Martín-Alfonso and co-workers [46, 47] wherein intricate relationship among the temperature, viscosity and pressure are provided for bitumen.

In order to deduce the model (2.10), the standard procedure in classical mechanics is to split the Cauchy stress tensor $\mathbf{T}$ additively as

$$\mathbf{T} = \mathbf{T}_C + \mathbf{T}_E, \tag{2.12}$$

where $\mathbf{T}_C$, the constraint stress, is assumed not to depend on the state variables (in the case of the classical fluid the velocity gradient) and $\mathbf{T}_E$, the so-called 'extra' stress, is constitutively prescribed, but is assumed to not depend on the constrained part $\mathbf{T}_C$. According to the the Constraint Principle of Truesdell and Noll [90], the further assumption that $\mathbf{T}_C$ does no work implies that

$$\mathbf{T}_C \cdot \mathbf{D} = 0 \quad \text{whenever } \operatorname{tr}\mathbf{D} = \mathbf{1} \cdot \mathbf{D} = 0.$$

This immediately leads to

$$\mathbf{T}_C = -p\mathbf{1},$$

p being a Lagrange multiplier. Importantly, $\mathbf{T}_E$ cannot depend on p, and thus quantities such as the viscosity cannot depend on the pressure. It is also important to note that the above procedure would be inapplicable if the constraint were nonlinear in $\mathbf{D}$. In any event, the standard procedure leads to the material function not depending on the constraint.

Let us consider an implicit relation of the form

$$\mathbf{f}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta}) = \mathbf{0}, \tag{2.13}$$

i.e. among the stress, the symmetric part of the velocity gradient, the temperature and the material derivative of the temperature. It then follows that

$$\frac{\partial \mathbf{f}}{\partial \mathbf{T}}\dot{\mathbf{T}} + \frac{\partial \mathbf{f}}{\partial \mathbf{D}}\dot{\mathbf{D}} + \frac{\partial \mathbf{f}}{\partial \theta}\dot{\theta} + \frac{\partial \mathbf{f}}{\partial \dot{\theta}}\ddot{\theta} = \mathbf{0},$$

where $\partial\mathbf{f}/\partial\mathbf{T}$ and $\partial\mathbf{f}/\partial\mathbf{D}$ are fourth-order tensors, $\partial\mathbf{f}/\partial\theta$ and $\partial\mathbf{f}/\partial\dot{\theta}$ are second-order tensors. We could also start with models of the form

$$\begin{aligned}[\mathbf{A}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta})]\dot{\mathbf{T}} + [\mathbf{B}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta})]\dot{\mathbf{D}} + \mathbf{C}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta})\dot{\theta} \\ + \mathbf{E}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta})\ddot{\theta} = \mathbf{0},\end{aligned} \tag{2.14}$$

where $\mathbf{A}$ and $\mathbf{B}$ are fourth-order tensor, $\mathbf{C}$ and $\mathbf{E}$ are second-order tensor. While the class of models defined through (2.14) is larger, in one sense, than that defined through (2.13) since not all models belonging to (2.14) belong to (2.13) as (2.14) may not be integrable, we note that (2.14) requires the stress $\mathbf{T}$, the symmetric velocity gradient $\mathbf{D}$ and the material time derivative of temperature $\dot{\theta}$ have time derivatives while (2.13) makes no such restriction. However, we shall be interested in sufficiently smooth functions $\mathbf{T}$, $\mathbf{D}$ and θ, so for such a class of functions (2.14) is more general than (2.13). Given an explicit model for the Cauchy stress tensor, since it can always be expressed in the form (2.13), we can express it in the form (2.14) by merely taking its derivative.

Suppose

$$\begin{cases} \mathbf{A}(\mathbf{T},\mathbf{D},\theta,\dot{\theta}) = \boldsymbol{\mathcal{I}} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1} - 2\frac{\partial\mu}{\partial \mathrm{tr}\mathbf{T}}(\mathrm{tr}\mathbf{T},\theta)\left[\mathbf{D} - \frac{\alpha(\theta)\dot{\theta}}{3}\mathbf{1}\right]\otimes\mathbf{1}, \\ \mathbf{B}(\mathbf{T},\mathbf{D},\theta,\dot{\theta}) = -2\mu(\mathrm{tr}\mathbf{T},\theta)\boldsymbol{\mathcal{I}}, \\ \mathbf{C}(\mathbf{T},\mathbf{D},\theta,\dot{\theta}) = -2\frac{\partial\mu}{\partial\theta}(\mathrm{tr}\mathbf{T},\theta)\left[\mathbf{D} - \frac{\alpha(\theta)\dot{\theta}}{3}\mathbf{1}\right] + \frac{2}{3}\mu(\mathrm{tr}\mathbf{T},\theta)\frac{\mathrm{d}\alpha}{\mathrm{d}\theta}(\theta)\dot{\theta}\mathbf{1}, \\ \mathbf{E}(\mathbf{T},\mathbf{D},\theta,\dot{\theta}) = \frac{2}{3}\mu(\mathrm{tr}\mathbf{T},\theta)\alpha(\theta)\mathbf{1}, \end{cases} \tag{2.15}$$

where $\boldsymbol{\mathcal{I}}$ denotes the fourth-order identity tensor, μ and α are sufficiently smooth functions, μ depending on both $\mathrm{tr}\mathbf{T}$ and θ, α only on θ. Furthermore, since we are interested in describing mechanically incompressible but thermally compressible fluids, we shall require that (2.9) is met.

From (2.14) and (2.15) it follows that

$$\begin{aligned} \dot{\mathbf{T}} =& \frac{1}{3}(\mathrm{tr}\dot{\mathbf{T}})\mathbf{1} + 2\left(\frac{\partial\mu}{\partial\mathrm{tr}\mathbf{T}}\mathrm{tr}\dot{\mathbf{T}} + \frac{\partial\mu}{\partial\theta}\dot{\theta}\right)\mathbf{D} + 2\mu(\mathrm{tr}\mathbf{T},\theta)\dot{\mathbf{D}} \\ &- \frac{2}{3}\mu(\mathrm{tr}\mathbf{T},\theta)\left[\frac{\mathrm{d}\alpha}{\mathrm{d}\theta}(\theta)\dot{\theta}^2 + \alpha(\theta)\ddot{\theta}\right]\mathbf{1} - \frac{2}{3}\alpha(\theta)\dot{\theta}\left(\frac{\partial\mu}{\partial\mathrm{tr}\mathbf{T}}\mathrm{tr}\dot{\mathbf{T}} + \frac{\partial\mu}{\partial\theta}\dot{\theta}\right)\mathbf{1}, \end{aligned}$$

which can be integrated to yield

$$\mathbf{T} = \frac{1}{3}(\mathrm{tr}\mathbf{T})\mathbf{1} + 2\mu(\mathrm{tr}\mathbf{T},\theta)\left[\mathbf{D} - \frac{1}{3}\alpha(\theta)\dot{\theta}\mathbf{1}\right] + \mathbf{T}_0$$

where $\mathbf{T}_0$ is some constant symmetric stress tensor. The further requirement that the stress be purely spherical when the fluid is at rest in isothermal processes leads to

$$\mathbf{T} = \frac{1}{3}(\mathrm{tr}\mathbf{T})\mathbf{1} + 2\mu(\mathrm{tr}\mathbf{T},\theta)\left[\mathbf{D} - \frac{1}{3}\alpha(\theta)\dot{\theta}\mathbf{1}\right]. \tag{2.16}$$

We notice that (2.16) automatically meets the constraint (2.9). We thus do not need to enforce the constraint (2.9) by using a Lagrange multiplier.

Let us define

$$p = -\frac{1}{3}\mathrm{tr}\mathbf{T},$$

then, by (2.9) and (2.16),

$$\mathbf{T} = -p\mathbf{1} + 2\mu(p,\theta)\left[\mathbf{D} - \frac{1}{3}(\mathrm{tr}\mathbf{D})\mathbf{1}\right]. \tag{2.17}$$

We now consider the implications of assuming that $\mathbf{f}$ defined through the relation (2.13) is an isotropic function. Then

$$\mathbf{f}(\mathbf{QTQ}^T, \mathbf{QDQ}^T, \theta, \dot{\theta}) = \mathbf{Qf}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta})\mathbf{Q}^T \quad \forall \mathbf{Q} \in \mathrm{Orth},$$

where Orth denotes the set of all orthogonal transformations. It then follows that (see Spencer [82])

$$\begin{aligned} &\alpha_0\mathbf{1} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{TD} + \mathbf{DT}) \\ &\quad + \alpha_6(\mathbf{T}^2\mathbf{D} + \mathbf{DT}^2) + \alpha_7(\mathbf{TD}^2 + \mathbf{D}^2\mathbf{T}) + \alpha_8(\mathbf{T}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}^2) = \mathbf{0}, \end{aligned} \tag{2.18}$$

where the material functions α_i, $i = 0, 1, \ldots, 8$, depend on θ, $\dot{\theta}$ and on the invariants

$$\mathrm{tr}\,\mathbf{T},\ \mathrm{tr}\,\mathbf{D},\ \mathrm{tr}\,\mathbf{T}^2,\ \mathrm{tr}\,\mathbf{D}^2,\ \mathrm{tr}\,\mathbf{T}^3,\ \mathrm{tr}\,\mathbf{D}^3,\ \mathrm{tr}\,(\mathbf{TD}),\ \mathrm{tr}\,(\mathbf{T}^2\mathbf{D}),\ \mathrm{tr}\,(\mathbf{TD}^2),\ \mathrm{tr}\,(\mathbf{T}^2\mathbf{D}^2).$$

When we consider fluid models of the form (2.18), if

$$\alpha_0 = -\frac{1}{3}\mathrm{tr}\mathbf{T} + \frac{2}{3}\mu(\mathrm{tr}\mathbf{T},\theta)\alpha(\theta)\dot{\theta}, \quad \alpha_1 = 1, \quad \alpha_2 = -2\mu(\mathrm{tr}\mathbf{T},\theta)$$

and, as we are interested in linearly viscous fluids, all the other α_i are identically zero, we obtain the model (2.17). Such a constitutive assumption, i.e. the special choice of the functions α_i ($i = 0, 1, \ldots, 8$), automatically implies that the fluid under consideration is mechanically incompressible but thermally compressible as it always meets the constraint (2.9). We may then conclude that we do not need to necessarily enforce the constraint via Lagrange multipliers or require that the constraint stress is workless while working with these implicit models.

We shall henceforth take (2.17) as model for the Cauchy stress tensor.

For a fluid it is customary to require constitutive equations for the heat flux vector $\mathbf{q}$, for the specific internal energy e and for the specific entropy η, and we assume these quantities as functions of

$$p, \theta, \mathbf{v}, \mathbf{L}, \nabla\theta.$$

The Principle of material frame indifference [95] reduces this set of variables to

$$p, \theta, \mathbf{D}, \nabla\theta,$$

and the representation theorems for linear isotropic functions lead us to consider the following constitutive fluid model

$$e = e(p,\theta) + u(p,\theta)\mathrm{tr}\mathbf{D}, \tag{2.19}$$

$$\eta = \eta(p,\theta) + h(p,\theta)\mathrm{tr}\mathbf{D}, \tag{2.20}$$

$$\mathbf{q} = -k(p,\theta)\nabla\theta, \tag{2.21}$$

where k is the *heat conductivity.*

Also the second law of thermodynamics places restrictions on the thermomechanical constitutive equations (2.17), (2.19) and (2.21). To this end we record the second law of thermodynamics in the form of the Clausius-Duhem inequality

$$\rho\dot{\eta} \geq \rho\frac{r}{\theta} - \mathrm{div}\left(\frac{\mathbf{q}}{\theta}\right). \tag{2.22}$$

Inequality (2.22) holds for all themodynamic processes, i.e. for all fields ρ, θ, $\mathbf{v}$ and p satisfying equations (2.5)-(2.7) and (2.9). Hence by Liu Lemma [27, 39, 53] there exist six Lagrange multipliers Λ^ρ, Λ^{v_i} $(i = 1,2,3)$, Λ^e and Λ^θ such that, denoting by $d_{<ij>}$ the components of the deviatoric velocity gradient $\mathbf{D} - [(\mathrm{tr}\mathbf{D})/3]\mathbf{1}$,

$$\begin{aligned}
&\rho\left[\frac{\partial\eta}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial\eta}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\eta}{\partial d_{ii}}\frac{\partial d_{ii}}{\partial t} + v_j\left(\frac{\partial\eta}{\partial p}\frac{\partial p}{\partial x_j} + \frac{\partial\eta}{\partial\theta}\frac{\partial\theta}{\partial x_j} + \frac{\partial\eta}{\partial d_{ii}}\frac{\partial d_{ii}}{\partial x_j}\right)\right] \\
&\quad - \rho\frac{r}{\theta} - \frac{1}{\theta}\frac{\partial k}{\partial p}\frac{\partial p}{\partial x_j}\frac{\partial\theta}{\partial x_j} - \frac{1}{\theta}\left(\frac{\partial k}{\partial\theta} - \frac{k}{\theta}\right)\left(\frac{\partial\theta}{\partial x_j}\right)^2 - \frac{k}{\theta}\frac{\partial^2\theta}{\partial x_j^2} \\
&\quad - \Lambda^\rho\left[\frac{\partial\rho}{\partial t} + v_j\frac{\partial\rho}{\partial x_j} + \rho\frac{\partial v_j}{\partial x_j}\right] \\
&\quad - \Lambda^{v_i}\left[\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) + \frac{\partial p}{\partial x_i} - 2\left(\frac{\partial\mu}{\partial p}\frac{\partial p}{\partial x_j} + \frac{\partial\mu}{\partial\theta}\frac{\partial\theta}{\partial x_j}\right)d_{<ij>}\right. \\
&\qquad\quad \left. - 2\mu\frac{\partial d_{<ij>}}{\partial x_j} - \rho b_i\right] \\
&\quad - \Lambda^e\left\{\rho\left[\frac{\partial e}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial e}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial e}{\partial d_{ii}}\frac{\partial d_{ii}}{\partial t} + v_j\left(\frac{\partial e}{\partial p}\frac{\partial p}{\partial x_j} + \frac{\partial e}{\partial\theta}\frac{\partial\theta}{\partial x_j} + \frac{\partial e}{\partial d_{ii}}\frac{\partial d_{ii}}{\partial x_j}\right)\right]\right. \\
&\qquad\quad \left. - \frac{\partial k}{\partial p}\frac{\partial p}{\partial x_j}\frac{\partial\theta}{\partial x_j} - \frac{\partial k}{\partial\theta}\left(\frac{\partial\theta}{\partial x_j}\right)^2 - k\frac{\partial^2\theta}{\partial x_j^2} + p\frac{\partial v_j}{\partial x_j} - 2\mu d_{<ij>}^2 - \rho r\right\} \\
&\quad - \Lambda^\theta\left[\frac{\partial v_j}{\partial x_j} - \alpha\left(\frac{\partial\theta}{\partial t} + v_j\frac{\partial\theta}{\partial x_j}\right)\right] \geq 0
\end{aligned}$$

for all fields ρ, θ, $\mathbf{v}$ and p, or, equivalently, such that

$$\left.\begin{array}{r} \dfrac{\partial\eta}{\partial p} - \Lambda^e \dfrac{\partial e}{\partial p} = 0, \\ \dfrac{\partial\eta}{\partial\theta} - \Lambda^e \dfrac{\partial e}{\partial\theta} + \dfrac{\Lambda^\theta}{\rho}\alpha = 0, \\ \dfrac{\partial\eta}{\partial d_{ii}} - \Lambda^e \dfrac{\partial e}{\partial d_{ii}} = 0, \\ \Lambda^\rho = 0 \\ \Lambda^{v_i} = 0 \quad \forall i = 1,2,3, \\ \Lambda^e = \dfrac{1}{\theta}, \\ \Lambda^\theta + \Lambda^e p = 0, \\ \dfrac{k}{\theta^2}\left(\dfrac{\partial\theta}{\partial x_j}\right)^2 + 2\Lambda^e \mu d^2_{<ij>} \geq 0 \quad \text{for all fields} \quad \rho,\ \theta,\ \mathbf{v} \text{ and } p. \end{array}\right\} \tag{2.23}$$

By (2.23) we readily deduce that the constitutive functions k and μ are non-negative,

$$\begin{aligned} e &= e(p,\theta), \quad \text{viz } u(p,\theta) = 0 \text{ in (2.19)}, \\ \eta &= \eta(p,\theta), \quad \text{viz } h(p,\theta) = 0 \text{ in (2.20)}, \end{aligned}$$

and

$$\frac{\partial e}{\partial p} = \theta\frac{\partial\eta}{\partial p} = -\theta\frac{\alpha}{\rho}, \quad \frac{\partial e}{\partial\theta} = c_p - p\frac{\alpha}{\rho}, \tag{2.24}$$

where $c_p = c_p(p,\theta) = \theta(\partial\eta/\partial\theta)_p$ is the specific heat at constant pressure.

2.3 Governing equations of fluid dynamics

We are now in position to derive from the equations of balance of mass, linear momentum, internal energy and from the constitutive fluid model introduced in the previous section the governing equation of fluid dynamics. We first introduce (2.9) into (2.5) and obtain

$$\frac{\dot\rho}{\rho} = -\alpha(\theta)\dot\theta \tag{2.25}$$

by which we deduce that

$$\alpha = -\frac{1}{\rho}\frac{\partial\rho}{\partial\theta}. \tag{2.26}$$

Next, introducing (2.17), (2.21) and (2.24) into the equations of balance (2.6) and (2.7) gives

$$\rho\dot{\mathbf{v}} = -\nabla p + \frac{\mu}{3}\nabla(\operatorname{div}\mathbf{v}) - \frac{2}{3}(\operatorname{div}\mathbf{v})\nabla\mu + 2\mathbf{D}\cdot\nabla\mu + \mu\Delta\mathbf{v} + \rho\mathbf{b} \tag{2.27}$$

and

$$\rho c_p \dot{\theta} - \alpha\theta\dot{p} = k\Delta\theta + \nabla k \cdot \nabla\theta + 2\mu \left[\|\mathbf{D}\|^2 - \frac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)^2 \right] + \rho r. \tag{2.28}$$

Equations (2.9), (2.25), (2.27) and (2.28) form the governing equations for the determination of the fields ρ, $\mathbf{v}$, θ and p. It is interesting to note that (2.28) is the equation for the determination of $\dot{p}$ since $\dot{\theta}$ is determined from (2.9).

2.4 Oberbeck-Boussinesq approximation

Few approximations in fluid mechanics have proved as useful and successful in predicting observed phenomena as the Oberbeck-Boussinesq approximation which has implications to a wide variety of flows within the context of astrophysical and geophysical fluid dynamics. The Oberbeck-Boussinesq approximation consists in keeping with a perturbation of the governing equations by identifying a small non-dimensional parameter and retaining terms of like order. While this is the popular wisdom concerning the approximation, this is not a true depiction of the state of affairs as this is not what is strictly carried out in order to obtain the Oberbeck-Boussinesq equations. These celebrated equations are not obtained by a standard perturbation technique. In order to justify the Oberbeck-Boussinesq by appealing to a perturbative approach many arguments have been put forth to justify the inclusion of terms that appear in the equations, but most of these arguments do not pass muster as explained below.

The approximate equations that have been used, and continue to be used, with great success, were first derived by Oberbeck [57, 58] and subsequently and independently derived by Boussinesq [6]. Oberbeck and Boussinesq were interested in obtaining the equations that would govern the flow of a classical linearly viscous fluid which undergoes isochoric motion in isothermal flows, but it could change its volume due to changes in temperature. As we have already seen, this implies that the det $\mathbf{F}$ is a constant in motion when the temperature is a constant, but the value of the det $\mathbf{F}$ could vary with temperature, $\mathbf{F}$ being the deformation gradient. If the motions are sufficiently smooth, this then implies that the div $\mathbf{v}$ vanishes when temperature is a constant but changes when the temperature changes, $\mathbf{v}$ being the velocity of the fluid.

Justification for the approximation due to Oberbeck and Boussinesq are too numerous to be listed and here we mention some of them. Important studies are due to Rayleigh [69], Jeffreys [31], Chandrasekhar [11], Spiegel and Veronis [83], Mihaljan [48], Roberts [76]; Roberts and Stewartson [77], Spiegel and Weiss [84], Hills and Roberts [28], Zeytounian [98]. Not all the above mentioned papers try to provide a rigorous justification for the approximation, some of them do try to provide some sort of rationale for the approximation, but they are not convincing for reasons discussed below. Recently, Rajagopal, Ruzika and Srinivasa [62] carried out an analysis in which they delineate the status of the Oberbeck-Boussinesq approximation based on a certain non-dimensional numbers that they introduce. However, their study implies that the approximation cannot be viewed as a proper perturbation in which terms of like order are retained and in their derivation they show that the Oberbeck-Boussinesq equations result as a consequence of mixing terms of different orders in a small parameter. They also provided higher order approximations to the problem. It might yet be possible to develop a proper perturbation scheme wherein the Oberbeck-Boussinesq equations are obtained as an approximation at a specific order of the perturbation; however at this juncture in time no such analysis is available.

We now discuss briefly some of the attempts to justify the Oberbeck-Boussinesq approximation; a more detailed critique of the various attempts can be found in [62]. Spiegel and Veronis [83] considered the motion of a compressible fluid and they introduced a small parameter ϵ related to the ratio of the variation in density in the absence of motion to the spatial average value of the density and then carried out a perturbation analysis. Spiegel and Veronis were fully aware that their approximation did not retain terms of the same order in the perturbation. In fact, in [83] they explicitly state "In equation (19) we have retained the term $g\epsilon(\rho'/\Delta\rho_0)\mathbf{k}$ even though it contains ϵ as a factor", and this is clearly unacceptable as they recognize. Another shortcoming of the approach of Spiegel and Veronis [83] is that the layer of fluid has to be sufficiently thin while the physical applications, especially in astrophysics and geophysics, require considerably thick layers.

A common problem with many of the justifications for the Oberbeck-Boussinesq approximation stems from the need to retain a term that is the product of the coefficient of thermal expansion and gravity. This product should be of order one, while the coefficient of thermal expansion has to tend to zero. This leads to the untenable requirement that gravity has to tend to infinity. As we saw above, Spiegel and Veronis [83] explicitly retain a term at first order in which the small parameter that appears for the perturbation appears and is multiplied by the acceleration due to gravity. Similarly, in the study by Mihaljan [48] which is often cited for giving a rigorous justification of the Oberbeck-Boussinesq approximation, we encounter a similar difficulty. Mihaljan uses two small parameters for perturbation and he carries out the perturbation analysis. Unfortunately, he does not recognize that when one of the small parameters goes to zero it immediately forces the other small parameter to tend to infinity. In effect he encounters the same problem as that faced by Spiegel and Veronis [83], but under a different guise. Hills and Roberts in their study [28], much in keeping with [83], require that the product of the coefficient of thermal expansion and the acceleration due to gravity be a constant while the coefficient of thermal expansion tends to zero, impossibility if gravity were to be finite. In fact, they recognize the problem with their approach and state explicitly that "As we shall see this last requirement is essential, because otherwise buoyancy forces are lost". Here, the requirement that they refer to is that the product of the coefficient of thermal expansion and the acceleration due to gravity be a constant as the coefficient of thermal expansion tends to zero.

Another attempt at providing a rationale for the Oberbeck-Boussinesq approximation is due to Gray and Giorgini [24]. After providing a very clear discussion of the subtle issues that need to be taken into account in order to obtain the approximation, they make certain ad hoc assumptions concerning the smallness of certain parameters to arrive at the Oberbeck-Boussinesq approximation. Though the study does not provide a rigorous basis for the approximation, their study is an interesting attempt at arriving at the same.

Our study here is similar in its approach as the study by Rajagopal, Ruzika and Srinivasa [62] for the celebrated Oberbeck-Boussinesq equations. However, since the viscosity, the specific heat at constant pressure and the heat conductivity are all functions of both the temperature and pressure and the coefficient of volumetric thermal expansion is temperature dependent, the analysis is much more complicated.

Let us consider a layer of fluid of thickness d, the top and the bottom surfaces of which being held at constant temperature T_2 and T_1 (say $T_1 > T_2$), respectively. In order to non-dimensionalize the equations (2.9), (2.25), (2.27) and (2.28) we choose convenient reference values π_0 and T_0 for pressure and temperature, respectively, and introduce the following dimensionless quantities:

$$\begin{gathered}\mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad \mathbf{v}^* = \frac{\mathbf{v}}{U}, \quad \rho^* = \frac{\rho}{\rho_0}, \quad t^* = \frac{U}{d}t, \\ p^* = \frac{p-\pi_0}{\rho_0 g d}, \quad \mathbf{b}^* = \frac{\mathbf{b}}{g}, \quad \theta^* = \frac{\theta - T_0}{\delta T_0}, \quad \mu^* = \frac{\mu}{\rho_0 U d}, \\ \alpha^* = \frac{\alpha}{\alpha_0}, \quad c_p^* = \frac{\delta T_0}{g d} c_p, \quad k^* = \frac{\delta T_0}{\rho_0 g U d^2} k, \quad r^* = \frac{d}{U^3} r,\end{gathered} \tag{2.29}$$

where

$$\delta T_0 = T_1 - T_2, \quad U = \sqrt{g d \alpha_0 \delta T_0}\,,$$

g is the acceleration due to gravity, ρ_0 and α_0 are the density and the thermal expansion at the reference temperature T_0, respectively. Introducing (2.29) into (2.9), (2.25), (2.27) and (2.28) leads to (omitting all asterisks)

$$\frac{\dot\rho}{\rho} = -F^2\alpha\dot\theta, \tag{2.30}$$

$$\operatorname{div}\mathbf{v} = F^2\alpha\dot\theta, \tag{2.31}$$

$$\begin{aligned} F^2\rho\dot{\mathbf{v}} = &- \nabla p + \frac{F^2}{3}\mu\nabla(\operatorname{div}\mathbf{v}) - \frac{2}{3}F^2(\operatorname{div}\mathbf{v})\nabla\mu \\ &+ 2F^2\nabla\mu\cdot\mathbf{D} + F^2\mu\Delta\mathbf{v} + \rho\mathbf{b} \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} \rho c_p\dot\theta - F^2\alpha\left(\theta + \frac{T_0}{\delta T_0}\right)\dot p = {}& k\Delta\theta + \nabla k\cdot\nabla\theta \\ &+ 2F^2\mu\left[\|\mathbf{D}\|^2 - \frac{1}{3}(\mathrm{tr}\mathbf{D})^2\right] + F^2\rho r, \end{aligned} \tag{2.33}$$

where

$$F = \frac{U}{\sqrt{gd}} = \sqrt{\alpha_0\delta T_0}$$

is the Froude number.

We now introduce the small parameter ϵ with respect to which we shall carry out our perturbance. Let

$$\epsilon = F^2 = \frac{U^2}{gd} \ll 1^{1}$$

and

$$\mathbf{v} = \sum_{n=0}^{+\infty}\epsilon^n\mathbf{v}_n, \quad \theta = \sum_{n=0}^{+\infty}\epsilon^n\theta_n, \quad p = \sum_{n=0}^{+\infty}\epsilon^n p_n \tag{2.34}$$

be the power series in ϵ of the physical quantities $\mathbf{v}$, θ and p. From now on we shall assume that α, c_p, k and μ are analytic functions and we shall limit our analysis to pressure and temperature departures from the reference state (π_0, T_0) for which we can write

$$\alpha(\theta) = \sum_{n=0}^{+\infty}\frac{1}{n!}\frac{\mathrm{d}^n\alpha}{\mathrm{d}\theta^n}(0)\theta^n, \tag{2.35}$$

$$c_p(p,\theta) = \sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}c_p}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p^{j_1}\theta^{j_2}, \tag{2.36}$$

[1]The non-dimensional parameter F^2 is known as the second Froude number.

$$k(p,\theta) = \sum_{j_1+j_2=0}^{+\infty} \frac{1}{j_1!j_2!} \frac{\partial^{(j_1+j_2)}k}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p^{j_1}\theta^{j_2} \tag{2.37}$$

and

$$\mu(p,\theta) = \sum_{j_1+j_2=0}^{+\infty} \frac{1}{j_1!j_2!} \frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p^{j_1}\theta^{j_2}. \tag{2.38}$$

Thus, from (2.30) and (2.35) we get

$$\rho = \exp\left[-\epsilon\sum_{n=0}^{+\infty} \frac{1}{(n+1)!}\frac{\mathrm{d}^n\alpha}{\mathrm{d}\theta^n}(0)\theta^{n+1}\right]$$

and hence

$$\rho(\theta) = 1 - \epsilon\left[\sum_{n=0}^{+\infty} \frac{1}{(n+1)!}\frac{\mathrm{d}^n\alpha}{\mathrm{d}\theta^n}(0)\theta^{n+1}\right] + o(\epsilon), \tag{2.39}$$

where $o(\epsilon)$ represents the terms in ϵ^n with $n \geq 2$.

Inserting (2.34)-(2.39) into (2.31)-(2.33) we get

$$\sum_{n=0}^{+\infty}\epsilon^n \mathrm{div}\,\mathbf{v}_n = \epsilon\sum_{j=0}^{+\infty}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\sum_{n=0}^{+\infty}\epsilon^n\left[\theta^j\left(\frac{\partial\theta_n}{\partial t} + \mathbf{v}\cdot\nabla\theta\right)\right]_n, \tag{2.40}$$

$$\begin{aligned}
&\epsilon\left[1-\epsilon\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\sum_{m=0}^{+\infty}\epsilon^m(\theta^{j+1})_m + o(\epsilon)\right] \\
&\times\sum_{n=0}^{+\infty}\epsilon^n\left[\frac{\partial\mathbf{v}_n}{\partial t} + (\mathbf{v}\cdot\nabla\mathbf{v})_n\right] = -\sum_{n=0}^{+\infty}\epsilon^n\nabla p_n \\
&+\frac{\epsilon}{3}\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left[p^{j_1}\theta^{j_2}\nabla(\mathrm{div}\,\mathbf{v})\right]_n \\
&-\frac{2\epsilon}{3}\sum_{j_1+j_2=1}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left[\mathrm{div}\,\mathbf{v}\nabla(p^{j_1}\theta^{j_2})\right]_n \\
&+2\epsilon\sum_{j_1+j_2=1}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n[\mathbf{D}\cdot\nabla(p^{j_1}\theta^{j_2})]_n \\
&+\epsilon\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left(p^{j_1}\theta^{j_2}\Delta\mathbf{v}\right)_n \\
&+\mathbf{b}\left[1-\epsilon\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\sum_{m=0}^{+\infty}\epsilon^m(\theta^{j+1})_m + o(\epsilon)\right],
\end{aligned} \tag{2.41}$$

$$
\begin{aligned}
&\left[1-\epsilon\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\sum_{m=0}^{+\infty}\epsilon^m(\theta^{j+1})_m+o(\epsilon)\right] \qquad (2.42)\\
&\quad\times\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}c_p}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left[p^{j_1}\theta^{j_2}\left(\frac{\partial\theta}{\partial t}+\mathbf{v}\cdot\nabla\theta\right)\right]_n\\
&\quad-\epsilon\sum_{m=0}^{+\infty}\frac{1}{m!}\frac{\mathrm{d}^m\alpha}{\mathrm{d}\theta}(0)\sum_{n=0}^{+\infty}\left[\theta^m\left(\theta+\frac{T_0}{\delta T_0}\right)\left(\frac{\partial p}{\partial t}+\mathbf{v}\cdot\nabla p\right)\right]_n=\\
&\quad\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}k}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left(p^{j_1}\theta^{j_2}\Delta\theta\right)_n\\
&\quad+\sum_{j_1+j_2=1}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}k}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n[\nabla\theta\cdot\nabla(p^{j_1}\theta^{j_2})]_n\\
&\quad+2\epsilon\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left(p^{j_1}\theta^{j_2}\|\mathbf{D}\|^2\right)_n\\
&\quad-\frac{2}{3}\epsilon\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n[p^{j_1}\theta^{j_2}(\mathrm{tr}\mathbf{D})^2]_n\\
&\quad+\epsilon r\left[1-\epsilon\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\sum_{m=0}^{+\infty}\epsilon^m(\theta^{j+1})_m+o(\epsilon)\right].
\end{aligned}
$$

We are now in position to equate the like powers of ϵ and obtain a sistematic hierarchy of equations. Collecting the terms of $O(1)$ in equations (2.40)-(2.42) we obtain

$$\operatorname{div}\mathbf{v}_0=0, \qquad (2.43)$$

$$-\nabla p_0+\mathbf{b}=\mathbf{0} \qquad (2.44)$$

and

$$c_p(p_0,\theta_0)\left(\frac{\partial\theta_0}{\partial t}+\mathbf{v}_0\cdot\nabla\theta_0\right)=k(p_0,\theta_0)\Delta\theta_0+\nabla[k(p_0,\theta_0)]\cdot\nabla\theta_0. \qquad (2.45)$$

We notice that the above equations are not sufficient to determine all the field variables at $O(1)$. Therefore, in order to attain closure, we proceed to obtain the equations at $O(\epsilon)$. Setting

$$G(\theta_0)=\int_0^{\theta_0}\alpha(\theta)\mathrm{d}\theta=\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\theta_0^{j+1},$$

from (2.41) we obtain

$$\frac{\partial \mathbf{v}_0}{\partial t}+\mathbf{v}_0\cdot\nabla\mathbf{v}_0 = -\nabla p_1 + \frac{1}{3}\mu(p_0,\theta_0)\nabla(\mathrm{div}\,\mathbf{v}_0) - \frac{2}{3}(\mathrm{div}\,\mathbf{v}_0)\nabla[\mu(p_0,\theta_0)] + 2\mathbf{D}_0\cdot\nabla[\mu(p_0,\theta_0)] + \mu(p_0,\theta_0)\Delta\mathbf{v}_0 - G(\theta_0)\mathbf{b}$$

which, in the light of (2.43), becomes

$$\frac{\partial \mathbf{v}_0}{\partial t}+\mathbf{v}_0\cdot\nabla\mathbf{v}_0 = -\nabla p_1 + 2\mathbf{D}_0\cdot\nabla[\mu(p_0,\theta_0)] + \mu(p_0,\theta_0)\Delta\mathbf{v}_0 - G(\theta_0)\mathbf{b}. \quad (2.46)$$

Now equations (2.43)-(2.46) form a closed system and it is interesting to remark that p_0 is the pressure due to the body forces acting on the fluid while p_1 is the pressure due to the thermal expansion of the fluid. Next, by means of (2.29) we re-dimensionalize equations (2.43)-(2.46) and obtain the equations governing the flows in a fluid layer at small second Froude number

$$\begin{cases} -\nabla p_0 + \rho_0\mathbf{b} = \mathbf{0} \\ \rho_0\left(\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0\cdot\nabla\mathbf{v}_0\right) = -\alpha_0(T_1-T_2)\nabla p_1 \\ \qquad +2\mathbf{D}_0\cdot\nabla[\mu(p_0,\theta_0)] + \mu(p_0,\theta_0)\Delta\mathbf{v}_0 - \rho_0 G(\theta_0)\mathbf{b} \\ \mathrm{div}\,\mathbf{v}_0 = 0 \\ \rho_0 c_p(p_0,\theta_0)\left(\frac{\partial \theta_0}{\partial t} + \mathbf{v}_0\cdot\nabla\theta_0\right) = k(p_0,\theta_0)\Delta\theta_0 \\ \qquad +\nabla[k(p_0,\theta_0)]\cdot\nabla\theta_0, \end{cases} \quad (2.47)$$

where the function G is now defined as

$$G(\theta_0) = \int_{T_0}^{\theta_0} \alpha(\theta)\mathrm{d}\theta.$$

It is easy to check that, if α, c_p, k and μ are assumed to depend only on temperature, system (2.47) simplifies to

$$\begin{cases} \rho_0\left(\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0\cdot\nabla\mathbf{v}_0\right) = -\nabla p + 2\mathbf{D}_0\cdot\nabla\mu(\theta_0) + \mu(\theta_0)\Delta\mathbf{v}_0 - \rho\mathbf{b} \\ \mathrm{div}\,\mathbf{v}_0 = 0 \\ \rho_0 c_p(\theta_0)\left(\frac{\partial \theta_0}{\partial t} + \mathbf{v}_0\cdot\nabla\theta_0\right) = k(\theta_0)\Delta\theta_0 + \nabla k(\theta_0)\cdot\nabla\theta_0, \end{cases} \quad (2.48)$$

where

$$p = p_0 + \alpha_0(T_1-T_2)p_1 \quad \text{and} \quad \rho = \rho_0[1-G(\theta_0)].$$

Finally, if α, c_p, k and μ are supposed to be constant, (2.48) reduces to the classical Oberbeck-Boussinesq equations [11, 17, 87]

$$\begin{cases} \rho_0\left(\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0\cdot\nabla\mathbf{v}_0\right) = -\nabla p + \mu\Delta\mathbf{v}_0 - \rho\mathbf{b} \\ \mathrm{div}\,\mathbf{v}_0 = 0 \\ \frac{\partial \theta_0}{\partial t} + \mathbf{v}_0\cdot\nabla\theta_0 = \kappa\Delta\theta_0, \end{cases}$$

where $\rho = \rho_0[1 - \alpha(\theta_0 - T_0)]$ and $\kappa = k/(\rho_0 c_p)$ is the *thermal diffusivity*.

2.5 Equations of magnetohydrodynamics

The objective of magnehydrodynamics (MHD) is the study of the ways in which magnetic fields can affect the behaviour of electrically conducting fluids. The electrical conductivity of the fluid and the embedding magnetic field contibute to effects of two kinds. First, as the electrically conducting fluid moves across the magnetic lines of force, electric currents are generated in the fluid (according to Faraday-Neumann-Lenz law) and the induced magnetic field contributes to change in the existing field. At the same time the fluid elements carrying currents transverse magnetic lines of force contribute to additional forces (Lorentz forces) which modify the motion and to additional supplies to internal energy due to Joule effect.

The equations governing the interactions between the electromagnetic field and the motion of an electrically conducting fluid are based upon the assumption of validity of Maxwell's equations. Since changes in time of electric and magnetic fields ($\mathbf{E}$ and $\mathbf{H}$, respectively) are determined by the instantaneous distribution of $\mathbf{E}$ and $\mathbf{H}$ and by the motion of the electric charges, irrespective of how this distribution and this motion are produced, Maxwell's equations are not formally altered by the fluid motion. Then, denoting by ϵ_e the dielectric constant of the fluid and by μ_e the magnetic permeability, we have

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \mathbf{D}_t \,, \tag{2.49}$$

$$\operatorname{curl} \mathbf{E} = -\mathbf{B}_t \,, \tag{2.50}$$

$$\operatorname{div} \mathbf{B} = 0 \,, \tag{2.51}$$

$$\operatorname{div} \mathbf{D} = \rho_e \,, \tag{2.52}$$

$$\mathbf{B} = \mu_e \mathbf{H} \quad \text{and} \quad \mathbf{D} = \epsilon_e \mathbf{E}, \tag{2.53}$$

where the vectors $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{J}$ are, respectively, the *magnetic induction*, the *electric induction* (or *displacement vector*) and the *current density*, $\mathbf{D}_t$ is the *displacement current* and the scalar quantity ρ_e represents the *electric charge density.* The current density $\mathbf{J}$, expressed through Ohm's law, is the sum of the *conduction current*

$$\sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

and the *convection current* $\rho_e \mathbf{v}$. The equation for the current density is therefore

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \rho_e \mathbf{v} \tag{2.54}$$

which is to be added to equations (2.49)-(2.53).

The governing equations of magnetohydrodynamics are obtained by coupling the equations of electromagnetism (2.49)-(2.54) and the equations of fluid dynamics (2.5), (2.27) and (2.28), the last two ones containing additional terms due to the interactions between the fluid and the electromagnetic field, viz

$$\rho\dot{\mathbf{v}} = -\nabla p + \frac{\mu}{3}\nabla(\operatorname{div}\mathbf{v}) - \frac{2}{3}(\operatorname{div}\mathbf{v})\nabla\mu + 2\mathbf{D}\cdot\nabla\mu + \mu\Delta\mathbf{v} + \rho\mathbf{b} + \mathbf{J}\times\mathbf{B} \quad (2.55)$$

and

$$\rho c_p\dot{\theta} - \alpha\theta\dot{p} = k\Delta\theta + \nabla k\cdot\nabla\theta + 2\mu\left[\|\mathbf{D}\|^2 - \frac{1}{3}(\operatorname{tr}\mathbf{D})^2\right] + \rho r + \frac{|\mathbf{J}|^2}{\sigma}. \quad (2.56)$$

$\mathbf{J}\times\mathbf{B}$ is the Lorentz force and $|\mathbf{J}|^2/\sigma$ is the heat produced by Joule effect.

Equations (2.5), (2.55) and (2.56) are invariant with respect to Galileian transformations whereas Maxwell's equations are invariant with respect to Lorentz's transformations. Thus, in order to obtain a coherent system of PDEs, as in most problems involving conductors, other than those concerned with rapid oscillations, the displacement current can be ignored so that, as it is well kown, also Maxwell's equations are invariant with respect to Galileian transformations (see [12]).

Let now L, t_0, V, E_0 and H_0 be typical values of lenght, time, velocity, electric and magnetic fields, respectively, and, by following Agostinelli [1], let us assume that

$$R_t = \frac{t_0 V}{L} \simeq 1, \quad (2.57)$$

$$R_e = \frac{E_0}{\mu_e H_0 V} \simeq 1 \quad (2.58)$$

and

$$\frac{V^2}{c^2} = V^2\epsilon_e\mu_e \ll 1. \quad (2.59)$$

By assumption (2.57) we do not consider high frequency phenomena. Condition (2.58) is a good approximation for fluids having a very large electric conductivity because, for $\sigma\to\infty$, from (2.54) one has $\mathbf{E} = -\mu_e\mathbf{v}\times\mathbf{H}$ and hence $|\mathbf{J}|$ is of order $\sigma\mu_e H_0 V$. Finally by (2.59) we assume that the fluid velocity is much smaller than the light speed in the fluid. As consequences of (2.57)-(2.59) we shall see that the displacement and the convection currents can be neglected. We first introduce the following scaling

$$\begin{gathered}\mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad t^* = \frac{t}{t_0}, \quad \mathbf{v}^* = \frac{\mathbf{v}}{V}\\ \mathbf{E}^* = \frac{\mathbf{E}}{E_0}, \quad \mathbf{H}^* = \frac{\mathbf{H}}{H_0}, \quad \mathbf{J}^* = \frac{\mathbf{J}}{\mu_e\sigma V H_0},\end{gathered} \quad (2.60)$$

and the magnetic Reynolds number

$$R_m = \frac{VL}{\eta},$$

$\eta = (\mu_e \sigma)^{-1}$ being the *magnetic visosity* (or *magnetic diffusivity*). Then, introducing the non dimensional quantities (2.60) into equation (2.49) yields (omitting all asterisks)

$$\frac{1}{R_m}\text{curl}\,\mathbf{H} = \mathbf{J} + \frac{R_c R_e}{R_t R_m}\mathbf{E}_t.$$

By assumptions (2.57)-(2.59) we can thus ignore the displacement current and so (2.49) becomes

$$\text{curl}\,\mathbf{H} = \mathbf{J}. \tag{2.61}$$

In a similar way, writing equation (2.54) by taking into account (2.52) and $(2.53)_2$, the dimensionless equation for the current density is

$$\mathbf{J} = (R_e\mathbf{E} + \mathbf{v}\times\mathbf{H}) + \frac{R_c R_e}{R_m}(\text{div}\,\mathbf{E})\mathbf{v}.$$

This equation shows that, since $\dfrac{R_c R_e}{R_m} \ll 1$, the convective current is negligible with respect to the conduction current, and so

$$\mathbf{J} = \sigma(\mathbf{E} + \mu_e\mathbf{v}\times\mathbf{H}). \tag{2.62}$$

We now observe that, by taking the curl of both sides of equation (2.61), by means of equations (2.50), (2.51), $(2.53)_1$ and (2.62) we get

$$\mathbf{H}_t + \text{curl}\,(\mathbf{H}\times\mathbf{v}) = \eta\Delta\mathbf{H}.$$

Moreover, by (2.62) and $(2.53)_1$, the Lorentz force is given by

$$\mathbf{J}\times\mathbf{B} = \mu_e\text{curl}\,\mathbf{H}\times\mathbf{H} = \mu_e\mathbf{H}\cdot\nabla\mathbf{H} - \frac{\mu_e}{2}\nabla|\mathbf{H}|^2,$$

$(\mu_e\nabla|\mathbf{H}|^2)/2$ being the *magnetic pressure*, and the heat produced by Joule effect is

$$\frac{|\mathbf{J}|^2}{\sigma} = \frac{|\text{curl}\,\mathbf{H}|^2}{\sigma}.$$

Therefore the governing equations of non relativistic MHD are

$$\begin{cases} \dot{\rho} + \rho \mathrm{div}\, \mathbf{v} = 0 \\ \rho \dot{\mathbf{v}} = -\nabla \left(p + \frac{\mu_e}{2} \nabla |\mathbf{H}|^2 \right) + \frac{\mu}{3} \nabla (\mathrm{div}\, \mathbf{v}) - \frac{2}{3} (\mathrm{div}\, \mathbf{v}) \nabla \mu \\ \qquad +2\mathbf{D} \cdot \nabla \mu + \mu \Delta \mathbf{v} + \rho \mathbf{b} + \mu_e \mathbf{H} \cdot \nabla \mathbf{H} \\ \rho c_p \dot{\theta} - \alpha \theta \dot{p} = k \Delta \theta + \nabla k \cdot \nabla \theta + 2\mu \left[\|\mathbf{D}\|^2 - \frac{1}{3} (\mathrm{tr} \mathbf{D})^2 \right] \\ \qquad\qquad + \rho r + \dfrac{|\mathrm{curl}\, \mathbf{H}|^2}{\sigma} \\ \mathbf{H}_t + \mathrm{curl}\, (\mathbf{H} \times \mathbf{v}) = \eta \Delta \mathbf{H} \\ \mathrm{div}\, \mathbf{H} = 0 \end{cases} \tag{2.63}$$

and form a coherent system of PDEs.

2.6 Porous media

By a *porous medium* we mean a material consisting of a solid matrix with interconnected void. We suppose that the solid matrix is rigid. The interconnectedness of the void (the *pores*) allows the flow of one or more fluids through the material. In the simplest situation (the single-phase flow) the void is saturated by a single fluid. In two-phase flow a liquid and a gas share the void space. Here we shall discuss the former situation.

In a natural porous medium the distribution of pores with respect to shape and size is irregular. Examples of natural porous media are beach sand, sandstone, limestone, wood and human lung. On the pore scales (the microscopic scale) the flow quantities (velocity, pressure, etc.) will clearly be irregular. But in typical experiments the quantities of interest are measured over volumes that contain many pores. Such space-averaged (macroscopic) quantities change in a regular manner with respect to space and time and hence are amenable to theoretical treatment.

The usual way of deriving the laws governing the macroscopic variables is to begin with the standard equations obeyed by the fluid and to obtain the macroscopic equations by averaging over volumes containing many pores. In this approach, a macroscopic variable is defined as an appropriate mean over a sufficiently large *representative elementary volume* (r.e.v.); this operation yields the value of that variable at the centre of the r.e.v.. It is assumed that the result is independent of the size of the representative elementary volume. The lenght scale of r.e.v. is much larger than the pore scale, but considerably smaller than the lenght scale of the macroscopic flow domain.

2.7 Porosity, seepage velocity and the equation of continuity

The porosity φ of a porous medium is defined as the fraction of the total volume of the medium that is occupied by void space, that is

$$\varphi = \frac{\text{total volume of the pores}}{\text{total volume of the medium}}.$$

Thus $1 - \varphi$ is the fraction that is occupied by the solid. For an isotropic medium the *surface porosity* (i.e. the fraction of void area to total area of a typical cross section) will normally be equal to φ.

For natural media, φ does not normally exceed 0.6. Nonuniformity of grain size tends to lead to smaller porosities than for uniform grains. For man-made materials such as metallic foams φ can approach the value 1.

We construct a continuum model for a porous medium based on the r.e.v. concept. We introduce a Cartesian frame of reference and consider volume elements that are sufficiently large compared with the pore volumes in order to obtain reliable volume averages. In other words, the averages are not sensitive to the choice of volume element. A distinction is made between an average taken with respect to a volume element V_m (incorporating both solid and fluid material), and one taken with respect to a volume V_f consisting of fluid only. For example, we denote the average of fluid velocity over V_m by $\mathbf{v}$ which is usually called the *seepage velocity.* Taking an average of the fluid velocity over a volume V_f we get the intrinsic average velocity $\mathbf{V}$, which is related to $\mathbf{v}$ by the Dupuit-Forchheimer relationship

$$\mathbf{v} = \varphi \mathbf{V}. \tag{2.64}$$

Once we have a continuum to deal with, we can apply the usual arguments of section 2.1 to derive differential equations expressing conservation laws. For instance, denoting by ρ_f the fluid density and considering an elementary unit volume of the medium V, the conservation of mass is expressed by

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \varphi\rho_f \mathrm{d}V &= \int_V \left[\frac{\partial(\varphi\rho_f)}{\partial t} + \mathrm{div}(\varphi\rho_f\mathbf{V})\right]\mathrm{d}V \qquad (2.65)\\ &= \int_V \left[\varphi\frac{\partial\rho_f}{\partial t} + \mathrm{div}(\rho_f\mathbf{v})\right]\mathrm{d}V, \end{aligned}$$

where we have taken into account (2.64) and that φ is independent of t. By (2.65) we then deduce the continuity equation in a porous medium

$$\varphi\frac{\partial\rho_f}{\partial t} + \mathrm{div}(\rho_f\mathbf{v}) = 0. \tag{2.66}$$

2.8 Linear momentum equation in a porous medium

Following the same arguments that lead to the equation of continuity in a porous medium, we shall derive the most general equation of balance of linear momentum when the porous medium is isotropic and homogeneous, i.e. the porosity φ is constant. Let V be an elementary unit volume of the medium and equate the rate of change of the linear momentum of the fluid within that volume to the net forces acting on the fluid into the volume V:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varphi\rho_f \mathbf{V}\mathrm{d}V = \int_{\partial V} \varphi \mathbf{T}\cdot\mathbf{n}\mathrm{d}A + \int_V \varphi\rho_f \mathbf{b}\mathrm{d}V + \int_V \varphi \mathbf{I}\mathrm{d}V, \tag{2.67}$$

where

$$\mathbf{T} = -p\mathbf{1} + 2\mu_f\left[\left(\frac{\partial \mathbf{V}}{\partial \mathbf{x}}\right) + \left(\frac{\partial \mathbf{V}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\right]$$

is the stress tensor in the fluid, μ_f being the fluid viscosity which, for simplicity, is now assumed to be a constant, $\mathbf{b}$ is the body force and $\mathbf{I}$ is the density of interaction forces between the fluid and the porous matrix. By Reynolds' transport Theorem (2.1), the arbitrariety of the volume V, the Dupuit-Forchheimer relationship (2.64) and the equation of continuity (2.66), (2.67) yields the local balance of momentum

$$\frac{\rho_f}{\varphi}\mathbf{v}_t + \frac{\rho_f}{\varphi^2}\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \frac{\mu_f}{\varphi}\left[\Delta\mathbf{v} + \nabla(\mathrm{div}\,\mathbf{v})\right] + \mathbf{I} + \rho_f\mathbf{b}. \tag{2.68}$$

We now discuss various approximated forms of the momentum equation (2.68) and the basic assumptions which justify them.

2.8.1 Darcy's law

Darcy's investigations on steady-state flow in a uniform porous medium [13] revealed an equation for the linear momentum of the type

$$\nabla p = -\frac{\mu}{\varphi K}\mathbf{v} + \rho_f\mathbf{b}, \tag{2.69}$$

where μ is the *dynamic viscosity* of the fluid and the coefficient K is independent of the nature of the fluid but it depends on the geometry of the medium. K has dimensions (lenght)^2 and is called the *permeability* of the medium.

Following Rajagopal [66], the basic assumptions leading to (2.69) are that:

(i) The solid is a rigid porous body and thus the balance of the linear momentum of the solid can be ignored.

(ii) The only interaction forces that come into play are due to frictional forces the fluid encounters at the boundaries of the pores. This can be modelled by a drag term proportional to the fluid velocity. The coefficient of proportionality being a constant.

(iii) The frictional effects within the fluid due to its viscosity can be neglected.

(iv) The flow is sufficiently slow that the inertial nonlinearities can be neglected.

(v) The flow is steady.

(vi) The density of the fluid is constant.

(vii) The stress for the fluid is that for an ideal Euler fluid as the frictional effects in the fluid can be neglected with respect to the frictional effects in the pore (which has already been incorporated in the interaction term).

Assumption (i) implies that we need to concern ourselves with only the balance of linear momentum for the fluid as the porous matrix is rigid and and does not deform. Thus on fixing the frame to the porous matrix the velocity of the solid is zero. Next, assumption (ii) implies that

$$\mathbf{I} = -\frac{\mu}{K}\mathbf{V},$$

where the *dynamic viscosity* μ is usually assumed to be a constant.

Assumptions (iv) and (v) imply that the inertial terms in the right-hand side of (2.68) can be ignored.

Assumption (iii) implies that as far as the response of the fluid is concerned, the effects of viscosity (frictional effects) can be neglected with respect to the friction that manifests itself due to the flow in the pores. This does not mean that the fluid has no viscosity. In fact, assumptions (ii) and (iii) together imply that the viscosity of the fluid and the roughness of the solid surface lead to far greater frictional resistance at the porous boundaries of the solid in comparison to the frictional dissipation in the fluid, but these two assumptions do not necessarily imply that the fluid stress tensor is that for an Euler fluid. Only by assumption (vii) we can approximate the Cauchy stress tensor of the fluid as $\mathbf{T} = -p\mathbf{1}$.

Finally, since ρ_f is constant, the fluid can undergo only isochoric motions and the equation of continuity (2.66) reduces to

$$\text{div}\,\mathbf{v} = 0. \tag{2.70}$$

Equations (2.69) and (2.70) constitute what is referred as Darcy's law. The subsequent generalizations of (2.69) [56] (such as that carried out by Forchheimer [21]) can be easily obtained by modifying the form of the interaction term.

2.8.2 Brinkman's equations

Let us now relax some of the assumptions (*i*)-(*vii*). We shall not enforce the assumptions (*iii*) and (*vii*) while we shall retain the other ones. We shall then include the frictional forces in the fluid when we consider the balance of linear momentum. The equation of balance of linear momentum (2.68) then becomes

$$-\nabla p + \frac{\mu_f}{\varphi}\Delta\mathbf{v} - \frac{\mu}{\varphi K}\mathbf{v} + \rho_f\mathbf{b} = \mathbf{0}. \tag{2.71}$$

Let us observe that whenever the lenght scale is much greater than $(\mu_f K/\mu)^{1/2}$, the Laplacian term in equation (2.71) is negligible in comparison with the term proportional to $\mathbf{v}$ so that the Brinkman's equation reduce to Darcy's equation. In fact, if we introduce the following dimensionless quantities

$$\mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad \mathbf{b}^* = \frac{\mathbf{b}}{g}, \quad \mathbf{v}^* = \frac{\mu}{\rho_f g K}\mathbf{v}, \quad p^* = \frac{p}{\rho_f g d}, \tag{2.72}$$

where d is the lenght scale of the porous medium and g is the acceleration due to gravity, and substitute (2.72) into (2.71), we obtain the non-dimensional Brinkman's equation (omitting all asterisks)

$$-\nabla p + \frac{\mu_f K}{\mu d^2}\Delta\mathbf{v} - \mathbf{v} + \mathbf{b} = \mathbf{0}. \tag{2.73}$$

Therefore if

$$\frac{\mu_f K}{\mu d^2} \ll 1$$

(2.73) reduces to the dimensionless version of (2.69).

If we do not require the flow to be steady but assume that it is sufficiently slow that inertial nonlinearities can be neglected we get the unsteady Brinkmann's equation

$$\frac{\rho_f}{\varphi}\mathbf{v}_t = -\nabla p + \frac{\mu_f}{\varphi}\Delta\mathbf{v} - \frac{\mu}{\varphi K}\mathbf{v} + \rho_f\mathbf{b}.$$

Neglecting the frictional effects in the fluid, the above equation will lead to the unsteady version of Darcy's equation.

2.9 Energy equation in a porous medium

We now focus on the equation that express the balance of internal energy in a porous medium. We concentrate our attention on the simplest situation in which the medium is isotropic, homogeneous and where radiative effects, viscous dissipation and the work done by the pressure changes are negligible. Very shortly we shall assume that there is local equilibrium so that $T_s = T_f = T$, where T_s and T_f are the temperature of the solid matrix and of the fluid, respectively. Moreover we also assume that heat conduction in the porous matrix and in the fluid takes place in parallel so that there is no net heat transfer from one constituent to the other. More complex situations are considered in the book of Nield and Bejan Chapter 2 and Section 6.5.

Taking averages over an elemental volume of the medium we have, for the solid matrix,

$$(1-\varphi)(\rho c)_s \frac{\partial T_s}{\partial t} = (1-\varphi)\mathrm{div}(k_s \nabla T_s) \tag{2.74}$$

and, for the fluid,

$$\varphi(\rho c_p)_f \frac{\partial T_f}{\partial t} + (\rho c_p)_f \mathbf{v}\cdot\nabla T_f = \varphi \mathrm{div}(k_f \nabla T_f). \tag{2.75}$$

Here the subscripts s and f refer to the solid matrix and to the fluid, respectively, c is the specific heat of the solid, c_p is the specific heat at constant pressure of the fluid and k is the heat conductivity.

In writing equations (2.74) and (2.75) we have assumed that the surface porosity is equal to the porosity. This is pertinent to the conduction terms. For instance, $-k_s\nabla T_s$ is the conductive heat flux through the solid and thus $\mathrm{div}(k_s\nabla T_s)$ is the net rate of heat conduction into a unit volume of the solid. In equation (2.74) this appears multiplied by the factor $1-\varphi$ which is the ratio of the cross-sectional area of the medium. The other term in equation (2.74) contains the factor $1-\varphi$ because this is the ratio of the volume occupied by the solid to the total volume of the element. In equation (2.75) there also appears a convective term, due to the seepage velocity. We recognize that $\mathbf{V}\cdot\nabla T_f$ is the rate of change of temperature in the elemental volume due to the convection of the fluid into it, so this, multiplied by $(\rho c_p)_f$, must be the rate of change of thermal energy, per unit volume of the fluid, due to the convection. Note that in writing equation (2.75) we have used the Dupuit-Forchheimer relationship (2.64).

Setting $T_s = T_f = T$ and adding equations (2.74) and (2.75) we have

$$(\rho c)_m \frac{\partial T}{\partial t} + (\rho c_p)_f \mathbf{v}\cdot\nabla T = \mathrm{div}(k_m \nabla T), \tag{2.76}$$

where

$$(\rho c)_m = (1 - \varphi)(\rho c)_s + \varphi(\rho c_p)_f$$

and

$$k_m = (1 - \varphi)k_s + \varphi k_f$$

are, respectively, the overall heat capacity per unit volume and the overall thermal conductivity of the medium.

If the work done by the pressure changes is not negligible, then a term $-\alpha_f T(\varphi \partial p/\partial t + \mathbf{v} \cdot \nabla p)$ needs to be added to the left hand side of equation (2.76). Here α_f is the coefficient of volumetric thermal expansion of the fluid defined in (2.26). In natural convection the work done by the pressure changes is negligible if

$$\frac{g\alpha_f d}{c_{pf}} \ll 1, \tag{2.77}$$

d being a characteristic lenght scale of the medium, as one can easiliy deduce from the non-dimensional analysis performed in section 2.4. In natural convection the condition (2.77) is usually verified.