Appendix B

Smooth domains and regularity properties of the distance function

In this Appendix we collect some regularity results of the distance function $r(x) = \mathrm{dist}(x, \partial\Omega)$, when $\partial\Omega$ is the boundary of a smooth open subset Ω of $\mathbb{R}^N$. These results are well-known in the case where Ω is bounded (see e.g. [26, section 14.6]), but most of them may be extended, without much effort, to the unbounded case, as it is shown below.

First we define open sets with uniformly $C^{2+\alpha}$ boundaries, for $0 \leq \alpha < 1$.

Definition B.0.15 *Let Ω be an open subset of $\mathbb{R}^N$. We say that $\partial\Omega$ is uniformly of class $C^{2+\alpha}$ if there exist a covering of $\partial\Omega$, at most countable, $\{U_j\}_{j\in\mathbb{N}}$, and a sequence of diffeomorphisms $\varphi_j : \overline{U}_j \to \overline{B}_1$ of class $C^{2+\alpha}$ such that*

$$\begin{aligned} \varphi_j(U_j \cap \Omega) &= \{y \in B_1 \,|\, y_N > 0\} \\ \varphi_j(U_j \cap \partial\Omega) &= \{y \in B_1 \,|\, y_N = 0\} \end{aligned}$$

and the following properties are satisfied:

(i) there exists $k \in \mathbb{N}$ such that $\bigcap_{j\in J} U_j = \emptyset$, if $|J| > k$;

(ii) there exists $0 < \varepsilon < 1$ such that $\{x \in \Omega \mid r(x) < \varepsilon\} \subseteq \bigcup_{j\in\mathbb{N}} V_j$, where $V_j = \varphi_j^{-1}(B_{1/2})$;

(iii) there exists $C > 0$ such that

$$\sup_{j\in\mathbb{N}} \sum_{0\leq|\beta|\leq 2+\alpha} \|D^\beta \varphi_j\|_\infty + \|D^\beta \varphi_j^{-1}\|_\infty \leq C.$$

Now we show that such a set Ω satisfies a *uniform interior sphere condition*, i.e. at each point $y_0 \in \partial\Omega$ there exists a ball B_{y_0} depending on y_0, contained in Ω and such that $\overline{B}_{y_0} \cap \partial\Omega = \{y_0\}$; moreover the radii of these balls are bounded from below by a positive constant.

Proposition B.0.16 *If $\partial\Omega$ is uniformly of class C^2, then it satisfies a uniform interior sphere condition.*

Proof. Using condition (iii) and taking into account that φ_j is a diffeomorphism from $\overline{U}_j$ into $\overline{B}_1$, it is easy to see that if $y \in V_j$ and $|x - y| < 1/(2C)$, then $x \in U_j$.

Let $y_0 \in \partial\Omega$ and let $\eta(y_0)$ denote the unit inward normal vector to $\partial\Omega$ at y_0. For $0 \leq t < 1/(2C)$ the point $x = y_0 + t\eta(y_0)$ belongs to U_j and ($\varphi_j^{(N)}$ denotes the N-th component of φ_j)

$$\varphi_j^{(N)}(x) = tD\varphi_j^{(N)}(y_0) \cdot \eta(y_0) + R(t)$$

with $|R(t)| \leq Ct^2/2$. Since $\varphi_j^{(N)} = 0$ on $U_j \cap \partial\Omega$, then $D\varphi_j^{(N)}(y_0) = k\eta(y_0)$, with $k \geq C^{-1}$, by (iii). This yields $\varphi_j^{(N)}(x) \geq tC^{-1} - Ct^2/2 > 0$ for $0 < t < 2/C^3 := \delta$.

Thus, we have proved that

$$y + t\eta(y) \in \Omega, \qquad y \in \partial\Omega,\ t \in]0, \delta[.$$

Now, let $y \in \partial\Omega$ and set $B = B(z, \delta/2)$, where $z = y + \eta(y)\delta/2$. Then, it is easy to see that $B \subset \Omega$ and $y \in \partial B$. If y is not the unique point in $\partial\Omega \cap \partial B$, then it suffices to replace the above ball with that of radius $\delta/4$, centered at $z = y + \eta(y)\delta/4$. □

We are now ready to prove the properties of the distance function used in this paper.

Proposition B.0.17 *Assume that $\partial\Omega$ is uniformly of class C^2 and let δ be a positive constant such that at each point of $\partial\Omega$ there exists a ball which satisfies the interior sphere condition at y_0 with radius greater or equal to δ. Then*

(a) for every $x \in \Omega_\delta = \{y \in \overline{\Omega} \,|\, r(y) < \delta\}$ there exists a unique $\xi = \xi(x) \in \partial\Omega$ such that $|x - \xi| = r(x)$;

(b) $r \in C_b^2(\Omega_\delta)$;

(c) $Dr(x) = \eta(\xi(x))$, for every $x \in \Omega_\delta$.

Proof. (a) The existence part is obvious. For the uniqueness assertion, let $x \in \Omega_\delta$ and $y \in \partial\Omega$ such that $r(x) = |x - y|$. From Proposition B.0.16 there exists a ball $B = B(z, \rho)$ such that $B \subset \Omega$ and $\overline{B} \cap \partial\Omega = \{y\}$. Moreover from the definition of δ, $x \in B$. It is easy to see that x and z lie on the normal direction $\eta(y)$ and that the balls $B(x, r(x))$ and $B(z, \rho)$ are tangent at y. Then $B(x, r(x))$ still verifies the interior sphere condition at y. It follows that for every $\overline{y} \in \partial\Omega \setminus \{y\}$, one has $\overline{y} \notin B(x, r(x))$, so that y is actually the unique point such that $|x - y| = r(x)$.

The proof of the last two assertions relies on the first statement and the implicit function theorem and it is completely similar to that of the case Ω bounded. We refer to [26, section 14.6].

□