Regularity properties for second order partial differential operators with unbounded coefficients

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Contents

Introduzione

Operatori differenziali lineari ellittici e parabolici con coefficienti limitati e regolari sono stati oggetto di uno studio vasto e accurato negli ultimi decenni, il quale ha prodotto una teoria completa ed esauriente che comprende risultati di esistenza, unicità e regolarità per le soluzioni delle equazioni associate in vari spazi funzionali, come spazi L^p, spazi di Hölder e altri. Al momento, la letteratura dimostra un crescente interesse verso operatori con coefficienti illimitati o singolari, che generalizzano in modo naturale quelli classici. Tale interesse è sicuramente motivato dalle applicazioni alla probabilità e specialmente alle equazioni differenziali stocastiche e alla matematica finanziaria. Tra l'altro, il prototipo di questi operatori, l'operatore di Ornstein-Uhlenbeck, proviene proprio dalla probabilità.

Bisogna osservare subito che i risultati forniti dalla teoria classica non si estendono in modo ovvio al caso di coefficienti illimitati. Per esempio, è ben noto che il Laplaciano con dominio $W^{2,p}(\mathbb{R}^N)$ genera un semigruppo analitico fortemente continuo in $L^p(\mathbb{R}^N)$ $(1 < p < \infty)$. Lo strumento principale per stabilire questo risultato è rappresentato dalla stima fondamentale di Calderon-Zygmund. Aggiungiamo al Laplaciano un termine di ordine zero illimitato, consideriamo pertanto un operatore di Schrödinger $A = \Delta - V$. Se V è in $L^2_{\text{loc}}(\mathbb{R}^N)$ ed è positivo, allora, mediante il metodo delle forme quadratiche non è difficile provare che A, con il dominio $D(A)$ dettato dalla forma associata, genera un semigruppo analitico in $L^2(\mathbb{R}^N)$. È naturale a questo punto chiedersi se $D(A)$ coincide con l'intersezione dei domini dei singoli addendi di A oppure no. Se il potenziale V verifica la condizione di oscillazione $|DV| \leq \gamma V^{3/2}$, con una costante γ abbastanza piccola, allora la risposta è affermativa e lo stesso risultato vale peraltro anche con $p \neq 2$. Ma c'è un esempio in [41] di un operatore di Schrödinger in $L^2(\mathbb{R}^3)$ il cui potenziale verifica la condizione precedente con una costante γ non sufficientemente piccola e che genera un semigruppo con dominio che contiene propriamente l'intersezione dei domini. Chiaramente la condizione $|DV| \leq \gamma V^{3/2}$ consente una crescita polinomiale, che non costituisce una piccola perturbazione della parte principale di A, ossia del Laplaciano.

L'esempio prodotto rivela il fatto che la teoria degli operatori a coefficienti illimitati presenta degli aspetti abbastanza diversi, e non ancora completamente chiari, da quelli della teoria classica.

L'obiettivo di questa tesi è lo studio di proprietà di regolarità di operatori ellittici del secondo ordine a coefficienti regolari, ma illimitati in $\mathbb{R}^N$ o in suoi sottoinsiemi aperti illimitati.

Nel primo capitolo consideriamo il seguente operatore ellittico in forma divergenza

$$A = \sum_{i,j=1}^{N} D_i(q_{ij} D_j) + \langle F, D \rangle - V,$$

in $L^p(\mathbb{R}^N)$, $1 < p < \infty$ e studiamo condizioni sui coefficienti che assicurano che l'operatore genera un semigruppo fortemente continuo in $L^p(\mathbb{R}^N)$ con caratterizzazione del dominio. In particolare, dimostriamo che, sotto opportune ipotesi sui coefficienti e sulle loro derivate, l'operatore $(A, \mathcal{D}_p)$

genera un semigruppo, dove $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$ è lo spazio di Banach così definito

$$\mathcal{D}_p := \{u \in W^{2,p}(\mathbb{R}^N) \,:\, \langle F, Du\rangle \in L^p(\mathbb{R}^N), Vu \in L^p(\mathbb{R}^N)\},$$
$$\|u\|_{\mathcal{D}_p} := \|u\|_{W^{2,p}(\mathbb{R}^N)} + \|\langle F, Du\rangle\|_{L^p(\mathbb{R}^N)} + \|Vu\|_{L^p(\mathbb{R}^N)}.$$

Ciò implica risultati di regolarità ottimale per le soluzioni dell'equazione ellittica $\lambda u - Au = f$, poichè supponendo soltanto $u, \lambda u - Au \in L^p(\mathbb{R}^N)$, si ricava che $u \in W^{2,p}(\mathbb{R}^N), \langle F, Du\rangle$, $Vu \in L^p(\mathbb{R}^N)$. Un passo fondamentale nella dimostrazione di questo risultato è costituito da stime a priori della forma

$$\|u\|_{\mathcal{D}_p} \leq C\big(\|u\|_{L^p(\mathbb{R}^N)} + \|Au\|_{L^p(\mathbb{R}^N)}\big),$$

con $u \in \mathcal{D}_p$ e C costante indipendente da u. Quella per le derivate seconde è esattamente l'analogo della stima di Calderon-Zygmund per il Laplaciano e come questa è delicata da provare. Le stime precedenti implicano facilmente la chiusura dell'operatore $(A, \mathcal{D}_p)$. Segue anche in modo semplice la quasi-dissipatività di A, cioè la dissipatività di $A - \omega$, per un'opportuna costante $\omega \in \mathbb{R}$. Per applicare il teorema di generazione di Hille-Yosida, rimane solo da verificare la suriettività di $\lambda - A$ da $\mathcal{D}_p$ su $L^p(\mathbb{R}^N)$, per λ abbastanza grande. Ciò è provato mediante un procedimento di approssimazione che sfrutta casi già noti in letteratura. Tale procedimento distingue il caso $p = 2$ da quello $p \neq 2$. Questo fatto tuttavia risulta abbastanza frequente. I risultati ottenuti sono ispirati dai lavori [41] e [37], ma offrono anche nuovi casi non presenti in letteratura.

Nel secondo capitolo l'attenzione è rivolta all'operatore in forma non divergenza

$$\mathcal{A} = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \langle F, D\rangle - V,$$

nello spazio delle funzioni continue e limitate in $\overline{\Omega}$, $C_b(\overline{\Omega})$, dove Ω è un aperto illimitato di $\mathbb{R}^N$. L'ambientazione in un aperto generico e non in tutto lo spazio costituisce un elemento di novità, giacchè il caso $\Omega = \mathbb{R}^N$ è quello più largamente studiato in letteratura. Risulta altresì significativo l'approccio puramente analitico, visto che spesso risultati affini sono ottenuti mediante metodi probabilistici. Lo scopo del capitolo è quello di fornire delle ipotesi sui coefficienti di $\mathcal{A}$ affinchè il problema di Neumann

$$\begin{cases} u_t(t,x) - \mathcal{A}u(t,x) = 0 & t > 0, \quad x \in \overline{\Omega} \\ \dfrac{\partial u}{\partial \eta}(t,x) = 0 & t > 0, \quad x \in \partial\Omega \\ u(0,x) = f(x) & \qquad\quad\; x \in \overline{\Omega} \end{cases} \tag{0.0.1}$$

abbia un'unica soluzione classica limitata il cui gradiente spaziale soddisfa delle stime opportune. Il metodo usato per provare l'esistenza di tale soluzione consiste nel considerare una successione di soluzioni di problemi di Neumann in aperti limitati invadenti $\overline{\Omega}$, e nel far vedere che tale successione converge. La scelta di condizioni al bordo di Neumann non permette di avere una successione monotona (contrariamente al caso di condizioni di Dirichlet). Dunque, lo strumento principale usato per provare la convergenza è rappresentato dalle stime classiche di Schauder. La funzione limite così ottenuta è l'unica soluzione classica limitata del problema (0.0.1) (l'unicità è assicurata dall'ipotesi che esista una funzione di Liapunov opportuna). Associando ad ogni dato iniziale la soluzione costruita, è possibile definire un semigruppo di operatori lineari e limitati $(P_t)_{t\geq 0}$ in $C_b(\overline{\Omega})$, non fortemente continuo in generale (questo fatto è tipico per semigruppi associati ad operatori con coefficienti illimitati). Pertanto non si può definire il generatore in senso classico. Tuttavia, si può introdurre il cosiddetto generatore "debole", che nella situazione considerata, coincide con l'operatore di partenza.

La parte più importante del capitolo consiste nel provare delle stime sul gradiente del semigruppo. La prima stima è

$$|DP_tf(x)| \leq \frac{C_T}{\sqrt{t}}\|f\|_\infty \qquad 0 < t < T,\ x \in \overline{\Omega}, \tag{0.0.2}$$

che viene provata con il metodo di Bernstein. Sostanzialmente, si tratta di applicare il principio del massimo alle funzioni $z_n = u_n^2 + at|Du_n|^2$, dove u_n è la successione approssimante P_tf e a è un opportuno parametro positivo. Per far questo, il punto cruciale consiste nell'assumere che Ω sia convesso per dimostrare che ogni z_n ha derivata normale non positiva al bordo. Si ottengono così delle stime per $|Du_n|$, che al tendere di n all'infinito forniscono la stima (0.0.2). Nel caso di un dato iniziale più regolare, lo stesso metodo produce anche la seguente stima

$$|DP_tf(x)| \leq C_T(\|f\|_\infty + \|Df\|_\infty) \qquad 0 \leq t \leq T,\ x \in \overline{\Omega}$$

la quale implica che il dominio del generatore è contenuto in $C_b^1(\overline{\Omega})$. A differenza del caso L^p, in questo contesto non abbiamo la caratterizzazione del dominio, di conseguenza, anche i risultati di regolarità che se ne possono dedurre sono parziali. Oltre a stime *uniformi* nello stesso capitolo dimostriamo anche stime *puntuali* per il gradiente di P_tf. Queste ultime sono utili nello studio della realizzazione del semigruppo negli spazi $L^p(\Omega, \mu)$, dove μ è la misura invariante di (P_t) (quando esiste).

Nel terzo capitolo proviamo stime uniformi per il gradiente delle soluzioni di problemi parabolici del tipo (0.0.1) in domini illimitati Ω, con condizioni al bordo di Dirichlet. Se da un lato è immediato provare l'esistenza della soluzione classica limitata, per approssimazione, come nel caso precedente, dall'altro è più delicato provare la stima (0.0.2). La difficoltà consiste nel fatto che non è dato conoscere il valore al bordo delle funzioni alle quali si applica il metodo di Bernstein. Per superare tale ostacolo, mediante il confronto con un operatore unidimensionale, proviamo dapprima una stima al bordo per il gradiente della soluzione del problema in tutto Ω. Quindi, con il metodo di Bernstein proviamo la stima anche all'interno di Ω. C'è da notare che questa procedura richiede delle ipotesi ulteriori di regolarità per la soluzione. Per trattare il caso generale ricorriamo ancora una volta ad una tecnica di approssimazione.

Nel capitolo successivo, studiamo l'operatore unidimensionale $Au = au'' + bu'$ in $C_b(\mathbb{R})$, spazio delle funzioni continue e limitate in $\mathbb{R}$ e in $C(\overline{\mathbb{R}})$, spazio delle funzioni continue aventi limiti finiti a $\pm\infty$. Il risultato principale dimostra, in ciascuno dei due casi, che l'operatore genera un semigruppo con dominio costituito dall'intersezione dei domini di ogni addendo dell'operatore. Purtroppo il metodo impiegato resta genuinamente unidimensionale e conferma la difficoltà di avere informazioni sul dominio quando $p = \infty$ e si è in più dimensioni.

Infine, l'ultimo capitolo raccoglie prevalentemente alcuni fatti noti su misure invarianti associate a semigruppi di Feller in $C_b(\mathbb{R}^N)$. La trattazione poteva essere fatta in maggiore generalità, ma è stato scelto un livello più vicino al caso concreto maggiormente ricorrente, che è quello di semigruppi di Feller generati da operatori differenziali ellittici del secondo ordine.

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Introduction

Linear elliptic and parabolic operators with regular and bounded coefficients have nowadays a satisfactory theory including existence, uniqueness and regularity for the solutions to the corresponding equations in several functional spaces, such as L^p spaces, Hölder spaces and so on. Recently, the literature shows an increasing interest towards operators with unbounded or singular coefficients. Motivations come from probability and in particular from stochastic analysis. Indeed, there is a strong connection between second order differential operators and Markov processes. We briefly describe it. Let $\Xi = \{\xi_t\}$ be a Markov process in a probability space $(\Omega, \mathcal{F}, P)$, with state space $\mathbb{R}^N$. The corresponding transition probabilities $p(t, x, B)$, for $t > 0, x \in \mathbb{R}^N$, B Borel set of $\mathbb{R}^N$, represent the probability that Ξ reaches the set B at time t starting from x at $t = 0$. Given the initial distribution μ of Ξ, in order to reconstruct the process it is sufficient to determine the family of measures $p(t, x, \cdot)$ since, by the formula of total probability, one has

$$P(\xi_t \in B) = \int_{\mathbb{R}^N} p(t, x, B)\mu(dx).$$

Setting $(U(t)\mu)(B) := \int_{\mathbb{R}^N} p(t, x, B)\mu(dx)$, one obtains a semigroup in the space of all positive finite Borel measures in $\mathbb{R}^N$. This fact leads to look for an equation satisfied by $p(t, x, B)$. Such an equation actually exists and it is known as Kolmogorov backward equation. Unfortunately, it requires strong regularity to the function $p(t, x, B)$. This is the reason why it is more convenient to consider the adjoint semigroup $(T(t))$ in the space of all bounded continuous functions in $\mathbb{R}^N$. Under suitable assumptions on the process Ξ, it turns out that the generator of $(T(t))$ is a second order differential operator A with unbounded coefficients. By means of A, we can reconstruct the semigroup $(T(t))$ and therefore, by duality, the transition probabilities $p(t, x, B)$. The prototype of differential operators with unbounded coefficients is the Ornstein-Uhlenbeck operator $\mathcal{A}u = \mathrm{Tr}\,(QD^2u) + \langle Bx, Du\rangle$, where Q is a real, symmetric and nonnegative matrix and B is a real, nonzero matrix. The associated Markov semigroup $(T(t))$ has an explicit representation formula, due to Kolmogorov (see [16]).

For such a class of operators, it is not obvious to derive existence, uniqueness or regularity results similar to the classical ones. The well-known Calderon Zygmund estimate shows that the Laplacian Δ, endowed with domain $W^{2,p}(\mathbb{R}^N)$, generates a strongly continuous analytic semigroup in $L^p(\mathbb{R}^N)$, for every $p \in]1, \infty[$. By adding an unbounded lower order term, the picture of the situation changes radically, since the new term cannot be treated as a small perturbation of the Laplacian. To be definite, we mention two quite meaningful cases. Let V be a nonnegative function in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and consider the Schrödinger operator $A = \Delta - V$. By making use of the theory of quadratic forms, one can show that A generates a strongly continuous analytic semigroup in $L^2(\mathbb{R}^N)$, which can be extended to $L^p(\mathbb{R}^N)$, for every $1 \leq p < \infty$. A natural question is whether the domain in $L^p(\mathbb{R}^N)$, when $p > 1$, coincides with the intersection of the domains of Δ and V, i.e. $W^{2,p}(\mathbb{R}^N) \cap D(V)$, where $D(V) = \{u \in L^p(\mathbb{R}^N) \mid Vu \in L^p(\mathbb{R}^N)\}$. This further information is not automatic as in the classical case, where V is bounded. In order to get it one needs to require an additional assumption on V, namely, the oscillation condition

$|DV| \leq \gamma V^{3/2}$, where γ is a sufficiently small positive constant (see [43], [41]). We remark that even for $p=2$ the domain of A as generator can be strictly larger than $W^{2,2}(\mathbb{R}^N) \cap D(V)$ if in the previous condition the constant γ is too big (see [41]). On the other hand, the potential $V(x,y) = x^2y^2$ does not satisfy $|DV| \leq \gamma V^{3/2}$, for any γ, nevertheless the domain of $\Delta - V$ is $W^{2,2}(\mathbb{R}^N) \cap D(V)$ (see [42]). Surprisingly enough, the situation is much better in $L^1(\mathbb{R}^N)$ where the domain of $\Delta - V$ is always the intersection of the domains.

Now, let us consider the case when the Laplacian is perturbed by adding a first order term. For simplicity, we consider the Ornstein Uhlenbeck operator in one dimension, $Au = u'' + xu'$. It is readily seen that, if $1 < \alpha p \leq p+1$, the function $u(x) = (x^2+1)^{-\frac{\alpha}{2}} \sin x$ belongs to $W^{2,p}(\mathbb{R}^N)$ but xu' is not in $L^p(\mathbb{R}^N)$. Therefore $W^{2,p}(\mathbb{R}^N)$, which is the domain of the Laplacian, is strictly larger than $\{u \in W^{2,p}(\mathbb{R}) \mid xu' \in L^p(\mathbb{R})\}$, which is the domain on which A generates a strongly continuous semigroup. The same one dimensional operator is also a counterexample to analyticity (see [40]).

We remark that also second order operators in the complete form, namely with both first and zero order terms, are object of investigation. For instance, the operator $\Delta - \langle D\Phi, D\rangle$ in the weighted space $L^p(\mathbb{R}^N, e^{-\Phi}dx)$ is isometric to a complete second order operator in the unweighted space $L^p(\mathbb{R}^N)$. Hence, several properties for the former can be deduced by studying the latter.

In this thesis, we focus our attention on regularity properties of solutions to partial differential equations involving second order elliptic operators with regular, (possibly) unbounded coefficients. Even though stochastic calculus is an useful tool to treat such operators, our approach is purely analytic. We cite the recent book of S. Cerrai [13] for an exhaustive analysis of what can be proved by stochastic methods.

We start in Chapter 1 by considering the following elliptic operator in divergence form

$$A = \sum_{i,j=1}^{N} D_i(q_{ij}D_j) + \langle F, D\rangle - V, \tag{0.0.3}$$

in $L^p(\mathbb{R}^N)$, with $1 < p < \infty$. The coefficients are always supposed to be real valued. If, in addition, q_{ij} are in $C^1_b(\mathbb{R}^N)$ and F_i, V are measurable and bounded, then it is well known that $(A, W^{2,p}(\mathbb{R}^N))$ generates a strongly continuous analytic semigroup. As a consequence, one obtains optimal regularity for the solutions to the resolvent equation $\lambda u - Au = f$, when λ is sufficiently large. This means that assuming only $u, \lambda u - Au \in L^p(\mathbb{R}^N)$, one deduces $u \in W^{2,p}(\mathbb{R}^N)$. Our first aim is to generalize such a result to the case where the lower order coefficients of the operator are unbounded. More precisely, we look for conditions on q_{ij}, F_i, V which allow to prove that the operator A endowed with its *natural* domain generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$. We consider as *natural* the domain given by the intersection of the domains of each addend of A, i.e. $\{u \in W^{2,p}(\mathbb{R}^N) \mid \langle F, Du\rangle,\ Vu \in L^p(\mathbb{R}^N)\}$. We have pointed out that such a domain may be strictly contained in $W^{2,p}(\mathbb{R}^N)$.

There are several approaches to show that elliptic operators with unbounded coefficients generate strongly continuous semigroups in L^p (see [11], [12], [19], [35], [37], [41] and the list of references therein), but only some of them give a precise description of the domain. Besides, in some cases the problem is investigated only for $p=2$ (see [17], [18] and in [50]). Our work gets inspiration essentially from [37] and [41]. In [37] the case $V=0$ and F globally Lipschitz continuous is considered. Under the further assumption $\langle F, Dq_{ij}\rangle \in L^\infty(\mathbb{R}^N)$, $i,j = 1, ..., N$, it is proved that the corresponding operator A, endowed with the domain $\{u \in W^{2,p}(\mathbb{R}^N)\colon\ \langle F, Du\rangle \in L^p(\mathbb{R}^N)\}$, generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$, for every $1 < p < \infty$. Here, the characterization of the domain follows from regularity results for the solution to the non homogenous Cauchy problem associated with A.

In [41], a second order operator in the complete form is considered and the description of the domain of the generator in $L^p(\mathbb{R}^N)$ is given assuming that V is strictly positive and that the following conditions hold: $|DV| \le \gamma V^{3/2}$, $|F| \le \kappa V^{1/2}$ and $\mathrm{div}F + \beta V \ge 0$, where γ, κ, β are sufficiently small constants. We observe that the first two assumptions are the same of Cannarsa and Vespri in [12], whereas the last one replaces an additional bound on the constant κ assumed in [12]. In [41], with a more direct approach, it is proved that A generates a strongly continuous analytic semigroup in $L^p(\mathbb{R}^N)$, $(1 < p < \infty)$, with domain $\{u \in W^{2,p}(\mathbb{R}^N) : Vu \in L^p(\mathbb{R}^N)\}$. An interpolatory estimate allows to control the L^p norm of $\langle F, Du\rangle$ by the L^p norms of Vu and D^2u. The assumption $|DV| \le \gamma V^{3/2}$ is the essential ingredient to determine the domain and, as observed at the beginning in the case of Schrödinger operators, it is optimal. The condition $|F| \le \kappa V^{1/2}$ is the best possible to yield analyticity. Finally, we observe that the cases $p = 1$ and $p = \infty$ are also considered.

We formulate new conditions on F, V and their first order derivatives to show that $(A, \mathcal{D}_p)$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$ $(1 < p < \infty)$, where $\mathcal{D}_p$ is defined as

$$\mathcal{D}_p := \{u \in W^{2,p}(\mathbb{R}^N) : \langle F, Du\rangle \in L^p(\mathbb{R}^N), Vu \in L^p(\mathbb{R}^N)\}.$$

We observe that for suitable choices of the parameters involved, our framework covers [37] or [41]. Thus, our results can be seen as a continuous interpolation between them. We also cover new cases. For instance, we allow the conditions $|F| \le \theta V$, $|DV| \le \alpha V$, $|DF| \le \beta V$.

The first step to achieve our aim consists of proving a priori estimates of the form

$$\|u\|_{W^{2,p}(\mathbb{R}^N)} + \|\langle F, Du\rangle\|_{L^p(\mathbb{R}^N)} + \|Vu\|_{L^p(\mathbb{R}^N)} \le C(\|u\|_{L^p(\mathbb{R}^N)} + \|Au\|_{L^p(\mathbb{R}^N)}), \tag{0.0.4}$$

for every $u \in \mathcal{D}_p$ and for some constant $C > 0$ independent of u. For every test function u we prove the corresponding estimates for $\|Vu\|_{L^p(\mathbb{R}^N)}$ and $\|Du\|_{L^p(\mathbb{R}^N)}$ using the variational method, which relies on suitable integrations by parts and other elementary tools. The same technique yields the estimate of the second order derivatives when $p = 2$, too, and therefore (0.0.4) is completely proved, since the last term $\|\langle F, Du\rangle\|_{L^2(\mathbb{R}^N)}$ can be estimated by difference. Of course, the method fails for $p \neq 2$. The Calderon Zygmund estimate cannot be proved by means of integrations by parts. Thus a different method has to be used, but it requires stronger assumptions. This is the reason why we treat the cases $p = 2$ and $p \neq 2$ separately. When $p \neq 2$, the idea is to get first local estimates. To this aim, we localize the equation $Au = f$ by multiplying it by cutoff functions supported in certain balls $B(x_0, r(x_0))$, and then we make a change of variables, which is determined by the potential. This technique produces a family of new operators $\{A_{x_0}\}$ which satisfy the assumptions of [37], up to a bounded perturbation. Then, to each operator A_{x_0} we can apply the a priori estimate for the second order derivatives proved in [37], so that in the original setting we find out the following local estimates

$$\int_{B(x_0,r(x_0))} |D^2u|^p \le C \int_{B(x_0,2r(x_0))} |u|^p + |Au|^p + |Vu|^p + |V^{1/2}Du|^p \tag{0.0.5}$$

A crucial point is to make the dependence of the constant C precise. In particular, in order to apply a covering argument and to obtain a global estimate starting with (0.0.5), we need C to be independent of x_0. In this way we deduce that

$$\int_{\mathbb{R}^N} |D^2u|^p \le C \int_{\mathbb{R}^N} |u|^p + |Au|^p + |Vu|^p + |V^{1/2}Du|^p,$$

and then, using known results

$$\|D^2u\|_{L^p(\mathbb{R}^N)} \le C(\|u\|_{L^p(\mathbb{R}^N)} + \|Au\|_{L^p(\mathbb{R}^N)}),$$

as required. The last estimate among (0.0.4), namely the one for $\langle F, Du\rangle$, follows by difference. Hence (0.0.4) are verified for every test function u. By density, they can be extended to $\mathcal{D}_p$ and this yields, without any further effort, the closedness of $(A, \mathcal{D}_p)$ in $L^p(\mathbb{R}^N)$. It is also an easy task to prove that $(A, \mathcal{D}_p)$ is quasi dissipative.

The second step of our procedure consists of proving the surjectivity of $\lambda - A$ from $\mathcal{D}_p$ onto $L^p(\mathbb{R}^N)$, for sufficiently large λ. This is done, once again, differently when $p = 2$ or $p \neq 2$. In the first case, we find the solution of the equation $\lambda u - Au = f$ in $\mathcal{D}_2$ as the limit of a sequence of solutions of the same equation in balls with increasing radii and Dirichlet boundary conditions. This argument does not work for $p \neq 2$. In this case, we check the surjectivity of $\lambda - A$ approximating A by a family of operators which belong to the class studied in [37]. At this point, we can apply the Hille Yosida generation theorem and we show that $(A, \mathcal{D}_p)$ generates a strongly continuous semigroup, which is positive, but not analytic in general.

The generation result just proved holds whenever $1 < p < \infty$. If $p = \infty$, in spaces of continuous functions, the situation is more delicate and the explicit description of the domain is more complicated. However, useful information can be obtained if *gradient estimates* hold. To be definite, in the second chapter, we consider the second order differential operator in non divergence form

$$\mathcal{A} = \sum_{i,j^1}^{N} q_{ij} D_{ij} + \sum_{i=1}^{N} F_i D_i - V, \tag{0.0.6}$$

in a smooth open unbounded subset Ω of $\mathbb{R}^N$. Ω may be the whole space $\mathbb{R}^N$, but in this case several results are already known. We deal with the Cauchy-Neumann problem

$$\begin{cases} u_t(t,x) - \mathcal{A}u(t,x) = 0 & t > 0,\ x \in \overline{\Omega}, \\ \dfrac{\partial u}{\partial \eta}(t,x) = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = f(x) & x \in \overline{\Omega}, \end{cases} \tag{0.0.7}$$

where f is a continuous and bounded function in $\overline{\Omega}$ and η is the outward unit normal vector to $\partial\Omega$. Our aim is to determine conditions on the coefficients of $\mathcal{A}$ and on the domain Ω such that (0.0.7) admits a unique bounded classical solution u, whose spatial gradient verifies the following estimate

$$|Du(t,x)| \leq \frac{C_T}{\sqrt{t}} \|f\|_\infty \tag{0.0.8}$$

$0 < t \leq T, x \in \overline{\Omega}$. Estimate (0.0.8) has been deeply investigated in the literature, expecially by means of probabilistic tools. Our approach is purely analytic and allows to treat unbounded domains which do not coincide with the whole space. We proceed as follows. We consider the solutions u_n of Cauchy-Neumann problems with initial datum f, in a nested sequence of bounded regular domains $\{\Omega_n\}$, whose union is Ω. Since Neumann boundary conditions do not imply monotonicity (unlike Dirichlet boundary conditions), the main tool to prove the convergence of (u_n) is given by the classical Schauder estimates together with a compactness argument. The limit function u is not yet the classical solution to (0.0.7), since the continuity at $(0, x)$, when $x \in \partial\Omega$, is not ensured. To solve this problem we prove sharp estimates for the gradient of u_n. More precisely, we consider the function $z_n = u_n^2 + at|Du_n|^2$ and we prove that the differential inequality $(D_t - \mathcal{A})z_n \leq 0$ holds for a suitable choice of the parameter a independent of n. To do this, we assume a dissipativity condition on the drift F, a bound from below for V and that V grows at most exponentially. Moreover, assuming that Ω is convex and choosing all the domains Ω_n to be convex, we deduce that z_n has nonpositive normal derivative on $\partial\Omega_n$. This is the crucial point of our procedure. The classical maximum principle implies that $|Du_n| \leq C_T t^{-1/2} \|f\|_\infty$, where C_T is a constant independent of n. This estimate leads to the continuity of u in $\{0\} \times \partial\Omega$ as well as to the gradient estimate (0.0.8), as soon as n tends to ∞. The method used is known as *Bernstein's method.* It was used by A.Lunardi in [34] to prove (0.0.8) in the whole $\mathbb{R}^N$, whereas the same result is proved in [13] by means of probabilistic tools. A Liapunov type condition ensures that a maximum principle holds, hence the function u, produced by the previous approximation argument, is the *unique* bounded classical solution to (0.0.7).

Setting $(P_t f)(x) = u(t, x)$, we obtain a semigroup of linear bounded operators in $C_b(\overline{\Omega})$. Such a semigroup is not strongly continuous, in general, hence we cannot consider the generator in the classical sense, but only the so called weak generator. We also note that (P_t) is neither analytic in $C_b(\overline{\Omega})$, otherwise estimate (0.0.8) could be deduced from the analyticity estimate $\|\mathcal{A}T(t)f\|_\infty \le C\,t^{-1}\|f\|_\infty$ by an interpolation argument. We show that, in our situation, the weak generator coincides with the realization of $\mathcal{A}$ in $C_b(\overline{\Omega})$ with homogeneous Neumann boundary conditions, i.e. with the operator $\mathcal{A}$ endowed with the domain

$$D(\mathcal{A}) = \left\{ u \in C_b(\overline{\Omega}) \cap \bigcap_{1\le p<\infty} W^{2,p}(\Omega \cap B_R) \text{ for all } R > 0 : \mathcal{A}u \in C_b(\overline{\Omega}), \frac{\partial u}{\partial \eta}\Big|_{\partial\Omega} = 0 \right\}.$$

The weak generator shares several properties with the generators of strongly continuous semigroups. In particular, since we assume $V \ge 0$, we have that $(0, +\infty) \subset \rho(\mathcal{A})$. Therefore, for every $f \in C_b(\overline{\Omega})$ and $\lambda > 0$ there exists a unique solution in $D(\mathcal{A})$ of the elliptic problem

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) = f(x) & x \in \Omega, \\ \dfrac{\partial u}{\partial \eta}(x) = 0 & x \in \partial\Omega. \end{cases}$$

A consequence of (0.0.8) is that the domain of $\mathcal{A}$ is contained in $C_b^1(\overline{\Omega})$. This can be interpreted as a partial regularity result for the solutions of the elliptic problem above.

Assuming that $V \equiv 0$, we derive further gradient estimates of pointwise type. More precisely, if $p > 1$ and $f \in C_b^1(\overline{\Omega})$ with $\partial f/\partial\eta = 0$ on $\partial\Omega$, then

$$|DP_t f(x)|^p \le e^{\sigma_p t} P_t(|Df|^p)(x), \tag{0.0.9}$$

where σ_p is a real constant depending on the coefficients of $\mathcal{A}$, N and p. If $q_{ij} = \delta_{ij}$, then the previous estimate is true also when $p = 1$. This case is almost optimal, since in [58] it is proved that (0.0.9) with $p = 1$ holds in $\mathbb{R}^N$ if and only if $D_k q_{ij} + D_j q_{ki} + D_i q_{kj} = 0$, for every i, j, k. Following the ideas of [7] for $p = 2$, we deduce that

$$|DP_t f(x)|^p \le \left(\frac{\sigma_2 \nu_0^{-1}}{2(1 - e^{-\sigma_2 t})} \right)^{\frac{p}{2}} P_t(|f|^p)(x), \tag{0.0.10}$$

for all $p \ge 2$, where ν_0 is the ellipticity constant of $\mathcal{A}$. An analogous estimate holds when $1 < p < 2$. Also in this case the thesis fails if $p = 1$, even for the heat semigroup. The previous estimate with $p = 2$ improves the first global gradient estimate (0.0.8), which can be now reformulated in the form

$$\|DP_t f\|_\infty \le \left(\frac{\nu_0^{-1}\sigma_2}{2(1 - e^{-\sigma_2\, t})} \right)^{\frac{1}{2}} \|f\|_\infty.$$

Therefore, if $\sigma_2 \le 0$, a Liouville type theorem for the operator $\mathcal{A}$ holds and it implies that if $f \in D(\mathcal{A})$ and $\mathcal{A}f = 0$, then f is constant. Other interesting consequences can be deduced if an invariant measure exists. We say that a probability Borel measure μ is invariant for (P_t) if for every bounded Borel function f and every $t \ge 0$

$$\int_\Omega P_t f\, d\mu = \int_\Omega f\, d\mu\ .$$

In this case, (P_t) can be extended to a strongly continuous semigroup in $L^p(\Omega,\mu)$ for every $1 \le p < \infty$ and, integrating estimate (0.0.10) with respect to μ, one gets an analogous estimate in the L^p norm. This implies that the domain of the generator of (P_t) in $L^p(\Omega,\mu)$ is continuously embedded in $W^{1,p}(\Omega,\mu)$.

Moreover, one can derive the hypercontractivity of P_t in the space $L^2(\Omega,\mu)$ and logarithmic Sobolev inequalities (this is the well known Bakry-Émery criterion). Finally, (0.0.9) with $p = 2$ and $\sigma_2 < 0$ implies the Poincaré inequality in $W^{1,2}(\Omega,\mu)$ and the spectral gap for the generator A_2 of (P_t) in $L^2(\Omega,\mu)$, which means that $\sigma(A_2) \setminus \{0\} \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le -C\}$, for some $C > 0$.

In the case $\Omega = \mathbb{R}^N$, estimates (0.0.9) and (0.0.10) with $p = 2$ and $q_{ij} = \delta_{ij}$ were proved respectively in [6] and [7] in the setting of abstract Markov generators, for functions belonging to a suitable algebra of smooth functions which is required to be invariant under the generator. Estimate (0.0.9) was proved also in [56] by probabilistic methods. A probabilistic approach is used in [49] too, for establishing estimate (0.0.9) in the case of a compact Riemannian manifold with convex boundary or of a complete manifold without boundary.

Dissipativity conditions are of crucial importance to get gradient estimates. Indeed, we give a counterexample to estimate (0.0.8) for an operator $\mathcal{A} = \Delta + \sum F_i D_i$ where F is not dissipative.

In the third chapter we deal with Cauchy-Dirichlet problems

$$\begin{cases} u_t(t,x) - \mathcal{A}u(t,x) = 0 & t > 0,\ x \in \Omega, \\ u(t,x) = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = f(x) & x \in \Omega, \end{cases} \tag{0.0.11}$$

where $\mathcal{A}$ is defined by (0.0.6) and f is continuous and bounded in Ω. Our aim is again to derive gradient estimates for bounded classical solutions to (0.0.11). Our approach is slightly different from the previous case. Indeed, if (u_n) is a sequence of solutions of Cauchy-Dirichlet problems in bounded domains, whose union is Ω, then it is not difficult to show that (u_n) converges to a solution of (0.0.11). But, if we set $z_n = u_n^2 + at|Du_n|^2$ and we try to apply the maximum principle to z_n, it is not clear what happens to z_n at $\partial\Omega$, even when Ω is a halfspace. To overcome this difficulty, we proceed in the following way. The existence of a bounded classical solution u to (0.0.11) can be proved completely by approximation. We note that in this case we do not expect that the solution is continuous at $(0,x)$, for $x \in \partial\Omega$. The uniqueness follows once again from a generalized version of the classical maximum principle. Afterwards, we observe that since $u = 0$ on $\partial\Omega$, only the normal derivative of u can be different from zero on $\partial\Omega$. This suggests us the comparison with certain one dimensional operators. In fact, following this idea and assuming a suitable control on F near to the boundary of Ω, we can prove the following estimate for Du on $\partial\Omega$

$$|Du(t,\xi)| \le \frac{C_T}{\sqrt{t}}\|f\|_\infty, \quad 0 < t \le T,\ \xi \in \partial\Omega.$$

Taking such an estimate into account, we can apply the maximum principle to the function $z = u^2 + at|Du|^2$ and we obtain

$$|Du(t,x)| \le \frac{C_T}{\sqrt{t}}\|f\|_\infty \quad 0 < t \le T,\ x \in \Omega. \tag{0.0.12}$$

We note that the technique allows to have a precise control of the constant C_T. Unfortunately, the last step works if one already knows that u is smooth enough. This is not our case, even though the initial datum f is smooth. Therefore, we use a trick, which consists in introducing an auxiliary potential W. We take W big enough to control the growth of the drift term F and then we consider the perturbed operators $A_\varepsilon = A - \varepsilon W$. We show that the realization of

A_ε with homogeneous Dirichlet boundary conditions generates a strongly continuous analytic semigroup $(T_{p,\varepsilon}(t))$ in $L^p(\Omega)$, for $p \geq 2$ and we characterize the domain. Choosing a large p and using Sobolev embeddings we prove that $u_\varepsilon(t,x) = T_{p,\varepsilon}(t)f(x)$ is the bounded classical solution of (0.0.11), with A replaced by A_ε and f smooth. Moreover, we are allowed to apply the previous gradient estimate to each u_ε, obtaining

$$|Du_\varepsilon(t,x)| \leq \frac{C_T}{\sqrt{t}}\|f\|_\infty, \quad 0 < t \leq T,\ x \in \Omega,$$

with C_T independent of ε. A suitable extracted sequence of u_ε converges to the bounded classical solution u of (0.0.11) and estimate (0.0.12) follows by taking the limit. Finally, by a standard approximation argument we prove estimate (0.0.12) for every continuous and bounded function f in Ω. We point out that we do not need convexity assumptions on Ω to carry out this program. At the moment, the same procedure seems to be useful to remove the convexity of Ω in the case where Neumann boundary condition is considered. But this is a work in progress. As far as local gradient estimates for (3.0.1) are concerned, we mention [54], which establishes them in the Riemannian setting, and [15], [53] for the case when Ω is an open subset of a Hilbert space and A is an Ornstein-Uhlenbeck operator. Moreover in [57], see also [31], connections between estimates (0.0.12) and some isoperimetric inequalities are investigated.

In Chapter 4 we study the second order ordinary differential operator $Au = au'' + bu'$ and we characterize the domains on which A generates semigroups in $C_b(\mathbb{R})$ and in $C(\overline{\mathbb{R}})$, the space of continuous functions having finite limits at $\pm\infty$. Unfortunately, the technique used cannot be extended to higher dimensions. We just cite [41], where a complete description of the domain is given in $C_0(\mathbb{R}^N)$ when the operator contains a potential term which balances the growth of the drift coefficient. We refer to [34] for the case of Hölder spaces.

Minimal assumptions on the coefficients of A guarantee that A endowed with the domain

$$D_{\max}(A) := \{u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R}) \mid Au \in C_b(\mathbb{R})\}$$

generates a semigroup in $C_b(\mathbb{R})$, which is not strongly continuous in general and A with domain

$$D_{\mathrm{m}}(A) := \{u \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid Au \in C(\overline{\mathbb{R}})\}$$

generates a strongly continuous semigroup in $C(\overline{\mathbb{R}})$. Hence, we have only to describe explicitly such domains. Our aim is to show that under suitable assumptions on a and b

$$D_{\max}(A) = \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}$$

and

$$D_{\mathrm{m}}(A) = \{u \in C^2(\overline{\mathbb{R}}) \mid bu' \in C(\overline{\mathbb{R}})\}.$$

As a consequence, we derive optimal regularity for the solutions to the elliptic equations $\lambda u - Au = f$ both in $C_b(\mathbb{R})$ and in $C(\overline{\mathbb{R}})$. Let us consider the first case. Set $D = \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}$. Since $D \subset D_{\max}(A)$ and $\lambda - A$ is bijective from $D_{\max}(A)$ onto $C_b(\mathbb{R})$, to prove the statement it is sufficient to show that $\lambda - A$ is surjective from D onto $C_b(\mathbb{R})$. Once again, a crucial point is represented by a priori estimates. More precisely, assuming that $ab' \leq c_1 + c_2 b^2$, we prove that for every $u \in C^2([-\alpha,\alpha])$ with $u'(\pm\alpha) = 0$, we have

$$\|bu'\|_{C([-\alpha,\alpha])} \leq C(\|Au\|_{C([-\alpha,\alpha])} + \|u\|_{C([-\alpha,\alpha])}), \tag{0.0.13}$$

with C independent of α. Then, we construct a solution $u \in D$ of the equation $\lambda u - Au = f$, for $f \in C_b(\mathbb{R})$, by approximation, considering the solutions of the equation $\lambda u - Au = f$ in $[-n,n]$

with Neumann boundary conditions and using (0.0.13). In a similar way, but requiring slightly stronger assumptions on a, b, we prove the statement in $C(\overline{\mathbb{R}})$.

The last chapter is devoted to the collection of some known results concerning invariant measures for Feller semigroups in $C_b(\mathbb{R}^N)$. We present this argument in a quite general context, which is not the most general possible, but is close to the main concrete situation where this concept arises, namely the theory of second order differential operators with unbounded coefficients. In the last section we study the operator

$$B = \mathrm{div}(qD) - \langle qD\Phi, D\rangle + \langle G, D\rangle$$

in the space $L^p(\mathbb{R}^N, \mu)$, $1 < p < \infty$, where $d\mu = e^{-\Phi}dx$. Via the transformation $v = e^{-\frac{\Phi}{p}}u$, the operator B on $L^p(\mathbb{R}^N, \mu)$ is similar to an operator A of the form (0.0.3) in the unweighted space $L^p(\mathbb{R}^N)$. Suitable assumptions on the coefficients q, Φ, G allow to apply the generation results of Chapter 1 to the transformed operator so that, via the inverse transformation, we can deduce that B, endowed with the domain

$$\mathcal{D}_\mu = \{u \in W^{2,p}(\mathbb{R}^N, \mu) \mid \langle G, Du\rangle \in L^p(\mathbb{R}^N, \mu)\} \tag{0.0.14}$$

generates a strongly continuous semigroup $(T(t))$ on $L^p(\mathbb{R}^N, \mu)$. We note that, in particular, the measure μ can be the invariant measure of $(T(t))$.

Lecce, 16 april 2004

Chapter 1

Elliptic operators in $L^p(\mathbb{R}^N)$: characterization of the domain

In this chapter we consider the following linear second order elliptic operator in divergence form

$$Au := A_0u + \langle F, Du\rangle - Vu, \tag{1.0.1}$$

where

$$A_0u := \sum_{i,j=1}^{N} D_i(q_{ij}D_ju)\,.$$

As usual, we will refer to F and V as the drift and the potential term, respectively, and neither F nor V will be assumed to be bounded.

Our aim is to prove a generation result for A in $L^p(\mathbb{R}^N)$ $(1 < p < +\infty)$ with respect to the Lebesgue measure, providing an explicit description of the domain of the generator. Precisely, we show that such a domain is the intersection of the domains of each addend of A in (1.0.1).

This problem is classical and well-known in the case of elliptic operators with regular and bounded coefficients. We refer to the book of Lunardi [32] for a detailed analysis of the subject. On the other hand, there are several approaches to show that elliptic operators with unbounded coefficients generate strongly continuous semigroups in L^p (see [11], [12], [19], [35], [37], [41] and the list of references therein), but only some of them give a precise description of the domain. Besides, in some cases the problem is investigated only for $p = 2$ (see [17], [18] and in [50]).

Here we prove that if $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$, with $1 < p < +\infty$, is the Banach space defined as

$$\begin{aligned}\mathcal{D}_p &:= \{u \in W^{2,p}(\mathbb{R}^N) \,:\, \langle F, Du\rangle \in L^p(\mathbb{R}^N), Vu \in L^p(\mathbb{R}^N)\}\,,\\ \|u\|_{\mathcal{D}_p} &:= \|u\|_{W^{2,p}(\mathbb{R}^N)} + \|\langle F, Du\rangle\|_{L^p(\mathbb{R}^N)} + \|Vu\|_{L^p(\mathbb{R}^N)}\,,\end{aligned}$$

then $(A, \mathcal{D}_p)$ generates a C_0-semigroup in $L^p(\mathbb{R}^N)$, if suitable growth conditions on F, V and their first order derivatives are assumed. As a by-product, one can deduce regularity results for the solution of the elliptic equation associated with A.

The precise description of the domain relies on a priori estimates of the form

$$\|u\|_{2,p} + \|\langle F, Du\rangle\|_p + \|Vu\|_p \le C(\|u\|_p + \|Au\|_p), \tag{1.0.2}$$

for every $p \in (1, \infty)$ and every test function u and for some constant $C > 0$ independent of u. We prove the estimates for $\|Vu\|_p$ and $\|Du\|_p$ using basically integrations by parts and other elementary tools. In the particular case $p = 2$, we also get an estimate for the second order derivatives of u (see Section 1.3). For $p \neq 2$, the variational method fails to estimate $\|D^2u\|_p$ and we have to employ a different technique, which works under stronger assumptions. This is done in Section 1.4, where we use an a priori estimate for the second order derivatives in the case where the involved operator has globally Lipschitz drift coefficient and bounded potential term (we prove such an estimate together with a generation result as a preliminary step in Section 1.2). Once the second order derivatives are estimated, the last term $\|\langle F, Du\rangle\|_p$ in (1.0.2) can be estimated easily by difference.

Using a density argument, (1.0.2) turns out to be true also for functions in $\mathcal{D}_p$. As a consequence, we establish the closedness of $(A, \mathcal{D}_p)$ in $L^p(\mathbb{R}^N)$. Moreover, it is easily seen that $(A, \mathcal{D}_p)$ is quasi dissipative in $L^p(\mathbb{R}^N)$. Therefore, in order to apply the Hille-Yosida generation theorem and to get the desired result, it remains to prove that $\lambda - A$ is surjective from $\mathcal{D}_p$ onto $L^p(\mathbb{R}^N)$, for λ large. Sections 1.5 and 1.6 are devoted to this aim. We proceed differently in the case $p = 2$ and $p \neq 2$. In the first case, we find the solution of the equation $\lambda u - Au = f$ in the whole space as the limit of a sequence of solutions of the same equation in balls with increasing radii and Dirichlet boundary conditions. In the second case, we check the surjectivity of $\lambda - A$ by approximating A with a family of operators whose drift coefficient is globally Lipschitz and whose potential term is bounded. We note that, once again, the first method works under weaker assumptions and this is the reason why we treat the case $p = 2$ separately.

Finally, in Section 1.7 we describe some properties of the above semigroups. We prove that they are positive, not analytic in general, consistent with respect to p. Moreover if V tends to $+\infty$ as $|x| \to +\infty$, then $(A, \mathcal{D}_p)$ has compact resolvent.

1.1 Assumptions and statement of the main results

In the following $q(x) = (q_{ij}(x))$ is a $N \times N$ symmetric real matrix such that $q_{ij} \in C^1_b(\mathbb{R}^N)$ and

$$\langle q(x)\xi, \xi\rangle := \sum_{i,j=1}^{N} q_{ij}(x)\,\xi_i\xi_j \geq \nu_0|\xi|^2, \qquad \nu_0 > 0, \tag{1.1.1}$$

for every $x, \xi \in \mathbb{R}^N$. Moreover, we consider $F \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ and $V \in C^1(\mathbb{R}^N)$ and we assume that V is bounded from below. Without loss of generality, we suppose that $V \geq 1$. We deal with the elliptic operator

$$Au := A_0u + \langle F, Du\rangle - Vu, \tag{1.1.2}$$

where $A_0u(x) := \sum_{i,j=1}^{N} D_i(q_{ij}(x)D_ju(x))$.

For $1 < p < +\infty$, we define the space $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$ as

$$\mathcal{D}_p := \{u \in W^{2,p}(\mathbb{R}^N) : \langle F, Du\rangle \in L^p(\mathbb{R}^N), Vu \in L^p(\mathbb{R}^N)\}, \tag{1.1.3}$$

$$\|u\|_{\mathcal{D}_p} := \|u\|_{2,p} + \|\langle F, Du\rangle\|_p + \|Vu\|_p\,. \tag{1.1.4}$$

We endow $\mathcal{D}_p$ also with the graph norm of the operator A, namely

$$\|u\|_A := \|Au\|_p + \|u\|_p\,.$$

In the case $p = 2$, besides the previous assumptions on the coefficients, we require that the following growth conditions hold

(H1) $|DV| \leq \alpha V^{3/2} + c_\alpha$, (H2) $\mathrm{div}F + \beta V \ge -c_\beta\,, \quad \sum_{i,j=1}^{N} D_iF_j(x)\xi_i\xi_j \le \tau\, V(x)|\xi|^2 + c_\tau|\xi|^2, \quad \xi, x \in \mathbb{R}^N,$

(H3) $\langle F, DV\rangle + \gamma V^2 \ge -c_\gamma,$

(H4) $|F(x)| \le \theta(1+|x|^2)^{1/2}V(x) + c_\theta$,

with $\alpha, \beta, \gamma, \tau, \theta > 0$ and $c_\alpha, c_\beta, c_\gamma, c_\tau, c_\theta \ge 0$ satisfying

$$1 - \frac{\beta}{2} - \tau > 0\,, \tag{1.1.5}$$

and

$$\frac{M}{4}\alpha^2 + \frac{\beta}{2} + \frac{\gamma}{2} < 1\,, \tag{1.1.6}$$

where $M := \sup_{x\in\mathbb{R}^N}\max_{|\xi|=1}\langle q(x)\xi, \xi\rangle$. We note that the second inequality in (H2) is a dissipativity condition for the function F.

The following generation result holds.

Theorem 1.1.1 (p=2) *Suppose that (H1), (H2), (H3), (H4), (1.1.5) and (1.1.6) hold. Then the operator $(A, \mathcal{D}_2)$ generates a C_0-semigroup on $L^2(\mathbb{R}^N)$. If $c_\beta = 0$, then the semigroup is contractive.*

In Section 1.6 we prove an analogous result in the general case $p > 1$. To this aim we use a different technique, which works under more restrictive assumptions on the coefficients of A. Precisely, we replace assumptions (H1), (H2) and (H4) with the following ones

(H1') $|DV(x)| \le \alpha \dfrac{V^{2-\sigma}(x)}{(1+|x|^2)^{\mu/2}},$

(H2') $|DF| \le \frac{1}{\sqrt{N}}(\beta V + c_\beta),$

(H4') $|F(x)| \le \theta(1+|x|^2)^{\mu/2}V^\sigma(x),$

respectively, where DF denotes the Jacobian matrix of F and $|DF|^2 = \sum_{k,i=1}^N |D_kF_i|^2$, $\alpha, \beta, \theta > 0$, $c_\beta \ge 0$, $\frac{1}{2} \le \sigma \le 1$ and $0 \le \mu \le 1$. Moreover, we suppose that for every $x \in \mathbb{R}^N$

(H5) $|\langle F(x), Dq_{ij}(x)\rangle| \le \kappa\, V(x) + c_\kappa,$

holds, with constants $\kappa > 0$ and $c_\kappa \ge 0$.

Analogously to the case $p = 2$, also in this case a smallness condition on the coefficients is required. Let

$$\omega := \begin{cases} \frac{M}{4}(p-1)\alpha^2\,, & \text{if } (\sigma,\mu) = \left(\frac{1}{2}, 0\right), \\ 0\,, & \text{otherwise.} \end{cases}$$

Then we assume that

$$\begin{aligned} &\omega + \sqrt{2}\,\frac{\beta + \sqrt{N}\alpha\theta}{p} + \alpha\theta\frac{p-1}{p} < 1\,, && \text{if } 1 < p < 2\,, \\ &\omega + \sqrt{2}\left(\beta + \sqrt{N}\alpha\theta\right)\left(\frac{1}{p} + \frac{1}{\sqrt{N}}\right) < 1\,, && \text{if } p \ge 2\,. \end{aligned} \tag{1.1.7}$$

The following generation result holds.

Theorem 1.1.2 (1<p<+∞) *Suppose that (H1'), (H2'), (H4'), (H5) and (1.1.7) are satisfied, for some $1 < p < \infty$. Then the operator $(A, \mathcal{D}_p)$ generates a C_0-semigroup on $L^p(\mathbb{R}^N)$, which turns out to be contractive if $c_\beta = 0$.*

Remark 1.1.3 We observe that (1.1.7) for $p \geq 2$ implies (1.1.7) for $1 < p < 2$, since

$$\sqrt{2}\,\frac{\beta + \sqrt{N}\alpha\theta}{p} + \alpha\theta\frac{p-1}{p} \leq \sqrt{2}\left(\beta + \sqrt{N}\alpha\theta\right)\left(\frac{1}{p} + \frac{1}{\sqrt{N}}\right), \qquad p > 1\,.$$

Moreover, we note that when $p = 2$, (1.1.7) is not equivalent to (1.1.6), but it is stronger. This fact relies on the different technique employed in the general case and, in particular, on the fact that we need that other suitable operators verify our assumptions. For further details we refer to Section 1.6. In any case, the two methods yield the same semigroup in $L^2(\mathbb{R}^N)$.
Finally, we point out that in Theorem 1.1.2 we do not explicitly assume (H3), since (H1') and (H4') imply

$$|\langle F, DV\rangle| \leq \alpha\theta V^2\,. \tag{1.1.8}$$

Remark 1.1.4 Hypothesis (H1) is essential to determine the domain. In fact in [41, Example 3.7] the authors exhibit a Schrödinger operator $A = \Delta - V$ in $L^2(\mathbb{R}^3)$ such that (H1) holds with a too large constant α and the domain is not $W^{2,2}(\mathbb{R}^3) \cap D(V)$. Moreover in [41] it is observed that (H1) holds for example for any polynomial whose homogenous part of maximal degree is positive definite. (H1) fails for the function $U = 1 + x^2y^2$.

Remark 1.1.5 We note that making particular choices of the parameters μ and σ, we may cover cases already known or discuss new ones. For example, if $\mu = 0$ and $\sigma = \frac{1}{2}$, then we get exactly the framework of [41]

$$|F| \leq \theta V^{1/2}, \qquad |DV| \leq \alpha V^{3/2}$$

and therefore of [12]. If we take V constant, then we reduce to the case where F is globally Lipschitz continuous studied in [37]. Setting $\mu = 0$ and $\sigma = 1$ we have the case

$$|F| \leq \theta V, \qquad |DV| \leq \alpha V,$$

which, according to our knowledge, seems to be new. From the second condition above, one deduces that V grows at most exponentially. In particular, we can treat in this way polynomials V as in Remark 1.1.4.

If we optimize assumption (H4') choosing $\mu = \sigma = 1$, analogously to (H4) in the case $p = 2$, then (H1') becomes $|DV(x)| \leq \alpha\frac{V(x)}{(1+|x|^2)^{1/2}}$, which is much more restrictive than (H1). This shows that the cases $p = 2$ and $p \neq 2$ are quite different. Such a difference is also confirmed by the fact that when $p = 2$ we do not require any condition on $\langle Dq_{ij}, F\rangle$.

The assumptions for $p \neq 2$ are determined by our approach to estimate the second order derivatives of a test function u in terms of u and Au. The idea is to get first local estimates. To this aim we change variables and localize the equation $Au = f$ in certain balls $B(x_0, r(x_0))$. The new operator produced by this technique (see (1.4.14)) has a globally Lipschitz continuous drift term and a bounded potential. The radius $r(x_0)$ has to grow at most linearly with respect to $|x_0|$ in order to use a covering argument and to obtain global estimates. So, roughly speaking, we must require that $r(x_0) \leq 1 + |x_0|$ and that $V(x)$ is "close" to $V(x_0)$ if $|x - x_0| < r(x_0)$. This is exactly guaranteed by assumptions (H4') (see (1.4.2)) and (H1') (see Lemma 1.4.3). The Lipschitz continuity of the transformed drift coefficient follows from (H2'). All the details are given in Section 1.6.

1.2 Operators with globally Lipschitz drift coefficient and bounded potential term

In this section we collect all the results concerning operators with globally Lipschitz drift coefficient and bounded potential term that will be used in the sequel. We first prove an a priori estimate for the second order derivatives of a test function u, using the same technique of [37] but specifying how the constants involved depend on the operator. Then, we show a generation result, giving an explicit description of the domain.

Let

$$B = \sum_{i,j=1}^{N} D_i(a_{ij}D_j) + \sum_{i=1}^{N} b_i D_i - c \tag{1.2.1}$$

and assume that

(i) $a_{ij} = a_{ji} \in C_b^1(\mathbb{R}^N)$, $\quad \sum_{i,j=1}^N a_{ij}\xi_i\xi_j \geq \nu_0|\xi|^2$,

(ii) $b = (b_1, ..., b_N)$ is globally Lipschitz in $\mathbb{R}^N$,

(iii) $c \in L^\infty(\mathbb{R}^N)$,

(iv) $\sup_{x\in\mathbb{R}^N} |\langle Da_{ij}(x), b(x)\rangle| < +\infty\,, \quad i,j = 1, ..., N.$

The following a priori estimate is a crucial point for our aims.

Theorem 1.2.1 *There exists a constant $C > 0$ depending on $p, N, \nu_0, \|a_{ij}\|_\infty, \|Da_{ij}\|_\infty, \|\langle Da_{ij}, b\rangle\|_\infty$, $\|c\|_\infty$ and the Lipschitz constant of b, denoted by $[b]_1$, such that for all $u \in C_c^\infty(\mathbb{R}^N)$*

$$\int_{\mathbb{R}^N} |D^2u|^p\,dx \leq C \int_{\mathbb{R}^N} (|Bu|^p + |u|^p)\,dx. \tag{1.2.2}$$

PROOF. We split the proof in two steps.

Step 1. We assume that the operator B is written in the non-divergence form

$$B = \sum_{i,j=1}^{N} a_{ij}D_{ij} + \sum_{i=1}^{N} b_i D_i - c$$

and that $b \in C^2(\mathbb{R}^N;\mathbb{R}^N)$ with bounded first and second order derivatives, besides assumptions (i), (ii), (iii) and (iv).

Let $u \in C_c^\infty(\mathbb{R}^N)$. Then u solves the equation

$$D_t u - Bu = f \quad \text{in } \mathbb{R}^{N+1},$$

with $f = -Bu$. Let us consider the ordinary Cauchy problem in $\mathbb{R}^N$

$$\begin{cases} \dfrac{d\xi}{dt} = b(\xi), & t \in \mathbb{R} \\ \xi(0) = x. \end{cases} \tag{1.2.3}$$

Since b is globally Lipschitz, for all $x \in \mathbb{R}^N$ there is a unique global solution $\xi(t,x)$ of (1.2.3) and the identity

$$x = \xi(t, \xi(-t,x)), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^N \tag{1.2.4}$$

holds. Moreover, from [36, Section 2.1] it follows that if ξ_x denotes the Jacobian matrix of the derivatives of ξ with respect to x, then

$$
\begin{aligned}
&|\xi_x(t,x)| \le e^{|t|\,\|Db\|_\infty}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N\\
&|\xi_{tx}(t,x)| \le \|Db\|_\infty e^{|t|\,\|Db\|_\infty}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N\\
&\left|\frac{\partial}{\partial t}\xi_x(t,\xi(-t,x))\right| \le \|Db\|_\infty e^{3|t|\,\|Db\|_\infty}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N.
\end{aligned} \tag{1.2.5}
$$

With analogous notation we have also that
(1.2.6)

$$
\begin{aligned}
&|\xi_{xx}(t,x)| \le e^{|t|\,\|Db\|_\infty}(e^{|t|\|Db\|_\infty}-1)\frac{\|D^2b\|_\infty}{\|Db\|_\infty}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N\\
&\left|\frac{\partial}{\partial x_i}\xi_x(t,\xi(-t,x))\right| \le e^{3|t|\,\|Db\|_\infty}(e^{|t|\|Db\|_\infty}-1)\frac{\|D^2b\|_\infty}{\|Db\|_\infty}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N,\ i=1,...N.
\end{aligned}
$$

In the case where b is constant, one should replace $\frac{e^{|t|\|Db\|_\infty}-1}{\|Db\|_\infty}$ by $|t|$. In particular, all the above functions are bounded in $[-T,T]\times\mathbb{R}^N$, for every $T>0$. Finally, the matrix ξ_x is invertible with determinant bounded away from zero in every strip $[-T,T]\times\mathbb{R}^N$.
Setting $v(t,y)=u(\xi(-t,y))$, a straightforward computation shows that

$$
D_t v - \widetilde{B}v = \widetilde{f}, \quad \text{in } \mathbb{R}^{N+1}
$$

with $\widetilde{f}(t,y)=f(\xi(-t,y))$ and

$$
\begin{aligned}
\widetilde{B} &= \sum_{i,j=1}^N \widetilde{a}_{ij}(t,y)D_{y_iy_j} + \sum_{i=1}^N \widetilde{b}_i(t,y)D_{y_i} - \widetilde{c},\\
\widetilde{a}_{ij}(t,y) &= \sum_{h,k=1}^N D_{x_h}\xi_i(t,\xi(-t,y))a_{hk}(\xi(-t,y))D_{x_k}\xi_j(t,\xi(-t,y))\\
\widetilde{b}_i(t,y) &= \sum_{h,k=1}^N D_{x_hx_k}\xi_i(t,\xi(-t,y))a_{hk}(\xi(-t,y)),\\
\widetilde{c}(t,y) &= c(\xi(-t,y)).
\end{aligned}
$$

Since the coefficients a_{ij} belong to $C_b^1(\mathbb{R}^N)$ and satisfy (iv), then $(t,y)\to a_{ij}(\xi(-t,y))$ is bounded and differentiable with bounded derivatives in $[-T,T]\times\mathbb{R}^N$. Taking into account (1.2.5) and (1.2.6) it follows that for all $(t,y)\in[-T,T]\times\mathbb{R}^N$ we have

$$
|\widetilde{a}_{ij}(t,y)| + |D_t\widetilde{a}_{ij}(t,y)| + |D_{y_k}\widetilde{a}_{ij}(t,y)| + |\widetilde{b}_i(t,y)| \le L, \quad i,j,k=1,...N,
$$

where L depends on $T, N, \|a_{ij}\|_\infty, \|Da_{ij}\|_\infty, \|\langle Da_{ij}, b\rangle\|_\infty, \|Db\|_\infty, \|D^2b\|_\infty$. Moreover

$$
\sum_{i,j=1}^N \widetilde{a}_{ij}(t,y)\eta_i\eta_j \ge \widetilde{\nu}_0|\eta|^2, \quad \eta,\, y\in\mathbb{R}^N,\, t\in[-T,T],
$$

with $\widetilde{\nu}_0$ depending on $\nu_0, T, \|Db\|_\infty$. Finally, the modulus of continuity of $\widetilde{a}_{ij}$ depends only on $T, N, \|a_{ij}\|_\infty, \|Da_{ij}\|_\infty, \|\langle Da_{ij}, b\rangle\|_\infty, \|Db\|_\infty, \|D^2b\|_\infty$. Therefore $D_t-\widetilde{B}$ is a uniformly parabolic operator in $[-T,T]\times\mathbb{R}^N$, for every $T>0$. Applying the classical L^p-estimates available from the theory of uniformly parabolic operators (see e.g. [30, Section IV.10]) we have that

$$
\int_{-1/2}^{1/2}\int_{\mathbb{R}^N}(|D_yv(t,y)|^p + |D_y^2v(t,y)|^p)dy\,dt \le K\int_{-1}^{1}\int_{\mathbb{R}^N}(|\widetilde{f}(t,y)|^p + |v(t,y)|^p)dy\,dt \tag{1.2.7}
$$

where K depends on $p, N, \widetilde{\nu}_0, \|\widetilde{a}_{ij}\|_\infty, \|D\widetilde{a}_{ij}\|_\infty, \|D_t\widetilde{a}_{ij}\|_\infty, \|\widetilde{b}_i\|_\infty, \|\widetilde{c}\|_\infty$, hence on p, N, ν_0, $\|a_{ij}\|_\infty$, $\|Da_{ij}\|_\infty$, $\|\langle Da_{ij}, b\rangle\|_\infty, \|Db\|_\infty, \|D^2 b\|_\infty, \|c\|_\infty$.
In order to come back to the function u, we observe that, setting $(S(t)\varphi)(x) = \varphi(\xi(t,x))$ then, for every fixed t, $S(t)$ maps $W^{2,p}(\mathbb{R}^N)$ into itself and

$$\begin{aligned}
\int_{\mathbb{R}^N} |(S(t)\varphi)(x)|^p dx &\leq \alpha_1(t) \int_{\mathbb{R}^N} |\varphi(y)|^p dy,\\
\int_{\mathbb{R}^N} |D_x(S(t)\varphi)(x)|^p dx &\leq \alpha_2(t) \int_{\mathbb{R}^N} |D_y\varphi(y)|^p dy,\\
\int_{\mathbb{R}^N} |D_x^2(S(t)\varphi)(x)|^p dx &\leq \alpha_3(t) \int_{\mathbb{R}^N} (|D_y^2\varphi(y)|^p + |D_y\varphi(y)|^p) dy,
\end{aligned}$$

with $\alpha_1(t), \alpha_2(t), \alpha_3(t)$ depending on $t, p, N, \sup_{\mathbb{R}^N} |\xi_x(-t,\cdot)|$ and $\alpha_3(t)$ depending also on $\sup_{\mathbb{R}^N}$ $|\xi_{xx}(-t,\cdot)|$. It follows that $t \mapsto \alpha_i(t)$, $i = 1,2,3$, are uniformly bounded in t in the interval $[-1,1]$. In the sequel we denote by α_i the respective upper bounds. Moreover, by (1.2.4) each $S(t)$ is invertible with $S(t)^{-1} = S(-t)$. Now, recalling that $u = S(t)v$, for every t, we have

$$\int_{\mathbb{R}^N} |D_x^2 u(x)|^p dx \leq \alpha_3 \int_{\mathbb{R}^N} (|D_y^2 v(t,y)|^p + |D_y v(t,y)|^p) dy.$$

Integrating from $-1/2$ to $1/2$ and taking into account (1.2.7) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |D_x^2 u(x)|^p dx &\leq \alpha_3\, K \int_{-1}^{1} \int_{\mathbb{R}^N} (|\widetilde{f}(t,y)|^p + |v(t,y)|^p) dy\, dt\\
&\leq 2\,\alpha_1\alpha_3 K \int_{\mathbb{R}^N} (|f(x)|^p + |u(x)|^p) dx,
\end{aligned}$$

which is the claim.

Step 2. Take B in the general form (1.2.1) and assume that the coefficients satisfy $(i), (ii), (iii)$ and (iv). Then we can write

$$B = \sum_{i,j=1}^N a_{ij} D_{ij} + \sum_{j=1}^N \left(\sum_{i=1}^N D_i a_{ij} + b_j \right) D_j - c.$$

Let $\eta \in C_c^\infty(\mathbb{R}^N)$, supp $\eta \subset B_1$, $\eta \geq 0$, $\int_{\mathbb{R}^N} \eta = 1$ and set $\hat{b} = b * \eta$. If we define

$$\hat{B} = \sum_{i,j=1}^N a_{ij} D_{ij} + \sum_{j=1}^N \hat{b}_j D_j - c,$$

then $\hat{B}$ satisfies all the assumptions of the previous step. Indeed, since b is Lipschitz continuous, $b - \hat{b}$ is bounded:

$$|b(x) - \hat{b}(x)| \leq \int_{\mathbb{R}^N} |b(x) - b(x-y)|\eta(y) dy \leq [b]_1 \int_{\mathbb{R}^N} |y|\eta(y) dy = c_\eta [b]_1.$$

Then

$$\begin{aligned}
|\langle Da_{ij}(x), \hat{b}(x)\rangle| &\leq |\langle Da_{ij}(x), b(x)\rangle| + |\langle Da_{ij}(x), b(x) - \hat{b}(x)\rangle|\\
&\leq \|\langle Da_{ij}, b\rangle\|_\infty + \|Da_{ij}\|_\infty c_\eta [b]_1,
\end{aligned}$$

and

$$\begin{aligned}
&\|D\hat{b}\|_\infty \leq [b]_1\\
&\|D^2\hat{b}\|_\infty \leq [b]_1 \|D\eta\|_1.
\end{aligned}$$

From the first step it follows that there exists a constant $C > 0$ depending on N,p,ν_0,$\|a_{ij}\|_\infty$, $\|Da_{ij}\|_\infty$, $\|\langle a_{ij}, b\rangle\|_\infty$,$[b]_1$,$\|c\|_\infty$ such that for all $u \in C_c^\infty(\mathbb{R}^N)$

$$\|D^2u\|_p \le C(\|\hat{B}u\|_p + \|u\|_p).$$

Therefore

$$\|D^2u\|_p \le C(\|Bu\|_p + \|Bu - \hat{B}u\|_p + \|u\|_p) \le C_1(\|Bu\|_p + \|Du\|_p + \|u\|_p),$$

with C_1 depending on the stated quantities. Using the interpolatory estimate $\|Du\|_p \le C_2\|u\|_p^{1/2} \cdot \|D^2u\|_p^{1/2}$ we conclude the proof. □

Next, we show that the operator B endowed with the domain

$$\mathcal{D} = \{u \in W^{2,p}(\mathbb{R}^N) \, : \, \langle b, Du\rangle \in L^p(\mathbb{R}^N)\}$$

generates a C_0-semigroup on $L^p(\mathbb{R}^N)$, $1 < p < +\infty$ (see also [37]). The following lemma is useful (see [37, Lemma 2.1]).

Lemma 1.2.2 *Let $1 < p < +\infty$ and $u \in W^{2,p}(B_R) \cap W_0^{1,p}(B_R)$. If $\eta \in C^1(\overline{B}_R)$ is nonnegative, then*

$$(p-1)\int_{B_R} \eta|u|^{p-2} \sum_{i,j=1}^N a_{ij}D_iuD_ju \; \chi_{\{u\neq 0\}} + \int_{B_R} u|u|^{p-2} \sum_{i,j=1}^N a_{ij}D_iuD_j\eta \le -\int_{B_R} \eta u|u|^{p-2} \sum_{i,j=1}^N D_i(a_{ij}D_ju). \tag{1.2.8}$$

Proof. Suppose first $p \ge 2$. In this case the function $u|u|^{p-2}$ belongs to $W^{1,q}(B_R)$, where q is the conjugate exponent of p. Indeed, it is obvious that $u|u|^{p-2}$ is in $L^q(B_R)$. Concerning the first order derivatives, we have $D(u|u|^{p-2}) = (p-1)|u|^{p-2}Du$. Then, using Hölder's inequality with exponent $\frac{p}{q} \ge 1$ we get

$$\begin{aligned} \int_{B_R} |u|^{q(p-2)}|Du|^q &\le \left(\int_{B_R} |Du|^p\right)^{\frac{q}{p}} \left(\int_{B_R} |u|^{\frac{pq(p-2)}{p-q}}\right)^{1-\frac{q}{p}} \\ &= \left(\int_{B_R} |Du|^p\right)^{\frac{q}{p}} \left(\int_{B_R} |u|^p\right)^{1-\frac{q}{p}}. \end{aligned}$$

Therefore, integration by parts is allowed in the right hand side of (1.2.8) and the statement is verified with equality.

Assume now $1 < p < 2$. Let first $u \in C^2(\overline{B}_R) \cap C_0(B_R)$. For every $\delta > 0$ we have

$$\begin{aligned} -\int_{B_R} \eta\, u(u^2+\delta)^{\frac{p}{2}-1} \sum_{i,j=1}^N D_i(a_{ij}D_ju) =& \int_{B_R} \eta(u^2+\delta)^{\frac{p}{2}-2}((p-1)u^2+\delta) \sum_{i,j=1}^N a_{ij}D_iuD_ju \\ &+ \int_{B_R} u(u^2+\delta)^{\frac{p}{2}-1} \sum_{i,j=1}^N a_{ij}D_iuD_j\eta. \end{aligned} \tag{1.2.9}$$

Then, from Fatou's Lemma we have

$$(p-1)\int_{B_R} \eta|u|^{p-2}\sum_{i,j=1}^N a_{ij}D_iuD_ju\ \chi_{\{u\neq 0\}}$$

$$\leq \liminf_{\delta\to 0}\Big(-\int_{B_R}\eta\, u(u^2+\delta)^{\frac{p}{2}-1}\sum_{i,j=1}^N D_i(a_{ij}D_ju) - \int_{B_R} u(u^2+\delta)^{\frac{p}{2}-1}\sum_{i,j=1}^N a_{ij}D_iuD_j\eta\Big)$$

$$= -\int_{B_R}\eta\, u|u|^{p-2}\sum_{i,j=1}^N D_i(a_{ij}D_ju) - \int_{B_R} u|u|^{p-2}\sum_{i,j=1}^N a_{ij}D_iuD_j\eta.$$

It follows that the function $\eta|u|^{p-2}\sum_{i,j=1}^N a_{ij}D_iuD_ju\ \chi_{\{u\neq 0\}}$ belongs to $L^1(B_R)$ and, letting $\delta\to 0$ in (1.2.9), by dominated convergence (1.2.8) holds with equality. In the general case where $u\in W^{2,p}(B_R)\cap W_0^{1,p}(B_R)$, we can consider a sequence (u_n) in $C^2(\overline{B}_R)\cap C_0(B_R)$ such that u_n converges to u in $W^{2,p}(B_R)$. In particular, we can find a subsequence (u_{n_k}) and functions $h_1,h_2,h_3\in L^p(B_R)$ such that $u_{n_k}, Du_{n_k}, D^2u_{n_k}$ converge to u, Du and D^2u, respectively, almost everywhere and

$$\begin{aligned} |u_{n_k}(x)| &\leq h_1(x),\\ |Du_{n_k}(x)| &\leq h_2(x),\\ |D^2u_{n_k}(x)| &\leq h_3(x) \end{aligned}$$

(see [10, Teorema IV.9]. Taking the previous step into account and applying again Fatou's Lemma, we get

$$(p-1)\int_{B_R}\eta|u|^{p-2}\sum_{i,j=1}^N a_{ij}D_iuD_ju\ \chi_{\{u\neq 0\}}$$

$$\leq \liminf_{k\to+\infty}\Big(-\int_{B_R}\eta\, u_{n_k}|u_{n_k}|^{p-2}\sum_{i,j=1}^N D_i(a_{ij}D_ju_{n_k})$$

$$-\int_{B_R} u_{n_k}|u_{n_k}|^{p-2}\sum_{i,j=1}^N a_{ij}D_iu_{n_k}D_j\eta\Big). \tag{1.2.10}$$

Using Young's inequality one has

$$\begin{aligned} \Big|u_{n_k}|u_{n_k}|^{p-2}\sum_{i,j=1}^N D_i(a_{ij}D_ju_{n_k})\Big| &\leq c_1|u_{n_k}|^{p-1}\Big(|Du_{n_k}|+|D^2u_{n_k}|\Big)\\ &\leq c_2\Big(|u_{n_k}|^p+\big(|Du_{n_k}|+|D^2u_{n_k}|\big)^p\Big)\\ &\leq c_3\Big(|u_{n_k}|^p+|Du_{n_k}|^p+|D^2u_{n_k}|^p\Big)\\ &\leq c_3\Big(h_1^p+h_2^p+h_3^p\Big)\in L^1(B_R), \end{aligned}$$

where c_3 depends on $\|a_{ij}\|_\infty, \|Da_{ij}\|_\infty$ and p. In the same way, one can estimate the remaining term, hence estimate (1.2.8) follows from (1.2.10) using dominated convergence. □

Proposition 1.2.3 *$(B,\mathcal{D})$ generates a strongly continuous semigroup $T(t)$ in $L^p(\mathbb{R}^N)$, $1<p<\infty$. Moreover, setting $\lambda_p := -\inf_{x\in\mathbb{R}^N}\Big(\frac{1}{p}\operatorname{div} b(x)+c(x)\Big)$, for all $\lambda>\lambda_p$ and $f\in L^p(\mathbb{R}^N)$, there exists a unique solution $u\in\mathcal{D}$ of $\lambda u - Bu = f$ and the estimate*

$$\|u\|_p \leq (\lambda-\lambda_p)^{-1}\|f\|_p \tag{1.2.11}$$

is satisfied.

Proof. It is sufficient to prove the statement when c is equal to zero, since in the general case, the thesis easily follows from a perturbation argument (see [21, III.1.3]).

Let us consider $(B, C_c^\infty(\mathbb{R}^N))$. Proceeding as in the forthcoming Lemma 1.3.1, it can be proved that $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathcal{D}$ with respect to its natural norm

$$\|u\|_{\mathcal{D}} = \|u\|_{2,p} + \|\langle b, Du\rangle\|_p.$$

The interpolatory estimate $\|Du\|_p \le k(\|u\|_p + \|D^2u\|_p)$ and estimate (1.2.2) yield immediately

$$\|Du\|_p \le C(\|u\|_p + \|Bu\|_p), \quad u \in C_c^\infty(\mathbb{R}^N).$$

Therefore, we have

$$\|\langle b, Du\rangle\|_p = \left\| \sum_{i,j=1}^N D_i(a_{ij}D_ju) - Bu \right\|_p \le C(\|D^2u\|_p + \|Du\|_p + \|Bu\|_p) \le C(\|u\|_p + \|Bu\|_p).$$

Collecting all the estimates so far, we have established that for every $u \in C_c^\infty(\mathbb{R}^N)$, hence, by density, for every $u \in \mathcal{D}$

$$\|u\|_{2,p} + \|\langle b, Du\rangle\|_p \le C(\|u\|_p + \|Bu\|_p).$$

Since the other inequality is obvious, we have that $\|\cdot\|_{\mathcal{D}}$ and the graph norm of B, $\|\cdot\|_B$, are equivalent. Therefore, $(\mathcal{D}, \|\cdot\|_B)$ is complete and as a consequence $(B, \mathcal{D})$ is closed in $L^p(\mathbb{R}^N)$.

Let us prove that $(B - \lambda_0, C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N)$, where

$$\lambda_0 = -\frac{1}{p} \inf_{\mathbb{R}^N} \operatorname{div} b.$$

In this case, we say that $(B, C_c^\infty(\mathbb{R}^N))$ is *quasi-dissipative*. Let $\lambda > \lambda_0$ and $u \in C_c^\infty(\mathbb{R}^N)$ be fixed. Multiplying the equation $\lambda u - Bu = f$ by $u|u|^{p-2}$ and integrating by parts we deduce

$$\lambda \int_{\mathbb{R}^N} |u|^p dx + (p-1) \int_{\mathbb{R}^N} |u|^{p-2} \sum_{i,j=1}^N a_{ij} D_iuD_ju\, dx + \frac{1}{p} \int_{\mathbb{R}^N} \operatorname{div} b\, |u|^p dx = \int_{\mathbb{R}^N} f\, u|u|^{p-2} dx$$

and then

$$\begin{aligned}(\lambda - \lambda_0) \int_{\mathbb{R}^N} |u|^p dx &\le \int_{\mathbb{R}^N} \left(\lambda + \frac{1}{p} \operatorname{div} b\right) |u|^p dx + \nu_0(p-1) \int_{\mathbb{R}^N} |Du|^2 |u|^{p-2} dx \\ &\le \left(\int_{\mathbb{R}^N} |f|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1-\frac{1}{p}}.\end{aligned}$$

Dividing by $\|u\|_p^{p-1}$ we get $(\lambda - \lambda_0)\|u\|_p \le \|\lambda u - Bu\|_p$, as claimed. Therefore, the operator $(B, C_c^\infty(\mathbb{R}^N))$ is quasi-dissipative.

The next step is to show that $(\lambda - B)C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for some large λ. Let q be the conjugate exponent of p and let $w \in L^q(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N} (\lambda\varphi - B\varphi) w\, dx = 0, \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^N). \tag{1.2.12}$$

We claim that $w = 0$. By a classical result concerning local regularity of distributional solutions to elliptic equations (see [5] and the references therein), it turns out that $w \in W^{2,q}_{\rm loc}(\mathbb{R}^N)$. Therefore we are allowed to integrate by parts in (1.2.12) and we deduce that

$$\int_{\mathbb{R}^N} \lambda w\, \varphi\, dx - \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_i(a_{ij}D_jw)\varphi\, dx + \int_{\mathbb{R}^N} \operatorname{div} b\, w\, \varphi\, dx + \int_{\mathbb{R}^N} \langle b, Dw\rangle \varphi\, dx = 0. \tag{1.2.13}$$

Using an approximation argument, it can be seen that the equation in this form is satisfied also by any function φ of $L^p(\mathbb{R}^N)$ with compact support. Indeed, if φ is such a function, set $\varphi_n = \varrho_n * \varphi$, where ϱ_n is a standard sequence of mollifiers. Then $\varphi_n \in C_c^\infty(\mathbb{R}^N)$ and φ_n converges to φ in $L^p(\mathbb{R}^N)$, as $n \to \infty$. Moreover, we can find $R > 0$ sufficiently large in such a way that $\mathrm{supp}\varphi_n$ and $\mathrm{supp}\varphi$ are contained in B_R, for every $n \in \mathbb{N}$. Each φ_n satisfies (1.2.13) and letting $n \to \infty$, we obtain that φ verifies (1.2.13), too.

Now, let η be in $C_c^\infty(\mathbb{R}^N)$ such that $\eta \equiv 1$ in B_1, $0 \le \eta \le 1$, $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_2$ and set $\eta_n(x) = \eta(\frac{x}{n})$. Plugging $w|w|^{q-2}\,\eta_n^2$ into (1.2.13) and using (1.2.8) we deduce

$$\begin{aligned}&\int_{\mathbb{R}^N} \lambda |w|^q\,\eta_n^2 + (p-1)\int_{\mathbb{R}^N} \eta_n^2 |w|^{q-2}\sum_{i,j=1}^N a_{ij} D_i w D_j w\ \chi_{\{w\neq 0\}}\\ &\quad +2\int_{\mathbb{R}^N} w|w|^{q-2}\eta_n \sum_{i,j=1}^N a_{ij}D_i w D_j \eta_n + \int_{\mathbb{R}^N} \mathrm{div} b\,|w|^q\,\eta_n^2 + \int_{\mathbb{R}^N} \langle b, Dw\rangle w|w|^{q-2}\,\eta_n^2\\ &\le \int_{\mathbb{R}^N} \lambda|w|^q\,\eta_n^2 - \int_{\mathbb{R}^N}\sum_{i,j=1}^N D_i(a_{ij}D_j w) w|w|^{q-2}\,\eta_n^2 + \int_{\mathbb{R}^N} \mathrm{div} b\,|w|^q\,\eta_n^2\\ &\quad + \int_{\mathbb{R}^N} \langle b, Dw\rangle w|w|^{q-2}\,\eta_n^2 = 0.\end{aligned}$$

Then, using the ellipticity condition and integrating by parts we get

$$\begin{aligned}&\int_{\mathbb{R}^N} \lambda|w|^q\,\eta_n^2 + \nu_0(p-1)\int_{\mathbb{R}^N}\eta_n^2|w|^{q-2}|Dw|^2\ \chi_{\{w\neq 0\}} + 2\int_{\mathbb{R}^N} w|w|^{q-2}\eta_n \sum_{i,j=1}^N a_{ij}D_i w D_j\eta_n\\ &\quad + \int_{\mathbb{R}^N} \mathrm{div} b\,|w|^q\,\eta_n^2 - \frac{1}{q}\int_{\mathbb{R}^N} \mathrm{div} b\,|w|^q\,\eta_n^2 - \frac{2}{q}\int_{\mathbb{R}^N}\langle b, D\eta_n\rangle |w|^q\,\eta_n \le 0.\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N}\left(\lambda + \frac{1}{p}\,\mathrm{div} b\right)|w|^q\,\eta_n^2 + \nu_0(p-1)\int_{\mathbb{R}^N}\eta_n^2|w|^{q-2}|Dw|^2\ \chi_{\{w\neq 0\}} \le I_1 + I_2, \tag{1.2.14}$$

where

$$\begin{aligned} I_1 &= -2\int_{\mathbb{R}^N} w|w|^{q-2}\eta_n \sum_{i,j=1}^N a_{ij}D_i w D_j\eta_n\,dx\\ I_2 &= \frac{2}{q}\int_{\mathbb{R}^N}\langle b, D\eta_n\rangle |w|^q\,\eta_n\,dx.\end{aligned}$$

From Hölder's inequality it follows that

$$\begin{aligned} |I_1| &\le 2\,NK\int_{\mathbb{R}^N}\eta_n\,|Dw|\,|D\eta_n|\,|w|^{q-1}\,dx\\ &\le \frac{2\,N\,\|D\eta\|_\infty K}{n}\int_{\mathbb{R}^N}\eta_n\,|Dw|\,|w|^{(q-2)/2}|w|^{q/2}\ \chi_{\{w\neq 0\}}\,dx \qquad (1.2.15)\\ &\le \frac{\|D\eta\|_\infty\,NK}{n}\int_{\mathbb{R}^N}\eta_n^2|Dw|^2|w|^{q-2}\ \chi_{\{w\neq 0\}}\,dx + \frac{\|D\eta\|_\infty\,NK}{n}\int_{\mathbb{R}^N}|w|^q\,dx,\end{aligned}$$

where $K = \max_{i,j}\|a_{ij}\|_\infty$. Concerning I_2, we observe that since b is Lipschitz continuous in $\mathbb{R}^N$, there exists a constant $L > 0$ such that $|b(x)| \le L(1+|x|)$, for every $x \in \mathbb{R}^N$. Therefore

$$\begin{aligned} |I_2| &\le \frac{2}{q}\int_{n\le|x|\le 2n}\eta_n(x)|b(x)|\,|D\eta_n(x)|\,|w(x)|^q\,dx \qquad (1.2.16)\\ &\le \frac{2\|D\eta\|_\infty L}{q}\int_{n\le|x|\le 2n}\frac{(1+|x|)}{n}|w(x)|^q\,dx\\ &\le \frac{6\,\|D\eta\|_\infty L}{q}\int_{n\le|x|\le 2n}|w(x)|^q\,dx.\end{aligned}$$

Taking (1.2.15) and (1.2.16) into account, (1.2.14) gives

$$\int_{\mathbb{R}^N}\left(\lambda+\frac{1}{p}\,\mathrm{div} b\right)|w|^q\,\eta_n^2+\left(\nu_0(p-1)-\frac{\|D\eta\|_\infty\,N\,K}{n}\right)\int_{\mathbb{R}^N}\eta_n^2|w|^{q-2}|Dw|^2\;\chi_{\{w\neq 0\}}$$
$$\leq\frac{\|D\eta\|_\infty\,N\,K}{n}\int_{\mathbb{R}^N}|w|^q\,dx+\frac{6\,\|D\eta\|_\infty L}{q}\int_{n\leq|x|\leq 2n}|w|^q\,dx.$$

For n large $\nu_0(p-1)-\frac{\|D\eta\|_\infty\,N\,K}{n}>0$ and if $\lambda>\lambda_0$ we have

$$(\lambda-\lambda_0)\int_{\mathbb{R}^N}|w|^q\,\eta_n^2\leq\frac{\|D\eta\|_\infty\,N\,K}{n}\int_{\mathbb{R}^N}|w|^q\,dx+\frac{6\,\|D\eta\|_\infty L}{q}\int_{n\leq|x|\leq 2n}|w|^q\,dx.$$

Letting $n\to+\infty$ we infer $w=0$.

From the Lumer Phillips Theorem [21, Theorem II.3.15] it follows that the closure $(\mathcal{B},D(\mathcal{B}))$ of $(B,\,C_c^\infty(\mathbb{R}^N))$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$. Since $(B,\mathcal{D})$ is closed and $C_c^\infty(\mathbb{R}^N)\subseteq\mathcal{D}$, we find that $(B,\mathcal{D})$ extends $(\mathcal{B},D(\mathcal{B}))$. Conversely, if $f\in\mathcal{D}$, then there exists a sequence (f_n) in $C_c^\infty(\mathbb{R}^N)$ such that f_n converges to f with respect to $\|\cdot\|_{\mathcal{D}}$, which is equivalent to $\|\cdot\|_B$. This implies, by definition, that $f\in D(\mathcal{B})$ and $Bf=\mathcal{B}f$. Therefore $(\mathcal{B},D(\mathcal{B}))$ coincides with $(B,\mathcal{D})$.

As far as the last part of the statement is concerned, we observe that as a consequence of the generation result, for λ large, the resolvent equation $\lambda u-Bu=f$ admits a unique solution $u\in\mathcal{D}$, for every $f\in L^p(\mathbb{R}^N)$. In order to determine the lower bound of λ, as before we have to multiply the equation $\lambda u-Bu=f$ by $u|u|^{p-2}$ and to integrate by parts. In this way we find that λ has to be strictly larger than $\lambda_p=-\inf\left(\frac{1}{p}\,\mathrm{div}\,b+c\right)$ and that estimate (1.2.11) holds, as stated. □

1.3 A priori estimates of $\|Vu\|_p$, $\|Du\|_p$ and $\|D^2u\|_2$

From now on, for clarity of exposition, we assume that $c_\alpha=c_\beta=c_\gamma=c_\tau=c_\theta=0$ in conditions (H1), (H2), (H3) and (H4). This is always possible, keeping the same constants α,β,γ,τ, just replacing V with $V+\lambda$ and choosing λ large enough (this implies possibly different constants in the statements).

In this section we provide, as a preliminary step, some a priori estimates for the solutions of the elliptic equation $\lambda u-Au=f$. Precisely, via integrations by parts and other elementary tools, we prove that for all $u\in\mathcal{D}_p$, the L^p-norms of Vu and Du may be estimated by the L^p-norms of Au and u itself, with constants independent of u. If $p=2$, we also deduce an analogous estimate for the second order derivatives of u.

Let us first show that $C_c^\infty(\mathbb{R}^N)$ is dense in $(\mathcal{D}_p,\|\cdot\|_{\mathcal{D}_p})$, $1<p<+\infty$, so that all our estimates will be proved on test-functions.

Lemma 1.3.1 *Suppose that (H4) holds. Then $C_c^\infty(\mathbb{R}^N)$ is dense in $(\mathcal{D}_p,\|\cdot\|_{\mathcal{D}_p})$.*

PROOF. Let η be a cut-off function such that $0\leq\eta\leq 1$, $\eta\equiv 1$ in B_1, $\mathrm{supp}\,\eta\subset\mathrm{B}_2$ and $|D\eta|^2+|D^2\eta|\leq L$. We write $\eta_n(x)$ in place of $\eta(x/n)$.

Suppose that $u\in\mathcal{D}_p$. It is easy to see that $\|\eta_n u-u\|_{\mathcal{D}_p}$, as $n\to\infty$. In fact, $\eta_n u\to u$ in $W^{2,p}(\mathbb{R}^N)$ and $V\eta_n u\to Vu$ in $L^p(\mathbb{R}^N)$, by dominated convergence. Moreover,

$$\langle F,D(\eta_n u)\rangle=\eta_n\langle F,Du\rangle+u\langle F,D\eta_n\rangle\,.$$

As before, the first term in the right hand side converges to $\langle F, Du\rangle$ in $L^p(\mathbb{R}^N)$, as n goes to infinity. The second term tends to 0 since from (H4) it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^p |\langle F, D\eta_n\rangle|^p\,dx &\le L^{p/2}\theta^p \int_{B_{2n}\setminus B_n} |Vu|^p \left(\frac{1+4n^2}{n^2}\right)^{p/2} dx \\ &\le 5^{p/2} L^{p/2}\,\theta^p \int_{\mathbb{R}^N\setminus B_n} |Vu|^p\,dx. \end{aligned} \tag{1.3.1}$$

This shows that the set of functions in $\mathcal{D}_p$ having compact support, denoted by $\mathcal{D}_{p,c}$, is dense in $\mathcal{D}_p$.

Suppose now that $u \in \mathcal{D}_{p,c}$. A standard convolution argument shows the existence of a sequence of smooth functions with compact support converging to u in $\mathcal{D}_p$. Thus, the density of $C_c^\infty(\mathbb{R}^N)$ in $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$ follows. □

We state that, under rather weak assumptions, the operator $(A, C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N)$, for any $1 < p < +\infty$.

Lemma 1.3.2 *Suppose that*

$$\operatorname{div} F + p\,V \ge 0. \tag{1.3.2}$$

Then $(A, C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N)$.

PROOF. We have to prove that for all $\lambda > 0$ and for all $u \in C_c^\infty(\mathbb{R}^N)$ one has

$$\|u\|_p \le \frac{1}{\lambda}\|\lambda u - Au\|_p. \tag{1.3.3}$$

Let $\lambda > 0$ be fixed. If $u \in C_c^\infty(\mathbb{R}^N)$ we set $u^* = u|u|^{p-2}$ and recall that

$$D(u^*) = (p-1)|u|^{p-2}Du, \quad D(|u|^p) = pu^* Du\,. \tag{1.3.4}$$

Set $\lambda u - Au = f$. Multiplying both sides of this equation by u^* and integrating by parts, we obtain

$$\lambda \int_{\mathbb{R}^N} |u|^p + (p-1)\int_{\mathbb{R}^N} \langle qDu, Du\rangle |u|^{p-2}\,dx + \frac{1}{p}\int_{\mathbb{R}^N} \operatorname{div} F |u|^p\,dx + \int_{\mathbb{R}^N} V|u|^p\,dx = \int_{\mathbb{R}^N} fu^*\,dx\,.$$

By (1.1.1) we get

$$(p-1)\int_{\mathbb{R}^N} \langle qDu, Du\rangle |u|^{p-2}\,dx \ge (p-1)\nu_0 \int_{\mathbb{R}^N} |Du|^2|u|^{p-2}\,dx \ge 0$$

and taking (1.3.2) into account it turns out that

$$\lambda\int_{\mathbb{R}^N} |u|^p \le \int_{\mathbb{R}^N} fu^*\,dx \le \left(\int_{\mathbb{R}^N} |f|^p\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |u|^p\,dx\right)^{1-\frac{1}{p}}.$$

Multiplying by $\|u\|_p^{1-p}$ we get (1.3.3). □

Remark 1.3.3 It is noteworthy observing that if (1.3.2) holds, $1 < p \le 2$ and $u \in C_c^\infty(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} |Du|^p \le c\int_{\mathbb{R}^N} (|Au|^p + |u|^p)\,dx\,, \tag{1.3.5}$$

where $c = c(\nu_0, p) > 0$. In fact, from the proof of Lemma 1.3.2, with $\lambda = 1$, we deduce that

$$
\begin{aligned}
(1.3.6) \qquad \int_{\mathbb{R}^N} |Du|^2 |u|^{p-2}\, dx &\leq \frac{1}{\nu_0(p-1)} \left(\int_{\mathbb{R}^N} |u - Au|^p\, dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |u|^p\, dx\right)^{1-\frac{1}{p}} \\
&\leq c \int_{\mathbb{R}^N} (|Au|^p + |u|^p)\, dx\,,
\end{aligned}
$$

where $c = c(\nu_0, p) > 0$. If $p = 2$, we are done. If $1 < p < 2$, Young's inequality with exponent $2/p$ yields

$$
\int_{\{u \neq 0\}} |Du|^p\, dx = \int_{\{u\neq 0\}} \left(|Du|^p |u|^{\frac{p(p-2)}{2}}\right) |u|^{-\frac{p(p-2)}{2}}\, dx \leq c_p \int_{\{u \neq 0\}} (|Du|^2|u|^{p-2} + |u|^p)\, dx
$$

and (1.3.5) follows by (1.3.6).

Remark 1.3.4 We note that condition (H2'), with $c_\beta = 0$, together with (1.1.7) implies condition (1.3.2), so that Lemma 1.3.2 still holds. If $c_\beta \neq 0$, then the same computations of Lemma 1.3.2 show that $(A - \frac{c_\beta}{p}, C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N)$, which means that operator $(A, C_c^\infty(\mathbb{R}^N))$ is quasi-dissipative. Explicitly, one has

$$
(1.3.7) \qquad \|u\|_p \leq \left(\lambda - \frac{c_\beta}{p}\right)^{-1} \|(\lambda - A)u\|_p\,, \quad u \in C_c^\infty(\mathbb{R}^N).
$$

In the following lemma we prove an estimate of the L^p-norm of Vu.

Lemma 1.3.5 *Let $1 < p < +\infty$. Assume that (H1), (H3) and*

$$
(1.3.8) \qquad \mathrm{div} F + \beta V \geq 0
$$

hold with

$$
(1.3.9) \qquad \frac{M}{4}(p-1)\alpha^2 + \frac{\beta}{p} + \gamma \frac{p-1}{p} < 1\,,
$$

where $M := \sup_{x \in \mathbb{R}^N} \max_{|\xi|=1} \langle q(x)\xi, \xi\rangle$.
If $u \in C_c^\infty(\mathbb{R}^N)$, then

$$
(1.3.10) \qquad \int_{\mathbb{R}^N} |Vu|^p\, dx \leq c \int_{\mathbb{R}^N} (|Au|^p + |u|^p)\, dx
$$

for some $c > 0$ depending only on p, M, ν_0 and on the constants in (H1), (H3) and (1.3.8).

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$. We recall that if $u^* = u|u|^{p-2}$, then (1.3.4) holds.
Integrating by parts one deduces

$$
\begin{aligned}
&\int_{\mathbb{R}^N} (A_0 u) V^{p-1} u^*\, dx = -\int_{\mathbb{R}^N} \langle qDu, D(V^{p-1}u^*)\rangle\, dx \\
&= -(p-1)\int_{\mathbb{R}^N} \langle qDu, Du\rangle V^{p-1}|u|^{p-2}\, dx - (p-1)\int_{\mathbb{R}^N} \langle qDu, DV\rangle V^{p-2}|u|^{p-2} u\, dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} V^{p-1}\langle F, Du\rangle\, u^*\, dx = \frac{1}{p}\int_{\mathbb{R}^N} V^{p-1}\langle F, D(|u|^p)\rangle\, dx \\
&= -\frac{1}{p}\int_{\mathbb{R}^N} V^{p-1} \mathrm{div} F |u|^p\, dx - \frac{p-1}{p}\int_{\mathbb{R}^N} V^{p-2}\langle F, DV\rangle |u|^p\, dx\,.
\end{aligned}
$$

Thus, multiplying (1.1.2) by $V^{p-1}u^*$ and integrating, we obtain

$$(1.3.11)\qquad \begin{aligned}&(p-1)\int_{\mathbb{R}^N}\langle qDu,Du\rangle V^{p-1}|u|^{p-2}\,dx+\int_{\mathbb{R}^N}|Vu|^p\,dx\\ &=-\int_{\mathbb{R}^N}(Au)V^{p-1}u^*\,dx-\frac{1}{p}\int_{\mathbb{R}^N}V^{p-1}\mathrm{div}F|u|^p\,dx\\ &-\frac{p-1}{p}\int_{\mathbb{R}^N}V^{p-2}\langle F,DV\rangle|u|^p\,dx-(p-1)\int_{\mathbb{R}^N}\langle qDu,DV\rangle V^{p-2}|u|^{p-2}u\,dx.\end{aligned}$$

Now, assumptions (1.3.8) and (H3) imply

$$(1.3.12)\qquad -\int_{\mathbb{R}^N}V^{p-1}\mathrm{div}F|u|^p\,dx\le\beta\int_{\mathbb{R}^N}|Vu|^p\,dx$$

and

$$(1.3.13)\qquad -\int_{\mathbb{R}^N}V^{p-2}\langle F,DV\rangle|u|^p\,dx\le\gamma\int_{\mathbb{R}^N}|Vu|^p\,dx\,,$$

respectively.
By (1.1.1) and (H1) the last term in (1.3.11) can be estimated as follows

$$(1.3.14)\qquad \begin{aligned}\int_{\mathbb{R}^N}\langle qDu,DV\rangle V^{p-2}|u|^{p-2}u\,dx &\le \int_{\mathbb{R}^N}\langle qDu,Du\rangle^{1/2}\langle qDV,DV\rangle^{1/2}V^{p-2}|u|^{p-1}\,dx\\ &\le \alpha\sqrt{M}\int_{\mathbb{R}^N}\langle qDu,Du\rangle^{1/2}V^{p-1/2}|u|^{p-1}\,dx\,.\end{aligned}$$

Setting $Q^2:=\int_{\mathbb{R}^N}\langle qDu,Du\rangle V^{p-1}|u|^{p-2}\,dx$ and $R^2:=\int_{\mathbb{R}^N}|Vu|^p\,dx$, from Hölder's inequality it follows

$$(1.3.15)\qquad \int_{\mathbb{R}^N}\langle qDu,Du\rangle^{1/2}V^{p-1/2}|u|^{p-1}\,dx\le QR.$$

Thus, collecting (1.3.11)–(1.3.14) we obtain

$$\begin{aligned}(p-1)Q^2+\left(1-\frac{\beta}{p}-\frac{\gamma(p-1)}{p}\right)R^2 &\le \alpha(p-1)\sqrt{M}QR+\left|\int_{\mathbb{R}^N}(Au)V^{p-1}u^*\,dx\right|\\ &\le (p-1)Q^2+\frac{(p-1)\alpha^2M}{4}R^2\\ &\quad+\left|\int_{\mathbb{R}^N}(Au)V^{p-1}u^*\,dx\right|.\end{aligned}$$

Since

$$\left|\int_{\mathbb{R}^N}(Au)V^{p-1}u^*\,dx\right|\le\int_{\mathbb{R}^N}|Au||Vu|^{p-1}\,dx\le\varepsilon R^2+c_\varepsilon\int_{\mathbb{R}^N}|Au|^p\,dx\,,$$

the thesis follows from (1.3.9) and by choosing ε small enough. □

The next result provides an L^p-estimate of $V|Du|$, with $p\ge 2$. In particular, since $V\ge 1$, it extends estimate (1.3.5) to the case $p>2$. We explicitly notice that we need a further assumption on F, namely the dissipativity condition.

Lemma 1.3.6 *Let $p\ge 2$. Assume that (H1), (H2), (H3) and (1.3.9) hold and that β satisfies also the inequality*

$$(1.3.16)\qquad 1-\frac{\beta}{p}-\tau>0\,.$$

If $u \in C_c^\infty(\mathbb{R}^N)$*, then*

$$\int_{\mathbb{R}^N} V|Du|^p\,dx + \int_{\mathbb{R}^N} |Du|^{p-2}|D^2u|^2\,dx \le c\int_{\mathbb{R}^N}(|Au|^p + |u|^p)\,dx\,, \tag{1.3.17}$$

with c *depending on* $N, p, \nu_0, \alpha, \beta, \tau, M, \|Dq_{ij}\|_\infty$.

Proof. We divide the proof in two steps: in the first step we consider the supplementary assumption that $q_{ij} \in C^2(\mathbb{R}^N)$, in the second one we remove this condition via an approximation procedure.

Step 1. Suppose that $q_{ij} \in C^2(\mathbb{R}^N) \cap C_b^1(\mathbb{R}^N)$, for every $1 \le i, j \le N$. Let $u \in C_c^\infty(\mathbb{R}^N)$ and define $f = \lambda u - Au$, with $\lambda > 0$ to be chosen later. With a fixed $k \in \{1, ..., N\}$, we differentiate with respect to x_k, so that

$$\begin{aligned}\lambda D_k u - \sum_{i,j=1}^N D_i(D_k q_{ij} D_j u) - \sum_{i,j=1}^N D_i(q_{ij} D_{jk} u) - \sum_{i=1}^N D_k F_i D_i u\\ - \sum_{i=1}^N F_i D_{ik} u + u D_k V + V D_k u = D_k f.\end{aligned} \tag{1.3.18}$$

Multiplying (1.3.18) by $D_k u|Du|^{p-2}$, summing over $k = 1, ..., N$ and integrating on $\mathbb{R}^N$ we get

$$\lambda\int_{\mathbb{R}^N}|Du|^p\,dx + I_1 + I_2 + I_3 + I_4 + I_5 + \int_{\mathbb{R}^N} V|Du|^p\,dx = \int_{\mathbb{R}^N}\langle Df, Du\rangle|Du|^{p-2}\,dx, \tag{1.3.19}$$

where

$$\begin{aligned}
I_1 &= -\int_{\mathbb{R}^N}\sum_{i,j,k=1}^N D_i(D_k q_{ij} D_j u)\,D_k u|Du|^{p-2}\,dx,\\
I_2 &= -\int_{\mathbb{R}^N}\sum_{i,j,k=1}^N D_i(q_{ij} D_{jk} u) D_k u|Du|^{p-2}\,dx,\\
I_3 &= -\int_{\mathbb{R}^N}\sum_{i,k=1}^N D_k F_i\; D_i u\; D_k u|Du|^{p-2}\,dx,\\
I_4 &= -\int_{\mathbb{R}^N}\sum_{i,k=1}^N F_i\; D_{ik} u\; D_k u|Du|^{p-2}\,dx,\\
I_5 &= \int_{\mathbb{R}^N}\langle DV, Du\rangle u|Du|^{p-2}\,dx\,.
\end{aligned}$$

Let us estimate the integrals above. Since $t \mapsto t|t|^{p-2}$ is in $C^1(\mathbb{R}^N;\mathbb{R}^N)$, integrating by parts and applying Hölder's and Young's inequalities we have

$$\begin{aligned}
|I_1| &= \left|\int_{\mathbb{R}^N}\sum_{i,j,k=1}^N D_k q_{ij}\; D_j u D_{ik} u|Du|^{p-2}\right.\\
&\quad \left. +(p-2)\int_{\mathbb{R}^N}\sum_{i,j,k,h=1}^N D_k q_{ij}\; D_j u D_k u D_h u D_{ih} u|Du|^{p-4}\right|\\
&\le c_1\int_{\mathbb{R}^N}|Du|^{p-1}|D^2u|\,dx = c_1\int_{\mathbb{R}^N}|Du|^{p/2}(|Du|^{(p-2)/2}|D^2u|)\,dx\\
&\le \frac{c_1}{\varepsilon}\int_{\mathbb{R}^N}|Du|^p\,dx + c_1\,\varepsilon\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\,,
\end{aligned}$$

where $c_1 = c_1\left(p, N, \|Dq_{ij}\|_\infty\right)$ and $\varepsilon > 0$ is arbitrary. Consequently

$$I_1 \geq -\frac{c_1}{\varepsilon}\int_{\mathbb{R}^N} |Du|^p\, dx - c_1\, \varepsilon \int_{\mathbb{R}^N} |Du|^{p-2}|D^2u|^2\, dx\,. \tag{1.3.20}$$

Assumption (1.1.1) allows to estimate the second integral, after an integration by parts; indeed

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^N} \sum_{i,j,k=1}^N q_{ij}\ D_{jk}u\ D_{ik}u|Du|^{p-2}\, dx \\
&\quad + \frac{p-2}{4}\int_{\mathbb{R}^N} \sum_{i,j=1}^N q_{ij}\ D_j(|Du|^2)\, D_i(|Du|^2)|Du|^{p-4}\, dx \\
&\geq \nu_0 \int_{\mathbb{R}^N} |D^2u|^2|Du|^{p-2}\, dx + \nu_0\frac{p-2}{4}\int_{\mathbb{R}^N} \left|D\big(|Du|^2\big)\right|^2 |Du|^{p-4}\, dx\,.
\end{aligned}$$

Since the last term is nonnegative we deduce that

$$I_2 \geq \nu_0 \int_{\mathbb{R}^N} |Du|^{p-2}|D^2u|^2\, dx. \tag{1.3.21}$$

From (H2) it follows immediately that

$$I_3 \geq -\tau \int_{\mathbb{R}^N} V|Du|^p\, dx. \tag{1.3.22}$$

As far as I_4 is concerned, integrating by parts, it turns out that

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}^N} \sum_{i,k=1}^N D_iF_i\ (D_ku)^2\ |Du|^{p-2}\, dx + \int_{\mathbb{R}^N} \sum_{i,k=1}^N F_i\ D_ku\ D_{ik}u\ |Du|^{p-2}\, dx \\
&\quad +(p-2)\int_{\mathbb{R}^N} \sum_{i,k,h=1}^N F_i\ (D_ku)^2\ D_hu\ D_{ih}u\ |Du|^{p-4}\, dx \\
&= \int_{\mathbb{R}^N} \mathrm{div}F\, |Du|^p\, dx - I_4 - (p-2)I_4
\end{aligned}$$

which implies by (H2) that

$$I_4 = \frac{1}{p}\int_{\mathbb{R}^N} \mathrm{div}F\, |Du|^p\, dx \geq -\frac{\beta}{p}\int_{\mathbb{R}^N} V|Du|^p\, dx. \tag{1.3.23}$$

Applying (H1) and Young's inequality, we get

$$\begin{aligned}
|I_5| &\leq \alpha \int_{\mathbb{R}^N} V^{\frac{3}{2}}|u||Du|^{p-1}\, dx = \alpha\int_{\mathbb{R}^N} (V\,|u|\,|Du|^{\frac{p-2}{2}})(V^{\frac{1}{2}}\ |Du|^{\frac{p}{2}})\, dx \\
&\leq \frac{\alpha}{\varepsilon}\int_{\mathbb{R}^N} |Vu|^2|Du|^{p-2}\, dx + \varepsilon\alpha \int_{\mathbb{R}^N} V|Du|^p\, dx \\
&\leq c_2 \int_{\mathbb{R}^N} |Vu|^p\, dx + c_2\int_{\mathbb{R}^N} |Du|^p\, dx + \varepsilon\alpha\int_{\mathbb{R}^N} V|Du|^p\, dx
\end{aligned}$$

with $c_2 = c_2(\varepsilon, p, \alpha)$. Then

$$I_5 \geq -c_2\int_{\mathbb{R}^N} |Vu|^p\, dx - c_2\int_{\mathbb{R}^N} |Du|^p\, dx - \varepsilon\alpha\int_{\mathbb{R}^N} V|Du|^p\, dx. \tag{1.3.24}$$

We are left to estimate the integral in the right hand side in (1.3.19). Integrating by parts and arguing as before we obtain

$$
\begin{aligned}
\left|\int_{\mathbb{R}^N}\langle Df,Du\rangle|Du|^{p-2}\,dx\right| &\le (p-1)\sum_{h,k=1}^N\int_{\mathbb{R}^N}|f|\,|Du|^{p-2}|D_{hk}u|\,dx\\
&= (p-1)\int_{\mathbb{R}^N}|f|\,|Du|^{\frac{p-2}{2}}|Du|^{\frac{p-2}{2}}\sum_{h,k=1}^N|D_{hk}u|\,dx\\
&\le c_3\int_{\mathbb{R}^N}|f|^2\,|Du|^{p-2}\,dx+\varepsilon(p-1)\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\,,
\end{aligned}
$$

with $c_3=c_3(p,N,\varepsilon)$. Applying Young's inequality we have finally

$$
\begin{aligned}
\left|\int_{\mathbb{R}^N}\langle Df,Du\rangle|Du|^{p-2}\,dx\right| &\le c_4\int_{\mathbb{R}^N}|f|^p\,dx+c_4\int_{\mathbb{R}^N}|Du|^p\,dx\\
&\quad+\varepsilon(p-1)\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\;,
\end{aligned}
\tag{1.3.25}
$$

with $c_4=c_4(p,N,\varepsilon)$. Collecting (1.3.20)–(1.3.25) from (1.3.19) we obtain

$$
\begin{aligned}
&\Big(\lambda-\frac{c_1}{\varepsilon}-c_2-c_4\Big)\int_{\mathbb{R}^N}|Du|^p\,dx\\
&+\Big(\nu_0-(c_1+p-1)\varepsilon\Big)\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\\
&+\left(1-\frac{\beta}{p}-\tau-\varepsilon\alpha\right)\int_{\mathbb{R}^N}V|Du|^p\,dx\\
&\le c_2\int_{\mathbb{R}^N}|Vu|^p\,dx+c_4\int_{\mathbb{R}^N}|f|^p\,dx\,.
\end{aligned}
$$

From (1.3.16) and (1.3.10), choosing first a small ε and then a large λ, we deduce that

$$
\int_{\mathbb{R}^N}(|Du|^p+V|Du|^p)\,dx+\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\le c\int_{\mathbb{R}^N}(|Au|^p+|u|^p)\,dx\,,
$$

where the constant c depends on $p,N,\nu_0,M,\|Dq_{ij}\|_\infty$ and the constants in (H1), (H2), (H3).

Step 2. Let φ be a standard mollifier and set, as usual, $\varphi_\varepsilon(x)=\varepsilon^{-N}\varphi\left(\frac{x}{\varepsilon}\right)$. If $q^\varepsilon_{ij}=q_{ij}*\varphi_\varepsilon$ and

$$
A^\varepsilon u=\sum_{i,j=1}^N D_i(q^\varepsilon_{ij}D_ju)+\langle F,Du\rangle-Vu\,,
$$

then by Step 1, noticing that $\|q^\varepsilon_{ij}\|_\infty\le\|q_{ij}\|_\infty$, $\|Dq^\varepsilon_{ij}\|_\infty\le\|Dq_{ij}\|_\infty$ and that (q^ε_{ij}) satisfy (1.1.1) with the same constant ν_0, it follows that

$$
\int_{\mathbb{R}^N}(|Du|^p+V|Du|^p)\,dx+\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\le c\int_{\mathbb{R}^N}(|A^\varepsilon u|^p+|u|^p)\,dx\,,
$$

with c independent of ε. Since $\|A^\varepsilon u-Au\|_p\to 0$ as ε goes to 0, we get the thesis. □

1.4 A priori estimates of $\|D^2u\|_p$, $\|\langle F,Du\rangle\|_p$

In the present section, we estimate the L^p norm of the second order derivatives of a solution $u\in\mathcal{D}_p$ of $Au=f$, $f\in L^p(\mathbb{R}^N)$. The proof is more involved than that of the case $p=2$ given in Section 1.3, since the variational method fails. Thus, we employ a different technique, which works under more restrictive assumptions on the coefficients of A, precisely we replace assumptions (H1) and (H4) with (H1') and (H4'), respectively. As noticed in Section 1.1, these assumptions imply (1.1.8). Moreover, (H5) is assumed.

The estimate of the second order derivatives is proved in Proposition 1.4.5. The idea is to define, via a change of variables and a localization argument, a family of operators, say $\{A_{x_0}\}_{x_0\in\mathbb{R}^N}$, with a globally Lipschitz drift coefficient and a bounded potential term. Then we apply Theorem 1.2.1 to each A_{x_0} to obtain local estimates of the L^p-norm of the second order derivatives of u. In order to get global estimates, we use a covering argument based on Besicovitch's Covering Theorem (see Proposition 1.4.1 below). We just note that the transformed operators $\{A_{x_0}\}$ turn out to be uniformly elliptic if and only if we require that $|F|\le\theta V^{1/2}$, which is the case of [41].

Once that the estimate of the second order derivatives is available, by difference we get the estimate for $\langle F, Du\rangle$.

Proposition 1.4.1 *Let $\mathcal{F}=\{B(x,\rho(x))\}_{x\in\mathbb{R}^N}$ be a collection of balls such that*

$$|\rho(x)-\rho(y)|\le L|x-y|,\quad x,y\in\mathbb{R}^N, \tag{1.4.1}$$

with $L<\frac{1}{2}$. Then there exist a countable subcovering $\{B(x_n,\rho(x_n))\}$ and a natural number $\zeta=\zeta(N,L)$ such that at most ζ among the doubled balls $\{B(x_n,2\rho(x_n))\}$ overlap.

The above proposition relies on the following version of the Besicovitch covering theorem, (see e.g. [4, Theorem 2.18]).

Proposition 1.4.2 *There exists a natural number $\xi(N)$ satisfying the following property. If $\Omega\subset\mathbb{R}^N$ is a bounded set and $\rho:\Omega\to(0,+\infty)$, then there is a set $S\subset\Omega$, at most countable, such that $\Omega\subset\bigcup_{x\in S}B(x,\rho(x))$ and every point of $\mathbb{R}^N$ belongs at most to $\xi(N)$ balls $B(x,\rho(x))$ centered at points of S.*

We turn now to the proof of Proposition 1.4.1.

Proof of Proposition 1.4.1. If $L=0$ then the radii are constant and the statement easily follows.

If $L>0$, we consider the sets

$$\begin{aligned}\Omega_n &:= B\Big(0,2\rho(0)(1+L)^n\Big)\setminus B\Big(0,2\rho(0)(1+L)^{n-1}\Big),\quad n\ge 1\\ \Omega_0 &:= B(0,2\rho(0)).\end{aligned}$$

Applying Proposition 1.4.2 we have that for all $n\in\mathbb{N}_0$ there exists a (at most) countable subset $S_n\subset\Omega_n$, such that $\Omega_n\subset\bigcup_{x\in S_n}B(x,\rho(x)) =: C_n$. Since (1.4.1) implies $\rho(x)\le\rho(0)+L|x|$, it is easy to prove that

$$C_n\subset B\Big(0,\rho(0)(2(1+L)^{n+1}+1)\Big)\setminus B\Big(0,\rho(0)(2(1-L)(1+L)^{n-1}-1)\Big),\quad n\ge 1.$$

Note that $2(1+L)^{n-1}(1-L)-1>0$ for all $n\ge 1$ because $L<\frac{1}{2}$. Since $1+L>1$, there exists $k=k(L)\in\mathbb{N}$ such that for all $n\ge k$

$$2(1-L)(1+L)^{n-1}-1>2(1+L)^{n-k+1}+1,$$

which implies that $C_n\cap C_{n-k}=\emptyset$. Hence the intersection of at most k among the sets C_n can be non-empty. Moreover, at most $\xi(N)$ among the balls centered at points of S_n overlap. It turns

out that $\mathcal{F}' = \{B(x,\rho(x)) \,:\, x\in S_n,\, n\in\mathbb{N}_0\} =: \{B(x_j,\rho_j)\}$ is a countable subcovering of $\mathbb{R}^N$ and if $\xi' = k\,\xi(N)$ then at most ξ' balls of $\mathcal{F}'$ overlap.
To estimate the number of overlapping doubled balls $\{B(x_j,2\rho_j)\}$ we proceed as in [41, Lemma 2.2]. Let $B(x_i,\rho_i)\in\mathcal{F}'$ be fixed and set $J(i) = \{j\in\mathbb{N} \,:\, B(x_i,2\rho_i)\cap B(x_j,2\rho_j)\neq\emptyset\}$. If $j\in J(i)$ it turns out that $|\rho_i-\rho_j|\le 2L(\rho_i+\rho_j)$, because $|x_i-x_j|\le 2(\rho_i+\rho_j)$, yielding $\frac{1-2L}{1+2L}\rho_i\le\rho_j\le\frac{1+2L}{1-2L}\rho_i$. Thus, the balls $B(x_j,\rho_j)$, $j\in J(i)$, are contained in $B(x_i,\frac{5+2L}{1-2L}\rho_i)$. Since at most ξ' of the balls $B(x_j,\rho_j)$ overlap, we obtain

$$\left(\frac{1-2L}{1+2L}\right)^N\rho_i^N\,\mathrm{card}\,J(i)\le\sum_{j\in J(i)}\rho_j^N\le\xi'\left(\frac{5+2L}{1-2L}\right)^N\rho_i^N\,,$$

which implies $\mathrm{card}\,J(i)\le\xi'\left(\frac{(5+2L)(1+2L)}{(1-2L)^2}\right)^N$, so that the number of overlapping doubled balls is an integer ζ, with $\zeta\le 1+\xi'\left(\frac{(5+2L)(1+2L)}{(1-2L)^2}\right)^N$. □

The following simple lemma is a straightforward consequence of assumption (H1') and it will be useful to prove Proposition 1.4.5 below.

Lemma 1.4.3 *Assume that (H1') holds. Then there exist $\varepsilon>0$ and two constants $a,b>0$, depending on α,σ,μ, such that for all $x_0\in\mathbb{R}^N$*

$$aV(x)\le V(x_0)\le bV(x),\quad \text{for every } x\in B(x_0,3\varepsilon r(x_0))\,,$$

with

$$r(x_0) := (1+|x_0|^2)^{\mu/2}V^{\sigma-1}(x_0). \tag{1.4.2}$$

Proof. We remark that from the choice of the parameters μ and σ and since $V\ge 1$ then

$$(1+|x|^2)^{\mu/2}V^{\sigma-1}(x)\le 1+|x|\,, \tag{1.4.3}$$

for every $x\in\mathbb{R}^N$. Moreover, (H1') is equivalent to one of the following inequalities

$$\begin{aligned} &|DV^{\sigma-1}(x)|\le\frac{\alpha(1-\sigma)}{(1+|x|^2)^{\mu/2}}, &&\sigma<1\,,\\ &|D\log V(x)|\le\frac{\alpha}{(1+|x|^2)^{\mu/2}}, &&\sigma=1\,. \end{aligned} \tag{1.4.4}$$

We prove the thesis assuming $\sigma<1$, the case $\sigma=1$ being analogous.

Fix $x_0\in\mathbb{R}^N$ and write r in place of $r(x_0)$.

Suppose first that $|x_0|<1$. From (1.4.3) and (1.4.2) it follows that $B(x_0,3\varepsilon r)\subset B(0,2)$, for every $0<\varepsilon\le 1/6$. Moreover, since V is a continuous function and $V\ge 1$, we have also that there exist $\omega_1,\omega_2>0$, independent of x_0, such that

$$\omega_1=\inf_{y\in B(0,2)}\frac{1}{V(y)}\le\inf_{y\in B(x_0,3\varepsilon r)}\frac{1}{V(y)}\le\frac{V(x_0)}{V(x)}\le\sup_{y\in B(0,2)}V(y)=\omega_2\,,\quad x\in B(x_0,3\varepsilon r).$$

Let us now deal with the case $|x_0|\ge 1$. By (1.4.3) one has $r(y)\le 1+|y|$, $y\in\mathbb{R}^N$, so that for every $0<\varepsilon\le 1/6$

$$\sup_{|y|\ge 1}\frac{1+|y|^2}{1+(|y|-3\varepsilon r)^2}<+\infty\,.$$

Therefore, there exist $\varepsilon\le 1/6$ and τ both independent of x_0, such that

$$\frac{3\varepsilon\alpha(1-\sigma)(1+|x_0|^2)^{\mu/2}}{(1+(|x_0|-3\varepsilon r)^2)^{\mu/2}}\le\tau<1\,,$$

where α and σ are as in (H1'). Thus, by the mean value theorem and (1.4.4) it follows that for every $x \in B(x_0, 3\varepsilon r)$

$$V^{\sigma-1}(x_0)(1-\tau) \le V^{\sigma-1}(x) \le V^{\sigma-1}(x_0)(1+\tau)$$

and, multiplying by $V^{1-\sigma}(x)V^{1-\sigma}(x_0)$,

$$V^{1-\sigma}(x)(1-\tau) \le V^{1-\sigma}(x_0) \le V^{1-\sigma}(x)(1+\tau)\,. \tag{1.4.5}$$

Therefore the statement is proved with $a = \inf\{\omega_1, (1-\tau)^{\frac{1}{1-\sigma}}\}$ and $b = \sup\{\omega_2, (1+\tau)^{\frac{1}{1-\sigma}}\}$. □

The following algebraic lemma is useful to prove Proposition 1.4.5.

Lemma 1.4.4 *If (H1') holds, with $(\sigma, \mu) \neq (\frac{1}{2}, 0)$, then for every $\delta > 0$ there exists $c_\delta > 0$ such that*

$$|DV| \le \delta V^{3/2} + c_\delta\,. \tag{1.4.6}$$

Proof. If $\frac{1}{2} < \sigma \le 1$, then (1.4.6) trivially follows by Young's inequality, with c_δ depending only on σ, α and c_α. If instead $\sigma = \frac{1}{2}$, then by assumption $\mu > 0$. For all $\delta > 0$ choose $R_\delta > 0$ such that $(1+|x|^2)^{\mu/2} \ge \alpha/\delta$ for every $x \in \mathbb{R}^N \setminus B_{R_\delta}$. Hence

$$|DV| \le \alpha \frac{V^{3/2}}{(1+|x|^2)^{\mu/2}} \le \delta V^{3/2} + \alpha \sup_{x \in B_{R_\delta}} V^{3/2}(x)\,.$$

□

In the following proposition we extend to the case $p \neq 2$ the estimate of the second order derivatives stated in (1.5.1) in the case $p = 2$.

Proposition 1.4.5 *Assume (H1'), (H2'), (H4'), (H5) with constants satisfying (1.1.7). If $u \in \mathcal{D}_p$ then*

$$\int_{\mathbb{R}^N} (|Vu|^p + |\langle F, Du\rangle|^p + |D^2u|^p)\,dx \le c \int_{\mathbb{R}^N} (|Au|^p + |u|^p)\,dx\,, \tag{1.4.7}$$

with c depending only on N, p, ν_0, M, $\|q_{ij}\|_\infty$, $\|Dq_{ij}\|_\infty$ and the constants in (H1'), (H2'), (H4') and (H5).

Proof. By Lemma 1.3.1 we may reduce to consider $u \in C_c^\infty(\mathbb{R}^N)$. Moreover, for the sake of simplicity and without loss of generality, we can prove the statement assuming $c_\beta = 0$.
Set $f = Au$. We claim that the assumptions of Lemma 1.3.5 hold. Since $|\operatorname{div} F| \le \sqrt{N}|DF|$ then (H2') implies

$$\operatorname{div} F + \beta V \ge 0 \tag{1.4.8}$$

with $\beta < p$ because of (1.1.7).
Moreover, (H1') and (H4') imply (1.1.8), that is

$$|\langle F, DV\rangle| \le \alpha\theta V^2\,.$$

If $(\sigma, \mu) = (\frac{1}{2}, 0)$, then (H1) trivially follows from (H1') and (1.1.8) implies (1.3.9). If instead $\sigma > \frac{1}{2}$ or $\mu > 0$, then by Lemma 1.4.4 (H1) holds, with α and c_α replaced by δ and c_δ, respectively, with δ arbitrarily small. Choose δ, depending only on N, p, M and on the constants in (H1'), (H2'), (H4') and (H5), such that

$$\frac{M}{4}(p-1)\delta^2 + \frac{\beta}{p} + \alpha\theta\frac{p-1}{p} < 1\,. \tag{1.4.9}$$

Thus, (1.3.9) holds and Lemma 1.3.5 implies

$$\int_{\mathbb{R}^N} |Vu|^p\,dx \le c\int_{\mathbb{R}^N} (|f|^p + |u|^p)\,dx\,. \tag{1.4.10}$$

It remains to estimate the L^p-norms of $|D^2u|$ and $\langle F, Du\rangle$. We begin by considering the second order derivatives of u. Then, by difference, we obtain the estimate of $\langle F, Du\rangle$.
For every $x_0 \in \mathbb{R}^N$, let ε and $r = r(x_0)$ be as in Lemma 1.4.3. We point out that ε is independent of x_0.
Define y_0 equal to λx_0, with $\lambda := V^{1/2}(x_0)$. We consider two cut-off functions η and φ in $C_c^\infty(\mathbb{R}^N)$, $0 \le \eta, \varphi \le 1$, satisfying the following conditions

$$\begin{aligned} &\eta \equiv 1 \text{ in } B(y_0, \varepsilon\lambda r)\,, \quad \operatorname{supp}\eta \subset B(y_0, 2\varepsilon\lambda r)\,,\\ &\varphi \equiv 1 \text{ in } B(y_0, 2\varepsilon\lambda r)\,, \quad \operatorname{supp}\varphi \subset B(y_0, 3\varepsilon\lambda r)\,,\\ &|D\eta|^2 + |D^2\eta| + |D\varphi|^2 + |D^2\varphi| \le \frac{L}{\lambda^2 r^2}\,, \end{aligned} \tag{1.4.11}$$

for some $L > 0$, depending on ε, but neither on x_0 nor on y_0. For every $x \in \mathbb{R}^N$, define $y = \lambda x$ and consider $v(y) = u\left(\frac{y}{\lambda}\right)$. Then v satisfies the equation

$$\sum_{i,j=1}^N D_{y_i}(\tilde q_{ij} D_{y_j} v)(y) + \frac{1}{\lambda}\langle \tilde F(y), D_y v(y)\rangle - \frac{1}{\lambda^2}\tilde V(y)v(y) = \frac{1}{\lambda^2}\tilde f(y), \quad y \in \mathbb{R}^N$$

with $\tilde q_{ij}(y) = q_{ij}\left(\frac{y}{\lambda}\right)$, $\tilde F(y) = F\left(\frac{y}{\lambda}\right)$, $\tilde V(y) = V\left(\frac{y}{\lambda}\right)$ and $\tilde f(y) = f(\frac{y}{\lambda})$.
Setting $w(y) = \eta(y)v(y)$ we deduce that

$$\sum_{i,j=1}^N D_{y_i}(\tilde q_{ij}(y) D_{y_j} w(y)) + \frac{1}{\lambda}\langle \tilde F(y), D_y w(y)\rangle - \frac{1}{\lambda^2}\tilde V(y)w(y) = g(y) \tag{1.4.12}$$

with g defined as follows

$$g(y) := \frac{1}{\lambda^2}\eta(y)\tilde f(y) + 2\langle \tilde q(y) D\eta(y), Dv(y)\rangle + \operatorname{div}(\tilde q D\eta)(y)v(y) + \frac{1}{\lambda}\langle \tilde F(y), D\eta(y)\rangle v(y), \tag{1.4.13}$$

$y \in \mathbb{R}^N$. Since $\operatorname{supp} w \subset B(y_0, 2\varepsilon\lambda r)$, equation (1.4.12) is equivalent to

$$\sum_{i,j=1}^N D_{y_i}(\tilde q_{ij}(y) D_{y_j} w(y)) + \frac{1}{\lambda}\varphi(y)\langle \tilde F(y), D_y w(y)\rangle - \frac{1}{\lambda^2}\varphi(y)\tilde V(y)w(y) = g(y), \quad y \in \mathbb{R}^N.$$

Now, let us define the operator

$$\tilde A = \sum_{i,j=1}^N D_{y_i}(\tilde q_{ij} D_{y_j}) + \frac{1}{\lambda}\varphi\,\langle \tilde F, D_y\rangle - \frac{1}{\lambda^2}\varphi\,\tilde V. \tag{1.4.14}$$

Claim 1. $\frac{1}{\lambda^2}\varphi\,\tilde V$ and $\left|\langle \frac{1}{\lambda}\varphi\tilde F, D\tilde q_{ij}\rangle\right|$ are bounded in $\mathbb{R}^N$ and $\frac{1}{\lambda}\varphi\,\tilde F$ is globally Lipschitz in $\mathbb{R}^N$ with $\left\|\frac{1}{\lambda^2}\varphi\,\tilde V\right\|_\infty$, $\left\|\langle \frac{1}{\lambda}\varphi\tilde F, D\tilde q_{ij}\rangle\right\|_\infty$ and the Lipschitz constant of $\frac{1}{\lambda}\varphi\,\tilde F$ independent of x_0.

Proof of claim 1. The main tool is Lemma 1.4.3. Recalling the definition of λ, $\tilde V$ and the relationship between y and x, from Lemma 1.4.3 it follows that

$$\sup_{y\in\mathbb{R}^N} \frac{1}{\lambda^2}\varphi(y)\,\tilde V(y) \le \sup_{x\in B(x_0,3\varepsilon r)} \frac{V(x)}{V(x_0)} \le \frac{1}{a},$$

Taking into account assumptions (H2'), (H4') and (1.4.11), we have that

$$\begin{aligned}
\sup_{y\in\mathbb{R}^N}\left|\frac{1}{\lambda}D_y\big(\varphi(y)\,\tilde F(y)\big)\right| &= \sup_{y\in B(y_0,3\varepsilon\lambda r)}\left|\frac{1}{\lambda^2}(D_xF)\left(\frac{y}{\lambda}\right)\varphi(y)+\frac{1}{\lambda}F\left(\frac{y}{\lambda}\right)D_y\varphi(y)\right|\\
&\le \sup_{x\in B(x_0,3\varepsilon r)}\frac{\beta V(x)}{V(x_0)}+L\sup_{x\in B(x_0,3\varepsilon r)}\frac{|F(x)|}{r\,V(x_0)}\\
&\le \beta\sup_{x\in B(x_0,3\varepsilon r)}\frac{V(x)}{V(x_0)}+L\theta\sup_{x\in B(x_0,3\varepsilon r)}\frac{(1+|x|^2)^{\frac{\mu}{2}}V^{\sigma}(x)}{(1+|x_0|^2)^{\frac{\mu}{2}}V^{\sigma}(x_0)}
\end{aligned}$$

Using Lemma 1.4.3 and equation (1.4.3) we infer that

$$\begin{aligned}
\sup_{y\in\mathbb{R}^N}\left|\frac{1}{\lambda}D_y\big(\varphi(y)\,\tilde F(y)\big)\right| &\le \frac{\beta}{a}+\frac{L\theta}{a^{\sigma}}\frac{[1+(|x_0|+3\varepsilon r)^2]^{\frac{\mu}{2}}}{(1+|x_0|^2)^{\frac{\mu}{2}}}\\
&\le \frac{\beta}{a}+\frac{L\,\theta\,8^{\frac{\mu}{2}}}{a^{\sigma}}
\end{aligned}$$

which implies that $\frac{1}{\lambda}\varphi\,\tilde F$ is globally Lipschitz in $\mathbb{R}^N$, uniformly with respect to x_0.
Finally, assumption (H5) yields

$$\begin{aligned}
\sup_{y\in\mathbb{R}^N}\left|\langle\frac{1}{\lambda}\varphi(y)\tilde F(y),D_y\tilde q_{ij}(y)\rangle\right| &\le \sup_{y\in B(y_0,3\varepsilon\lambda r)}\left|\langle\frac{1}{\lambda}\tilde F(y),D_y\tilde q_{ij}(y)\rangle\right|\\
&\le \sup_{x\in B(x_0,3\varepsilon r)}\frac{1}{\lambda^2}\,|\langle F(x),Dq_{ij}(x)\rangle|\\
&\le \kappa\sup_{x\in B(x_0,3\varepsilon r)}\frac{V(x)}{V(x_0)}+c_\kappa\sup_{x\in B(x_0,3\varepsilon r)}\frac{1}{V(x_0)}\le\frac{\kappa}{a}+c_\kappa\,,
\end{aligned}$$

because of Lemma 1.4.3 and $V\ge 1$.

Claim 2. The function g in (1.4.13) satisfies the estimate

(1.4.15)
$$\int_{\mathbb{R}^N}|g(y)|^p\,dy\le\frac{C}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}\Big(|u(x)|^p+|f(x)|^p+|V(x)u(x)|^p+|V^{1/2}(x)Du(x)|^p\Big)\,dx\,,$$

for some C depending on ε, but not on x_0.

Proof of claim 2. We separately consider each term of g. The constants occurring in the estimates may depend on ε.
The first term in (1.4.13) is the easiest to estimate, in fact

(1.4.16) $$\int_{\mathbb{R}^N}\left|\frac{1}{\lambda^2}\eta(y)f\left(\frac{y}{\lambda}\right)\right|^p\,dy\le\frac{1}{\lambda^{2p}}\int_{B(y_0,2\varepsilon\lambda r)}\left|f\left(\frac{y}{\lambda}\right)\right|^p\,dy=\frac{1}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}|f(x)|^p\,dx\,.$$

Using (1.4.11) we can estimate the L^p-norm of the next two terms as follows

$$\begin{aligned}
&\int_{\mathbb{R}^N}|2\langle\tilde q(y)D_y\eta(y),D_yv(y)\rangle|^p\,dy\le\frac{C_1}{\lambda^{2p}r^p}\int_{B(y_0,2\varepsilon\lambda r)}\left|Du\left(\frac{y}{\lambda}\right)\right|^p\,dy\\
&=\frac{C_1}{\lambda^{2p-N}r^p}\int_{B(x_0,2\varepsilon r)}|Du(x)|^p\,dx=\frac{C_1}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}\frac{V^{p(1-\sigma)}(x_0)}{(1+|x_0|^2)^{p\mu/2}}|Du(x)|^p\,dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\mathrm{div}(\tilde qD\eta)(y)v(y)|^p\,dy\le\frac{C_2}{\lambda^{2p}\,r^{2p}}\int_{B(y_0,2\varepsilon\lambda r)}|v(y)|^p\,dy\\
&=\frac{C_2}{\lambda^{2p-N}r^{2p}}\int_{B(x_0,2\varepsilon r)}|u(x)|^p\,dx=\frac{C_2}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}\frac{V^{2p(1-\sigma)}(x_0)}{(1+|x_0|^2)^{p\mu}}|u(x)|^p\,dx\,,
\end{aligned}$$

with C_1 and C_2 independent of x_0.
Recalling that $V \geq 1$, $\sigma \geq \frac{1}{2}$, $\mu \geq 0$ and using Lemma 1.4.3, we obtain

$$\begin{aligned}\int_{B(x_0,2\varepsilon r)} \frac{V^{p(1-\sigma)}(x_0)}{(1+|x_0|^2)^{p\mu/2}}|Du(x)|^p\,dx &\leq \int_{B(x_0,2\varepsilon r)} |V^{1/2}(x_0)Du(x)|^p\,dx \\ &\leq b^{p/2}\int_{B(x_0,2\varepsilon r)} |V^{1/2}(x)Du(x)|^p\,dx\end{aligned}$$

and

$$\begin{aligned}\int_{B(x_0,2\varepsilon r)} \frac{V^{2p(1-\sigma)}(x_0)}{(1+|x_0|^2)^{p\mu}}|u(x)|^p\,dx &\leq \int_{B(x_0,2\varepsilon r)} |V(x_0)u(x)|^p\,dx \\ &\leq b^{p}\int_{B(x_0,2\varepsilon r)} |V(x)u(x)|^p\,dx\,.\end{aligned}$$

Hence, there exists C_3 independent of x_0 such that the following inequality holds

$$\begin{aligned}(1.4.17)\qquad &\int_{\mathbb{R}^N} \left(|2\langle\tilde{q}(y)D_y\eta(y), D_yv(y)\rangle|^p + |\mathrm{div}(\tilde{q}D\eta)(y)v(y)|^p\right)\,dy \leq \\ &\leq \frac{C_3}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)} (|V(x)u(x)|^p + |V^{1/2}(x)Du(x)|^p)\,dx\,.\end{aligned}$$

Concerning the last term in (1.4.13), we use again assumption (H4') and we get

$$\begin{aligned}(1.4.18)\qquad &\int_{\mathbb{R}^N} \left|\frac{1}{\lambda}\langle\tilde{F}(y), D\eta(y)\rangle v(y)\right|^p\,dy \leq \frac{c}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)} \frac{|F(x)|^p|u(x)|^p}{r^p}\,dx \\ &\leq \frac{c\,\theta^p}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)} \left|\frac{(1+|x|^2)^{\mu/2}V^{\sigma-1}(x)}{(1+|x_0|^2)^{\mu/2}V^{\sigma-1}(x_0)}\right|^p |V(x)u(x)|^p\,dx \\ &\leq \frac{C_4}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)} |V(x)u(x)|^p\,dx\end{aligned}$$

where C_4 is not depending on x_0. Thus, the claim is proved since collecting (1.4.16)-(1.4.18), inequality (1.4.15) follows.

Let us now prove (1.4.7). Applying Theorem 1.2.1 with B replaced by $\tilde{A}$, we have

$$\int_{\mathbb{R}^N} |D^2w(y)|^p\,dy \leq K\int_{\mathbb{R}^N} (|w(y)|^p + |g(y)|^p)\,dy\,,$$

with K independent of x_0. By the definition of w it follows that

$$\int_{B(y_0,\varepsilon\lambda r)} |D^2v(y)|^p\,dy \leq K\int_{B(y_0,2\varepsilon\lambda r)} (|v(y)|^p + |g(y)|^p)\,dy$$

and consequently, since $y = \lambda x$,

$$\begin{aligned}&\frac{1}{\lambda^{2p-N}}\int_{B(x_0,\varepsilon r)} |D^2u|^p\,dx \leq \\ &\leq K_1\,\lambda^N\int_{B(x_0,2\varepsilon r)} |u|^p\,dx + K_1\,\frac{1}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)} \left(|u|^p + |f|^p + |Vu|^p + |V^{1/2}Du|^p\right)dx\,.\end{aligned}$$

Multiplying both sides of the previous inequality by λ^{2p-N} and recalling that $\lambda = V^{1/2}(x_0)$ we obtain

$$\begin{aligned}&\int_{B(x_0,\varepsilon r)} |D^2u|^p\,dx \leq \\ &\leq K_1\int_{B(x_0,2\varepsilon r)} |V(x_0)u(x)|^p\,dx + K_1\int_{B(x_0,2\varepsilon r)} \left(|u|^p + |f|^p + |Vu|^p + |V^{1/2}Du|^p\right)\,dx\,,\end{aligned}$$

which implies

$$
\int_{B(x_0,\varepsilon r)} |D^2u|^p\,dx \le K_2 \int_{B(x_0,2\varepsilon r)} \Big(|u|^p + |f|^p + |Vu|^p + |V^{1/2}Du|^p\Big)\,dx\,, \tag{1.4.19}
$$

because of Lemma 1.4.3. Now, in order to apply Proposition 1.4.1 we need to verify the Lipschitz continuity of the radius $\varepsilon\, r$ with respect to x_0. To this aim, we remark that from assumption (H1') it follows that

$$
\begin{aligned}
|D(\varepsilon r)(x)| &= \varepsilon \Big|\mu(1+|x|^2)^{\frac{\mu}{2}-1} x V^{\sigma-1}(x) + (\sigma-1)(1+|x|^2)^{\frac{\mu}{2}} V^{\sigma-2}(x) DV(x)\Big| \\
&\le \varepsilon \left\{ \frac{1}{(1+|x|^2)^{\frac{1-\mu}{2}} V^{1-\sigma}(x)} + (1-\sigma)(1+|x|^2)^{\frac{\mu}{2}} V^{\sigma-2}(x)|DV(x)| \right\} \\
&\le \varepsilon\,\{1+(1-\sigma)\alpha\}
\end{aligned}
$$

which is less than 1/2, choosing a smaller ε if necessary. Let $\{B(x_j,\varepsilon r_j)\}$ be the covering of $\mathbb{R}^N$ yielded by Proposition 1.4.1. Applying (1.4.19) to each x_j and summing over j, it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |D^2u|^p\,dx \le \sum_{j\in\mathbb{N}} \int_{B(x_j,\varepsilon r_j)} |D^2u|^p\,dx \\
&\le K_2 \sum_{j\in\mathbb{N}} \int_{B(x_j,2\varepsilon r_j)} \Big(|u|^p + |f|^p + |Vu|^p + |V^{1/2}Du|^p\Big)\,dx \\
&= K_2 \int_{\mathbb{R}^N} \Big(|u(x)|^p + |f(x)|^p + |V(x)u(x)|^p + |V^{1/2}(x)Du(x)|^p\Big) \sum_{j\in\mathbb{N}} \chi_{B(x_j,2\varepsilon r_j)}(x)\,dx \\
&\le \zeta\, K_2 \int_{\mathbb{R}^N} \Big(|u|^p + |f|^p + |Vu|^p + |V^{1/2}Du|^p\Big)\,dx\,,
\end{aligned}
$$

where ζ is given by Proposition 1.4.1. Now, [41, Proposition 2.3] yields two constants $\gamma_0, c > 0$ (independent of u) such that for all $0 < \gamma \le \gamma_0$

$$
\|V^{1/2}Du\|_p \le \gamma \|D^2u\|_p + \frac{c}{\gamma}\|Vu\|_p.
$$

Choosing γ sufficiently small and taking into account (1.4.10) it turns out that

$$
\int_{\mathbb{R}^N} |D^2u|^p\,dx \le c \int_{\mathbb{R}^N} (|f|^p + |u|^p)\,dx\,,
$$

for some $c > 0$ depending on the stated quantities.
Once that the estimate of the second order derivatives is available, by difference we get the estimate for $\langle F, Du\rangle$, that is

$$
\int_{\mathbb{R}^N} |\langle F, Du\rangle|^p\,dx \le c \int_{\mathbb{R}^N} (|f|^p + |u|^p)\,dx\,.
$$

□

1.5 Generation of a C_0-semigroup in $L^2(\mathbb{R}^N)$

In this section we prove Theorem 1.1.1, which states that the operator $(A, \mathcal{D}_2)$ (see (1.1.3)) generates a C_0-semigroup in $L^2(\mathbb{R}^N)$, which turns out to be contractive if $c_\beta = 0$.
The proof goes as follows. As a by-product of Lemma 1.3.1 we deduce that the a priori estimates proved in Section 1.3, with $p = 2$ extend to $\mathcal{D}_2$. More precisely, it follows from Lemma 1.3.1,

Remark 1.3.3, Lemmas 1.3.5 and 1.3.6 that if $u \in \mathcal{D}_2$ and (H1), (H2), (H3), (H4), (1.1.5) and (1.1.6) hold, then

$$\int_{\mathbb{R}^N} (|Du|^2 + |Vu|^2 + |D^2u|^2)\, dx \le c \int_{\mathbb{R}^N} (|Au|^2 + |u|^2)\, dx\,, \tag{1.5.1}$$

for some c depending only on $N, \nu_0, \alpha, \beta, \tau, M, \|Dq_{ij}\|_\infty$. By difference, since Au is in $L^2(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} |\langle F, Du\rangle|^2\, dx \le c \int_{\mathbb{R}^N} (|Au|^2 + |u|^2)\, dx\,, \tag{1.5.2}$$

with a possibly different c.
Estimates (1.5.1) and (1.5.2) allow to prove that $(A, \mathcal{D}_2)$ is closed in $L^2(\mathbb{R}^N)$. Clearly, it is densely defined. If $c_\beta = 0$, then $(A, \mathcal{D}_2)$ is also dissipative. In order to apply the Hille-Yosida Theorem, it remains to prove that $\lambda - A : \mathcal{D}_2 \to L^2(\mathbb{R}^N)$ is bijective for sufficiently large λ. This is proved through a standard procedure, namely by approximating the solution of the elliptic equation $\lambda u - Au = f$, $f \in L^2(\mathbb{R}^N)$, with a sequence of solutions of the same equation in balls with increasing radii and satisfying Dirichlet boundary conditions.

Lemma 1.5.1 *Suppose that (H1), (H2), (H3), (H4), (1.1.5) and (1.1.6) hold. Then $(A, \mathcal{D}_2)$ is closed in $L^2(\mathbb{R}^N)$. Moreover, $(A - \frac{c_\beta}{2}, \mathcal{D}_2)$ is dissipative.*

Proof. If $u \in \mathcal{D}_2$, then $\|u\|_A \le c_1 \|u\|_{\mathcal{D}_2}$, $\|\cdot\|_A$ being the graph norm of A, for some positive c_1 depending on $\|q_{ij}\|_\infty$ and $\|Dq_{ij}\|_\infty$. Moreover, from (1.5.1) and (1.5.2) there exists $c_2 > 0$ such that $\|u\|_{\mathcal{D}_2} \le c_2 \|u\|_A$. This proves that $\|\cdot\|_{\mathcal{D}_2}$ is equivalent to $\|\cdot\|_A$; since $\mathcal{D}_2$ is obviously complete with respect to the former, it turns out that $\mathcal{D}_2$ is also complete with respect to the latter, which just means that $(A, \mathcal{D}_2)$ is closed.
Finally, taking into account Remark 1.3.4 and Lemma 1.3.1, we conclude that $(A - \frac{c_\beta}{2}, \mathcal{D}_2)$ is dissipative. □

In the proposition below we study the surjectivity of the operator $\lambda - A$, for positive λ. We remark that the injectivity for $\lambda > \frac{c_\beta}{2}$ follows from the dissipativity stated in Lemma 1.5.1.

Proposition 1.5.2 *Suppose that (H1), (H2), (H3), (H4), (1.1.5) and (1.1.6) hold. Then for every $f \in L^2(\mathbb{R}^N)$ and for every $\lambda > c_\beta/2$, there exists a solution $u \in \mathcal{D}_2$ of*

$$\lambda u - Au = f, \quad \textit{in } \mathbb{R}^N. \tag{1.5.3}$$

Moreover,

$$\|u\|_2 \le \left(\lambda - \frac{c_\beta}{2}\right)^{-1} \|f\|_2\,. \tag{1.5.4}$$

Proof. We deal with the case $c_\beta = 0$ only, since the remaining case $c_\beta \neq 0$ is analogous.
For each $\rho > 0$ consider the Dirichlet problem

$$\begin{cases} \lambda u - Au = f, & \text{in } B_\rho \\[2ex] u = 0, & \text{on } \partial B_\rho\,, \end{cases} \tag{1.5.5}$$

with $\lambda > 0$ and $f \in L^2(\mathbb{R}^N)$. According to [26, Theorem 9.15] there exists a unique solution u_ρ of (1.5.5) in $W^{2,2}(B_\rho) \cap W^{1,2}_0(B_\rho)$. Let us prove that the dissipativity estimate

$$\lambda \|u_\rho\|_{L^2(B_\rho)} \le \|f\|_{L^2(\mathbb{R}^N)}$$

holds. Multiplying

$$\lambda u_\rho - Au_\rho = f \tag{1.5.6}$$

by u_ρ and integrating by parts with similar estimates as in the proof of Lemma 1.3.2, taking into account that $u_\rho = 0$ on ∂B_ρ, we get

$$\lambda \int_{B_\rho} u_\rho^2\,dx + \nu_0 \int_{B_\rho} |Du_\rho|^2\,dx + \frac{1}{2}\int_{B_\rho} \mathrm{div}F\, u_\rho^2\,dx + \int_{B_\rho} V u_\rho^2\,dx \le \int_{B_\rho} f u_\rho\,dx$$

and by (H2) it follows

$$\lambda \int_{B_\rho} u_\rho^2\,dx + \nu_0 \int_{B_\rho} |Du_\rho|^2\,dx + \left(1-\frac{\beta}{2}\right)\int_{B_\rho} V u_\rho^2\,dx \le \left(\int_{B_\rho} u_\rho^2\,dx\right)^{1/2}\left(\int_{B_\rho} f^2\,dx\right)^{1/2}.$$

Then we have

$$\|u_\rho\|_{L^2(B_\rho)} \le \lambda^{-1}\|f\|_{L^2(\mathbb{R}^N)}\,, \quad \|Du_\rho\|_{L^2(B_\rho)} \le \nu_0^{-1/2}\lambda^{-1/2}\|f\|_{L^2(\mathbb{R}^N)}\,. \tag{1.5.7}$$

Multiplying (1.5.6) by Vu_ρ, with analogous estimates as in the proof of Lemma 1.3.5 we get the inequality

$$\|Vu_\rho\|_{L^2(B_\rho)} \le c\|f\|_{L^2(\mathbb{R}^N)}\,, \tag{1.5.8}$$

with c independent of ρ.
Let $\rho_1 < \rho_2 < \rho$. By [26, Theorem 9.11] and (1.5.7) we obtain

$$\|u_\rho\|_{W^{2,2}(B_{\rho_1})} \le c_1\left(\|f\|_{L^2(B_{\rho_2})} + \|u_\rho\|_{L^2(B_{\rho_2})}\right) \le c_2\|f\|_{L^2(\mathbb{R}^N)},$$

with c_1 and c_2 independent of ρ. Thus, $\{u_\rho\}$ is bounded in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$, hence there is a sequence $\{u_{\rho_n}\}$, $\rho_n < \rho_{n+1}$, weakly convergent to u in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Actually, $\{u_{\rho_n}\}$ strongly converges to u in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$. In fact, fixed s and t, $0 < s < t$, for every n, m such that $\rho_n, \rho_m > t$, by [26, Theorem 9.11] again,

$$\|u_{\rho_n} - u_{\rho_m}\|_{W^{2,2}(B_s)} \le c(s,t)\|u_{\rho_n} - u_{\rho_m}\|_{L^2(B_t)}\,,$$

since both u_{ρ_n} and u_{ρ_m} satisfy $\lambda u - Au = f$ in B_t. The convergence of $\{u_{\rho_n}\}$ to u in $L^2(B_t)$ proves that $\{u_{\rho_n}\}$ is a Cauchy sequence in $W^{2,2}(B_s)$ and so the assertion follows. As a consequence, u is a solution of (1.5.3) for a.e. $x \in \mathbb{R}^N$.

In order to conclude, it remains to prove that $u \in \mathcal{D}_2$. First, we prove that $u \in W^{1,2}(\mathbb{R}^N)$ and $Vu \in L^2(\mathbb{R}^N)$, then that $\langle F, Du\rangle \in L^2(\mathbb{R}^N)$. Finally, by difference from (1.5.3) and using classical L^2-regularity, it follows that $u \in W^{2,2}(\mathbb{R}^N)$.
By (1.5.7) and (1.5.8) we get that, fixed $R < \rho_n$,

$$\int_{B_R} u_{\rho_n}^2\,dx \le \int_{B_{\rho_n}} u_{\rho_n}^2\,dx \le \lambda^{-2}\int_{\mathbb{R}^N} f^2\,dx\,,$$

$$\int_{B_R} |Du_{\rho_n}|^2\,dx \le \int_{B_{\rho_n}} |Du_{\rho_n}|^2\,dx \le \nu_0^{-1}\lambda^{-1}\int_{\mathbb{R}^N} f^2\,dx$$

and

$$\int_{B_R} (Vu_{\rho_n})^2\,dx \le \int_{B_{\rho_n}} (Vu_{\rho_n})^2\,dx \le c\int_{\mathbb{R}^N} f^2\,dx\,.$$

Since c does not depend on ρ_n and R, letting first $n \to +\infty$ and then $R \to +\infty$, we get (1.5.4) and

$$\int_{\mathbb{R}^N} (|Du|^2 + |Vu|^2)\, dx \le c \int_{\mathbb{R}^N} f^2\, dx\,.$$

In particular, $u \in W^{1,2}(\mathbb{R}^N)$ and $Vu \in L^2(\mathbb{R}^N)$.
Now, let $\eta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ in B_1, $\operatorname{supp} \eta \subset \mathrm{B}_2$ and $|D\eta|^2 + |D^2\eta| \le L$. Set $\eta_n(x) = \eta(x/n)$. We have

$$A(\eta_n u) - \eta_n A u = \sum_{i,j=1}^N q_{ij} D_j u D_i \eta_n + D_i(q_{ij} u D_j \eta_n) + \langle F, D\eta_n\rangle u\,. \tag{1.5.9}$$

Observe that $A(\eta_n u) - \eta_n Au \to 0$ as $n \to +\infty$ in the L^2-norm. In fact, $\sum_{i,j=1}^N (q_{ij} D_j u D_i \eta_n + D_i(q_{ij} u D_j \eta_n))$ goes to 0 in the L^2-norm, since $u \in W^{1,2}(\mathbb{R}^N)$ and, arguing as in (1.3.1), we obtain the convergence to 0 for the last term in (1.5.9), too. Since $\eta_n Au \to Au$ in L^2, then $A(\eta_n u) \to Au$, too. Being $\eta_n u \in \mathcal{D}_2$, by the equivalence of the two norms $\|\cdot\|_{\mathcal{D}_2}$ and $\|\cdot\|_A$ proved in Lemma 1.5.1 we get

$$\|\langle F, Du\rangle \eta_n\|_{L^2(\mathbb{R}^N)} \le c\left(\|A(\eta_n u)\|_{L^2(\mathbb{R}^N)} + \|\eta_n u\|_{L^2(\mathbb{R}^N)}\right) + \|\langle F, D\eta_n\rangle u\|_{L^2(\mathbb{R}^N)}.$$

Letting $n \to +\infty$, one then establishes

$$\|\langle F, Du\rangle\|_{L^2(\mathbb{R}^N)} \le c\left(\|Au\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)}\right)\,.$$

By difference, $\sum_{i,j=1}^N D_i(q_{ij} D_j u)$ belongs to $L^2(\mathbb{R}^N)$. Thus, by (1.1.1) and L^2 elliptic regularity the second order derivatives of u are in L^2, which implies that $u \in W^{2,2}(\mathbb{R}^N)$ and $u \in \mathcal{D}_2$. □

The proof that the operator $(A, \mathcal{D}_2)$ generates a strongly continuous semigroup in $L^2(\mathbb{R}^N)$ is now a straightforward consequence of the above results.

Proof of Theorem 1.1.1. It is easily seen that $(A, \mathcal{D}_2)$ is densely defined, then the assertion follows from the Hille-Yosida Theorem (see [21, Theorem II.3.5]). If $c_\beta = 0$ then $(A, \mathcal{D}_2)$ is dissipative and therefore the generated semigroup is contractive. □

1.6 Generation of a C_0-semigroup in $L^p(\mathbb{R}^N)$

The present section is devoted to the proof of Theorem 1.1.2. As in the case $p = 2$ treated in Section 1.5, the a priori estimates given by Proposition 1.4.5 allow to prove that $\|\cdot\|_{\mathcal{D}_p}$ and $\|\cdot\|_A$ are equivalent norms. This easily implies the closedness of $(A, \mathcal{D}_p)$. Moreover, it is readily seen that $(A, \mathcal{D}_p)$ is quasi dissipative. It remains to show that $\lambda - A$ is surjective for λ large and this is, actually, the main result of the section. The proof is different from that of Proposition 1.5.2, which does not work for $p \neq 2$. Here we approximate the coefficients of the operator A. Moreover, we clarify the reason why we require assumption (1.1.7), which is stronger than the corresponding one for $p = 2$. In fact, also the operators A_ε defined in the proof of Proposition 1.6.2 must satisfy our hypotheses.

The proof of the following Lemma is the same as the one of Lemma 1.5.1 and we omit it.

Lemma 1.6.1 *Suppose that (H1'), (H2'), (H4') and (H5) hold, with constants satisfying (1.1.7). Then $(A, \mathcal{D}_p)$ is closed in $L^p(\mathbb{R}^N)$. Moreover, $(A - \frac{c_\beta}{p}, \mathcal{D}_p)$ is dissipative.*

Proposition 1.6.2 *Suppose that (H1'), (H2'), (H4') and (H5) hold, with constants satisfying (1.1.7). Then for every $f \in L^p(\mathbb{R}^N)$ and for every $\lambda > \frac{c_\beta}{p}$ a unique solution $u \in \mathcal{D}_p$ of*

$$\lambda u - Au = f, \quad \textit{in } \mathbb{R}^N$$

exists. Moreover,

$$\|u\|_p \le \left(\lambda - \frac{c_\beta}{p}\right)^{-1} \|f\|_p \, . \tag{1.6.1}$$

PROOF. Uniqueness and estimate (1.6.1) immediately follow from (1.3.7). As far as the existence is concerned, for fixed $\varepsilon > 0$, let us define $F_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ and $V_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ as

$$F_\varepsilon := \frac{F}{1+\varepsilon V}\,, \qquad V_\varepsilon := \frac{V}{1+\varepsilon V}\,.$$

It is easy to prove that (H1'), (H2'), (H4') and (H5) imply

(H$_\varepsilon$1) $|DV_\varepsilon(x)| \le \alpha \frac{V_\varepsilon^{2-\sigma}(x)}{(1+|x|^2)^{\mu/2}}$,

(H$_\varepsilon$2) $|DF_\varepsilon| \le \sqrt{2}(\frac{\beta}{\sqrt{N}} + \alpha\theta)V_\varepsilon + \sqrt{\frac{2}{N}}\, c_\beta$,

(H$_\varepsilon$4) $|F_\varepsilon(x)| \le \theta(1+|x|^2)^{\mu/2} V_\varepsilon^\sigma(x)$,

(H$_\varepsilon$5) $|\langle F_\varepsilon(x), Dq_{ij}(x)\rangle| \le \kappa\, V_\varepsilon(x) + c_\kappa$,

respectively.
Assumptions (H$_\varepsilon$1), (H$_\varepsilon$2) and (H$_\varepsilon$4) yield

$$\operatorname{div} F_\varepsilon + \sqrt{2}(\beta + \sqrt{N}\alpha\theta)V_\varepsilon + \sqrt{2}\, c_\beta \ge 0\,, \qquad |\langle F_\varepsilon, DV_\varepsilon\rangle| \le \alpha\theta V_\varepsilon^2 \tag{1.6.2}$$

and

$$\sum_{i,j=1}^N D_i F_\varepsilon^j(x)\xi_i\xi_j \le \sqrt{2}\left(\frac{\beta}{\sqrt{N}} + \alpha\theta\right) V_\varepsilon(x)|\xi|^2 + \sqrt{\frac{2}{N}}\, c_\beta |\xi|^2\,, \qquad \xi, x \in \mathbb{R}^N\,.$$

Notice that V_ε is bounded and F_ε is globally Lipschitz in $\mathbb{R}^N$. Precisely,

$$||V_\varepsilon||_\infty \le \frac{1}{\varepsilon}\,, \quad \text{and} \quad ||D_i F_\varepsilon^j||_\infty \le \frac{1}{\varepsilon}\left(\frac{\beta}{\sqrt{N}} + \alpha\theta\right) + \frac{c_\beta}{\sqrt{N}}\,, \qquad 1 \le i,j \le N\,.$$

Moreover, if $(\sigma,\mu) \ne \left(\frac{1}{2}, 0\right)$ arguing as in the proof of Lemma 1.4.4 and observing that $V_\varepsilon \le V$, we have that for every $\delta > 0$ there exists $c_\delta \ge 0$ such that

$$|DV_\varepsilon| \le \delta V_\varepsilon^{3/2} + c_\delta\,, \quad \text{for every } \varepsilon > 0\,. \tag{1.6.3}$$

Therefore, the above inequality and (1.1.7) imply that there exists $\delta > 0$ independent of ε such that

$$\frac{M}{4}(p-1)\delta^2 + \sqrt{2}\frac{\beta + \sqrt{N}\alpha\theta}{p} + \alpha\theta\frac{p-1}{p} < 1\,. \tag{1.6.4}$$

Let us consider the operator

$$A_\varepsilon := A_0 + \langle F_\varepsilon, D\rangle - V_\varepsilon$$

where, as previously defined, A_0 stands for $\sum_{i,j=1}^N D_i(q_{ij}D_j)$.
Define $\mathcal{D}_{p,\varepsilon}$ and its norms $\|\cdot\|_{\mathcal{D}_{p,\varepsilon}}$ and $\|\cdot\|_{A_\varepsilon}$ analogously to $\mathcal{D}_p$, $\|\cdot\|_{\mathcal{D}_p}$ and $\|\cdot\|_A$, respectively, that is

$$\begin{aligned}
\mathcal{D}_{p,\varepsilon} &:= \left\{u \in W^{2,p}(\mathbb{R}^N) \,:\, \langle F_\varepsilon, Du\rangle \in L^p(\mathbb{R}^N)\right\}\,,\\
\|u\|_{\mathcal{D}_{p,\varepsilon}} &:= \|u\|_{2,p} + \|V_\varepsilon u\|_p + \|\langle F_\varepsilon, Du\rangle\|_p\,,\\
\|u\|_{A_\varepsilon} &:= \|A_\varepsilon u\|_p + \|u\|_p\,.
\end{aligned}$$

Since the constants involved in ($H_\varepsilon 1$), ($H_\varepsilon 2$), ($H_\varepsilon 4$), ($H_\varepsilon 5$) and (1.6.4) are independent of ε, from Lemma 1.6.1 we get that there exist k_1 and k_2, independent of ε, such that

$$k_1\|u\|_{A_\varepsilon} \leq \|u\|_{\mathcal{D}_{p,\varepsilon}} \leq k_2\|u\|_{A_\varepsilon}\,. \tag{1.6.5}$$

Since the operator A_ε satisfies the assumptions of Proposition 1.2.3, for every $\lambda > \sqrt{2}\frac{c_\beta}{p}$ one has $\lambda \in \rho(A_\varepsilon)$ and $\|R(\lambda, A_\varepsilon)\| \leq \left(\lambda - \sqrt{2}\frac{c_\beta}{p}\right)^{-1}$. In fact, using the inequality $V_\varepsilon \geq (1+\varepsilon)^{-1}$, the first estimate in (1.6.2) and noting that (1.1.7) implies $\sqrt{2}\,\frac{\beta+\sqrt{N}\alpha\theta}{p} < 1$, we get

$$-\inf_{x\in\mathbb{R}^N}\left(\frac{1}{p}\,\mathrm{div}F_\varepsilon(x) + V_\varepsilon(x)\right) \leq \frac{1}{1+\varepsilon}\left(\sqrt{2}\,\frac{\beta+\sqrt{N}\alpha\theta}{p} - 1\right) + \sqrt{2}\frac{c_\beta}{p} < \sqrt{2}\frac{c_\beta}{p}\,.$$

Therefore, if $\lambda > \sqrt{2}\,\frac{c_\beta}{p}$ then for every $f \in L^p(\mathbb{R}^N)$ and for all $\varepsilon > 0$, there exists a unique $u_\varepsilon \in \mathcal{D}_{p,\varepsilon}$ such that

$$\lambda u_\varepsilon - A_\varepsilon u_\varepsilon = f, \quad \text{in } \mathbb{R}^N \tag{1.6.6}$$

and

$$\|u_\varepsilon\|_p \leq \left(\lambda - \sqrt{2}\frac{c_\beta}{p}\right)^{-1}\|f\|_p\,. \tag{1.6.7}$$

Using (1.6.5), (1.6.6) and (1.6.7) we obtain

$$\|u_\varepsilon\|_{\mathcal{D}_{p,\varepsilon}} \leq k_2\left(\|A_\varepsilon u_\varepsilon\|_p + \|u_\varepsilon\|_p\right) \leq k_2\left(1 + \frac{\lambda+1}{\lambda - \sqrt{2}\frac{c_\beta}{p}}\right)\|f\|_p\,. \tag{1.6.8}$$

In particular, we have that $\{u_\varepsilon\}$ is bounded in $W^{2,p}(\mathbb{R}^N)$, thus there exist $u \in W^{2,p}(\mathbb{R}^N)$ and a sequence $\{u_{\varepsilon_n}\}$ converging to u weakly in $W^{2,p}(\mathbb{R}^N)$ and strongly in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore, up to a subsequence, $u_{\varepsilon_n} \to u$ and $Du_{\varepsilon_n} \to Du$ a.e. in $\mathbb{R}^N$. From (1.6.8) we obtain in particular that $\|V_{\varepsilon_n}u_{\varepsilon_n}\|_p + \|\langle F_{\varepsilon_n}, Du_{\varepsilon_n}\rangle\|_p \leq c\|f\|_p$, which implies, using Fatou's Lemma, that

$$\|Vu\|_p + \|\langle F, Du\rangle\|_p \leq c\|f\|_p\,.$$

Thus, $u \in \mathcal{D}_p$.

It remains to prove that u solves $\lambda u - Au = f$ a.e. in $\mathbb{R}^N$. From (1.6.6) and the definition of A_{ε_n} we infer that

$$\lambda u_{\varepsilon_n} - A_0 u_{\varepsilon_n} = f_{\varepsilon_n},$$

where $f_{\varepsilon_n} = f + \langle F_{\varepsilon_n}, Du_{\varepsilon_n}\rangle - V_{\varepsilon_n}u_{\varepsilon_n} \in L^p(\mathbb{R}^N)$. Applying the classical local L^p-estimates (see [26, Theorem 9.11]) it follows that for every $0 < \rho_1 < \rho_2$

$$\|u_{\varepsilon_n}\|_{W^{2,p}(B_{\rho_1})} \leq C(\|f_{\varepsilon_n}\|_{L^p(B_{\rho_2})} + \|u_{\varepsilon_n}\|_{L^p(B_{\rho_2})}), \tag{1.6.9}$$

with C depending on ρ_1, ρ_2 but independent of n. Since u_{ε_n} and f_{ε_n} converge to u and $f + \langle F, Du\rangle - Vu$, respectively, in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to \infty$, by applying (1.6.9) to the difference $u_{\varepsilon_n} - u_{\varepsilon_m}$ we get that $\{u_{\varepsilon_n}\}$ is a Cauchy sequence in $W^{2,p}(B_{\rho_1})$. This implies that u_{ε_n} converges to u in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and then, letting $n \to \infty$ in the equation solved by u_{ε_n}, it follows that u satisfies $\lambda u - Au = f$ a.e. in $\mathbb{R}^N$.

To conclude the proof it remains to show that $\lambda - A$ is surjective also when $\lambda > \frac{c_\beta}{p}$. This follows from the dissipativity of the operator $A - \frac{c_\beta}{p}$, stated in Lemma 1.6.1, and the fact that $\lambda - (A - \frac{c_\beta}{p})$ is surjective for $\lambda > (\sqrt{2}-1)c_\beta/p$. Thus $\lambda - (A - \frac{c_\beta}{p})$ is also surjective for $\lambda > 0$, which means that $\lambda - A$ is surjective for $\lambda > \frac{c_\beta}{p}$, as claimed. □

We are ready to prove Theorem 1.1.2.

Proof of Theorem 1.1.2. Since $C_c^\infty(\mathbb{R}^N) \subset \mathcal{D}_p \subset L^p(\mathbb{R}^N)$, it follows that $\mathcal{D}_p$ is a dense subset in $L^p(\mathbb{R}^N)$. Moreover, $(A, \mathcal{D}_p)$ is closed, by Lemma 1.6.1. By Proposition 1.6.2 and (1.6.1), for every $\lambda > \frac{c_\beta}{p}$, $\lambda - A : \mathcal{D}_p \to L^p(\mathbb{R}^N)$ is bijective and

$$\|(\lambda - A)^{-1} f\|_p \leq \left(\lambda - \frac{c_\beta}{p}\right)^{-1} \|f\|_p .$$

The thesis follows from the Hille-Yosida Theorem. □

1.7 Comments and consequences

In this final section we establish some further properties of the semigroup $T_p(\cdot)$ generated by $(A, \mathcal{D}_p)$ on $L^p(\mathbb{R}^N)$. We note that since all the assumptions of Theorem 1.1.2 for $p = 2$ imply those of Theorem 1.1.1, the semigroup $T_2(\cdot)$ is uniquely determined.
We point out that the semigroups given by Theorem 1.1.2 are not analytic, in general. A counterexample is the Ornstein-Uhlenbeck semigroup, as shown below (see e.g. [35, Example 4.4]).

Example 1.7.1 Let $Au = u'' + xu'$ be the Ornstein Uhlenbeck operator in one dimension. We prove that the semigroup $T(t)$ generated by A with domain $D(A) = \{u \in W^{2,p}(\mathbb{R}) \mid xu' \in L^p(\mathbb{R})\}$ in $L^p(\mathbb{R})$ is not differentiable and hence, *a fortiori*, it is not analytic. To this aim it is sufficient to prove that $T(t)$ is not continuous from $L^p(\mathbb{R})$ in $D(A)$. For every $u \in L^p(\mathbb{R})$, $t > 0$ and $x \in \mathbb{R}$, the Ornstein Uhlenbeck semigroup can be represented by

$$(T(t)u)(x) = \frac{1}{\sqrt{2\pi(e^{2t}-1)}} \int_{\mathbb{R}} e^{-\frac{y^2}{2(e^{2t}-1)}} u(e^t x - y) dy.$$

Let $u_n = \chi_{[n,n+1]}$. Then

$$(T(t)u_n)(x) = \frac{1}{\sqrt{2\pi(e^{2t}-1)}} \int_{e^t x - n - 1}^{e^t x - n} e^{-\frac{y^2}{2(e^{2t}-1)}} dy$$

and consequently

$$\frac{d}{dx}(T(t)u_n)(x) = \frac{e^t}{\sqrt{2\pi(e^{2t}-1)}} \Big(e^{-\frac{(e^t x - n)^2}{2(e^{2t}-1)}} - e^{-\frac{(e^t x - n - 1)^2}{2(e^{2t}-1)}} \Big),$$

$$\frac{d^2}{dx^2}(T(t)u_n)(x) = \frac{e^{2t}}{\sqrt{2\pi(e^{2t}-1)^3}} \Big(-(e^t x - n) e^{-\frac{(e^t x - n)^2}{2(e^{2t}-1)}} + (e^t x - n - 1) e^{-\frac{(e^t x - n - 1)^2}{2(e^{2t}-1)}} \Big).$$

It follows that

$$\begin{aligned}
\left\| \frac{d^2}{dx^2}(T(t)u_n) \right\|_p &= \frac{e^{2t}}{\sqrt{2\pi(e^{2t}-1)^3}} \left(\int_{\mathbb{R}} \left| y\, e^{-\frac{y^2}{2(e^{2t}-1)}} - (y-1)\, e^{-\frac{(y-1)^2}{2(e^{2t}-1)}} \right|^p e^{-t} dy \right)^{\frac{1}{p}} \\
&\leq \frac{e^{2t} 2^{1-\frac{1}{p}}}{\sqrt{2\pi(e^{2t}-1)^3}} \left(\int_{\mathbb{R}} |y|^p\, e^{-\frac{p\, y^2}{2(e^{2t}-1)}} e^{-t} dy \right. \\
&\quad \left. + \int_{\mathbb{R}} |y-1|^p\, e^{-\frac{p(y-1)^2}{2(e^{2t}-1)}} e^{-t} dy \right)^{\frac{1}{p}} \\
&\leq \frac{2\, e^{2t - \frac{t}{p}}}{\sqrt{2\pi(e^{2t}-1)^3}} \left(\int_{\mathbb{R}} |y|^p\, e^{-\frac{p\, y^2}{2(e^{2t}-1)}} dy \right)^{\frac{1}{p}} \\
&= c_p \frac{e^{t\left(2 - \frac{1}{p}\right)}}{(e^{2t}-1)^{1-\frac{1}{2p}}}.
\end{aligned}$$

Hence $\left\| \frac{d^2}{dx^2}(T(t)u_n) \right\|_p$ can be estimated indipendently of n. Moreover we have

$$\left\| x\frac{d}{dx}(T(t)u_n) \right\|_p^p = \frac{1}{(2\pi(e^{2t}-1))^{\frac{p}{2}}} \int_{\mathbb{R}} |y+n|^p \left| e^{-\frac{y^2}{2(e^{2t}-1)}} - e^{-\frac{(y-1)^2}{2(e^{2t}-1)}} \right|^p e^{-t} dy.$$

Since $y^2 \le (y-1)^2$ if $y \le 1/2$, by Fatou's Lemma we deduce that

$$\liminf_{n\to+\infty} \left\| x\frac{d}{dx}(T(t)u_n) \right\|_p^p \ge \frac{e^{-t}}{(2\pi(e^{2t}-1))^{\frac{p}{2}}} \int_{\{0\le y\le \frac{1}{2}\}} \liminf_{n\to+\infty} (y+n)^p \, e^{-\frac{p\,y^2}{2(e^{2t}-1)}} dy = +\infty.$$

Thus we have found a sequence (u_n) in $L^p(\mathbb{R})$ such that $\|u_n\|_p = 1$ but $\lim_{n\to+\infty} \|AT(t)u_n\|_p = +\infty$, for every fixed $t > 0$.

In the following proposition we prove the consistency of $T_p(\cdot)$.

Proposition 1.7.2 *Assume that the assumptions of Theorem 1.1.2 hold for some p and q, with $1 < p, q < +\infty$. If $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ then $T_p(t)f = T_q(t)f$, for all $t \ge 0$.*

Proof. By [21, Corollary III.5.5] we have only to prove that the resolvent operators of $(A, \mathcal{D}_p)$, $(A, \mathcal{D}_q)$ are consistent, for λ large, i.e. that for every $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ there exists $u \in W^{2,p}(\mathbb{R}^N) \cap W^{2,q}(\mathbb{R}^N)$ such that $\lambda u - Au = f$. This follows from the proofs of Proposition 1.6.2 and [37, Theorem 2.2] since the same property holds for uniformly elliptic operators. □

Now we prove the positivity of T_p.

Proposition 1.7.3 *$T_p(\cdot)$ is positive, i.e. if $f \in L^p(\mathbb{R}^N)$, $f \ge 0$, then $T_p(t)f \ge 0$, for all $t \ge 0$.*

Proof. The positivity of the semigroup T_p is equivalent to the positivity of the resolvent $(\lambda - A)^{-1}$ for all λ sufficiently large. By the proof of Proposition 1.6.2 this last property turns out to be true once that each A_ε is shown to have a positive resolvent. From [37, Theorem 2.2] this holds because the operators A_ε can be approximated by uniformly elliptic operators. □

In the following proposition we show the compactness of the resolvent of $(A, \mathcal{D}_p)$ assuming that the potential V tends to infinity as $|x| \to +\infty$. This result is similar to [41, Proposition 6.4] and we give the proof for the sake of completeness.

Proposition 1.7.4 *If $\lim_{|x|\to+\infty} V(x) = +\infty$ then the resolvent of $(A, \mathcal{D}_p)$ is compact.*

Proof. Let us prove that $\mathcal{D}_p$ is compactly embedded into $L^p(\mathbb{R}^N)$. Let $\mathcal{F}$ be a bounded subset of $\mathcal{D}_p$. By the assumption, given $\varepsilon > 0$ there exists $R > 0$ such that $V(x) \ge \varepsilon^{-1}$, if $|x| \ge R$. It follows that

$$\int_{|x|>R} |f(x)|^p \, dx \le \varepsilon^p \int_{|x|>R} |V(x)f(x)|^p \, dx \le \varepsilon^p C = \varepsilon' \tag{1.7.1}$$

for every $f \in \mathcal{F}$. Since the embedding of $W^{2,p}(B_R)$ into $L^p(B_R)$ is compact, the set $\mathcal{F}' = \{f_{|B_R} \,|\, f \in \mathcal{F}\}$, which is bounded in $W^{2,p}(B_R)$, is totally bounded in $L^p(B_R)$. Therefore there exist $r \in \mathbb{N}$ and $g_1, ..., g_r \in L^p(B_R)$ such that

$$\mathcal{F}' \subseteq \bigcup_{i=1}^r \{g \in L^p(B_R) \,|\, \|g - g_i\|_{L^p(B_R)} < \varepsilon'\}. \tag{1.7.2}$$

Set

$$\tilde{g}_i = \begin{cases} g_i & \text{in } B_R \\ 0 & \text{in } \mathbb{R}^N \setminus B_R. \end{cases}$$

Then $\tilde{g}_i \in L^p(\mathbb{R}^N)$ and from (1.7.1) and (1.7.2) it follows that

$$\mathcal{F} \subseteq \bigcup_{i=1}^r \{g \in L^p(\mathbb{R}^N) \,|\, \|g - \tilde{g}_i\|_p < 2\varepsilon'\}.$$

This implies that $\mathcal{F}$ is relatively compact in $L^p(\mathbb{R}^N)$ and the proof is complete. □

Finally, as a corollary of the estimates proved in the previous sections we prove an interpolatory estimate for the functions in $\mathcal{D}_p$.

Corollary 1.7.5 *For every $u \in \mathcal{D}_p$ the following estimate*

$$\|Du\|_p \le c\|u\|_p^{1/2}\|\lambda u - Au\|_p^{1/2}$$

holds for every λ sufficiently large.

PROOF. By density it is sufficient to consider $u \in C_c^\infty(\mathbb{R}^N)$. The thesis easily follows from (1.4.7), (1.6.1) and the inequality

$$\|Du\|_p \le c\|u\|_p^{1/2}\|D^2u\|_p^{1/2}\,.$$

□

Chapter 2

Gradient estimates in Neumann parabolic problems in convex regular domains

In the present chapter we study, by means of purely analytic tools, existence, uniqueness and gradient estimates of the solutions to the Neumann problems

$$\begin{cases} u_t(t,x) - \mathcal{A}u(t,x) = 0 & t>0,\ x\in\Omega, \\ \dfrac{\partial u}{\partial \eta}(t,x) = 0 & t>0,\ x\in\partial\Omega, \\ u(0,x) = f(x) & x\in\Omega, \end{cases} \tag{2.0.1}$$

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) = f(x) & x\in\Omega, \\ \dfrac{\partial u}{\partial \eta}(x) = 0 & x\in\partial\Omega, \end{cases} \tag{2.0.2}$$

where Ω is a regular convex open subset of $\mathbb{R}^N$, η is the unitary outward normal vector to $\partial\Omega$, f is a continuous and bounded function in $\overline{\Omega}$ and $\mathcal{A}$ is the linear second order elliptic operator

$$\mathcal{A} = \sum_{i,j=1}^{N} q_{ij}D_{ij} + \sum_{i=1}^{N} F_i\, D_i - V,$$

whose coefficients are supposed to be regular, possibly unbounded, in $\overline{\Omega}$. The set Ω may be unbounded. Obviously, if $\Omega = \mathbb{R}^N$ we do not require any boundary condition.

Problems (2.0.1) and (2.0.2) are classical in the theory of partial differential equations and they are well understood if the coefficients of $\mathcal{A}$ are bounded. If the coefficients are unbounded and $\Omega = \mathbb{R}^N$, several results of existence, uniqueness and regularity are known, (see [13], [27], [28], [34], [52]) and the overview [38]. Stochastic calculus is a useful tool ([13], [52], [56]); in particular the recent book of Sandra Cerrai [13] contains a deep and exhaustive analysis of what can be proved by stochastic methods.

We consider problem (2.0.1) and we prove that there exists a unique bounded classical solution $u(t,x)$. To do that, we consider the solutions u_n of Neumann problems in a nested sequence Ω_n of bounded domains whose union is Ω, and we prove that u_n converges to a solution of (2.0.1). We remark that one could approximate the solution with solutions of suitable mixed boundary value problems in Ω_n in such a way that for nonnegative initial data the approximating sequence is increasing. This was done by Seizo Itô in his pioneering paper [27]. Although this further property could be of much help in some steps, our techniques to get the gradient bounds do not work with such boundary conditions. Therefore we consider the Neumann boundary condition in each Ω_n. The solution u constructed in such a way is unique, since we assume a Lyapunov type condition which ensures that a maximum principle holds.

Setting $(P_t f)(x) = u(t,x)$, P_t turns out to be a semigroup of linear operators in the space $C_b(\overline{\Omega})$ of the continuous and bounded functions in $\overline{\Omega}$. We remark that in general P_t is not strongly continuous either in $C_b(\overline{\Omega})$ or in its subspace $BUC(\overline{\Omega})$ of the uniformly continuous and bounded functions. This is a typical fact for semigroups associated with elliptic operators with unbounded coefficients. Therefore the generator can not be defined in the classical way. In the literature there are several alternative definitions of generator; here we consider the weak generator introduced by E. Priola (see [48] and also Section 5.2). We prove that it coincides with the realization of $\mathcal{A}$ in $C_b(\overline{\Omega})$ with homogeneous Neumann boundary conditions (see Proposition 2.2.4). In this way, we can prove that the elliptic problem (2.0.2) admits a unique solution, whose second order derivatives exist only in the sense of distributions and are locally p summable for every p.

After we have ensured existence and uniqueness for problems (2.0.1) and (2.0.2), our next step consists in proving gradient estimates. We start by showing that

$$|DP_t f(x)| \le \frac{C_T}{\sqrt{t}} \|f\|_\infty \quad 0 < t < T,\ x \in \overline{\Omega},\quad f \in C_b(\overline{\Omega}), \tag{2.0.3}$$

$$|DP_t f(x)| \le C_T(\|f\|_\infty + \|Df\|_\infty) \quad 0 \le t \le T,\ x \in \overline{\Omega},\quad f \in C^1_\eta(\overline{\Omega}), \tag{2.0.4}$$

where

$$C^1_\eta(\overline{\Omega}) = \left\{ u \in C^1_b(\overline{\Omega}) :\ \frac{\partial u}{\partial \eta}(x) = 0,\quad x \in \partial\Omega \right\}. \tag{2.0.5}$$

We prove (2.0.3) and (2.0.4) using the Bernstein method, *i. e.* we apply the maximum principle to the equation satisfied by $z_n = u_n^2 + t|Du_n|^2$ (respectively $z_n = u_n^2 + |Du_n|^2$), that gives a bound for z_n independent of n, and then we obtain (2.0.3) (respectively (2.0.4)) letting $n \to \infty$. We observe that the convexity assumption on Ω is crucial at this point, since it leads to the condition $\frac{\partial z_n}{\partial \eta} \le 0$ at the boundary (see Lemma 2.1.3). In the case $\Omega = \mathbb{R}^N$ the previous estimates were proved in [34] with the same method and in [13] with probabilistic methods. As a consequence of (2.0.3) we have an elliptic regularity result, since we can show that the domain of the weak generator of P_t is contained in $C^1_b(\overline{\Omega})$.

Assuming $V \equiv 0$, we prove further gradient estimates. In the case $q_{ij} \equiv \delta_{ij}$ we show that

$$|DP_t f(x)|^p \le e^{k_0 p t} P_t(|Df|^p)(x) \quad t \ge 0,\ x \in \overline{\Omega},\quad f \in C^1_\eta(\overline{\Omega}). \tag{2.0.6}$$

for all $p \ge 1$, where $k_0 \in \mathbb{R}$ is determined by the dissipativity condition

$$\sum_{i,j=1}^N D_i F_j(x) \xi_i \xi_j \le k_0 |\xi|^2, \quad x \in \Omega,\ \xi \in \mathbb{R}^N. \tag{2.0.7}$$

If the coefficients q_{ij} are not constant we prove the similar estimate

$$|DP_t f(x)|^p \le e^{\sigma_p t} P_t(|Df|^p)(x) \quad t \ge 0,\ x \in \overline{\Omega},\quad f \in C^1_\eta(\overline{\Omega}), \tag{2.0.8}$$

for all $p > 1$, where $\sigma_p \in \mathbb{R}$ is a suitable constant. These estimates have interesting consequences. First, if there exists an invariant measure for P_t, that is a probability measure such that

$$\int_\Omega P_t f d\mu = \int_\Omega f d\mu, \quad t \ge 0,\ f \in C_b(\overline{\Omega}),$$

estimates (2.0.6) and (2.0.8) are of much help in the study of the realization of P_t in the spaces $L^p(\Omega,\mu)$, $1 \le p < \infty$ (see Remark 2.4.5 for such consequences and Chapter 5 for the main properties of invariant measures).

Second, we deduce the pointwise estimates

$$
\begin{aligned}
|DP_tf(x)|^p &\le \left(\frac{\sigma_2\nu_0^{-1}}{2(1-e^{-\sigma_2 t})}\right)^{\frac{p}{2}} P_t(|f|^p)(x), \quad t>0,\ p\ge 2,\\
|DP_tf(x)|^p &\le \frac{c_p\sigma_p\nu_0^{-1}}{t^{p/2-1}(1-e^{-\sigma_p t})}\, P_t(|f|^p)(x), \quad t>0,\ 1<p<2,
\end{aligned} \tag{2.0.9}
$$

for $f \in C_b(\overline{\Omega})$, where $c_p > 0$ is a suitable constant. Estimates (2.0.9) give the optimal constant in (2.0.3); moreover integrating over Ω with respect to the invariant measure μ we get the corresponding estimates for DP_tf in $L^p(\Omega,\mu)$, when $f \in L^p(\Omega,\mu)$.

Dissipativity conditions of the type (2.0.7) are of crucial importance to get gradient estimates. Indeed, in section 2.4 we give a counterexample to estimate (2.0.3) for an operator $\mathcal{A} = \Delta + \sum F_iD_i$ where F does not satisfy (2.0.7). Concerning estimate (2.0.6), in the case of variable coefficients q_{ij} the constant σ_p blows up as $p \to 1$, and we do not expect that (2.0.6) holds also for $p = 1$. Estimate (2.0.9) too fails in general for $p = 1$, as we show in the case of the heat semigroup. Finally we show an example related with the Ornstein-Uhlenbeck operator.

2.1 Assumptions and preliminary results

First we state our assumptions that will be kept throughout the chapter. $\Omega \subset \mathbb{R}^N$ is a convex open set with $C^{2+\alpha}$ boundary (see Definition B.0.15). The coefficients of the operator $\mathcal{A}$ are real-valued and belong to $C^{1+\alpha}_{\mathrm{loc}}(\overline{\Omega})$ and satisfy the following conditions:

$$
q_{ij} = q_{ji}, \quad \sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \ge \nu(x)|\xi|^2, \quad x\in\Omega,\ \xi\in\mathbb{R}^N, \quad \inf_{x\in\Omega}\nu(x) = \nu_0 > 0, \tag{2.1.1}
$$

$$
|Dq_{ij}(x)| \le M\nu(x), \quad x\in\Omega,\ i,j=1,...,N, \tag{2.1.2}
$$

$$
\sum_{i,j=1}^N D_iF_j(x)\,\xi_i\,\xi_j \le (\beta V(x)+k_0)\,|\xi|^2, \quad x\in\Omega,\ \xi\in\mathbb{R}^N, \tag{2.1.3}
$$

$$
V(x)\ge 0, \qquad |DV(x)| \le \gamma(1+V(x)), \quad x\in\Omega, \tag{2.1.4}
$$

for some constants $M,\gamma \ge 0$, $k_0,\beta \in \mathbb{R}$, $\beta < 1/2$. Moreover, we suppose that there exist a positive function $\varphi \in C^2(\overline{\Omega})$ and $\lambda_0 > 0$ such that

$$
\lim_{|x|\to+\infty}\varphi(x) = +\infty, \qquad \sup_{\overline{\Omega}}(\mathcal{A}\varphi - \lambda_0\varphi) < +\infty, \qquad \frac{\partial\varphi}{\partial\eta}(x) \ge 0, \quad x\in\partial\Omega. \tag{2.1.5}
$$

We introduce the following realization of operator $\mathcal{A}$ with homogeneous Neumann boundary condition

$$
D(\mathcal{A}) = \left\{u\in C_b(\overline{\Omega}) \cap \bigcap_{1\le p<\infty} W^{2,p}(\Omega\cap B_R) \text{ for all } R>0:\ \mathcal{A}u\in C_b(\overline{\Omega}),\ \frac{\partial u}{\partial\eta}\Big|_{\partial\Omega} = 0\right\}.
$$

We remark that if $\Omega = \mathbb{R}^N$ our results can be generalized to operators with locally Hölder continuous coefficients satisfying suitable assumptions by a standard convolution approximation, see Remark 2.3.4.

In this section we collect some preliminary results which are the main tools for the study of problems (2.0.1) and (2.0.2). We start by stating maximum principles for such problems, and consequent uniqueness results. For the proofs we refer to Appendix A.

Proposition 2.1.1 *Let $z \in C([0,T]\times\overline{\Omega})\cap C^{0,1}(]0,T]\times\overline{\Omega})\cap C^{1,2}(]0,T]\times\Omega)$ be a bounded function satisfying*

$$\begin{cases} z_t(t,x) - \mathcal{A}z(t,x) \le 0, & 0 < t \le T,\ x \in \Omega, \\ \dfrac{\partial z}{\partial \eta}(t,x) \le 0, & 0 < t \le T,\ x \in \partial\Omega, \\ z(0,x) \le 0 & x \in \Omega. \end{cases}$$

Then $z \le 0$. In particular there exists at most one bounded classical solution of problem (2.0.1).

Proposition 2.1.2 *Let $u \in C_b(\overline{\Omega}) \cap W^{2,p}(\Omega \cap B_R)$ for all $R > 0$ and $p < \infty$, be such that $\mathcal{A}u \in C_b(\overline{\Omega})$ and*

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) \le 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \eta}(x) \le 0, & x \in \partial\Omega, \end{cases} \tag{2.1.6}$$

for some $\lambda \ge \lambda_0$. Then $u \le 0$. In particular, there exists at most one solution in $D(\mathcal{A})$ of problem (2.0.2).

The following lemma is of crucial importance for our estimates; it holds for convex domains and this is the reason why we have assumed that Ω is convex.

Lemma 2.1.3 *Let Λ be a convex open set with C^1 boundary, not necessarily bounded. Let $u \in C^2(\overline{\Lambda})$ such that $\partial u/\partial\eta(x) = 0$ for all $x \in \partial\Lambda$. Then the function $v := |Du|^2$ satisfies*

$$\frac{\partial v}{\partial \eta}(x) \le 0, \quad x \in \partial\Lambda.$$

Proof. Let us introduce the notation $\dfrac{\partial\eta}{\partial\tau} = \left(\dfrac{\partial\eta_1}{\partial\tau}, \cdots, \dfrac{\partial\eta_N}{\partial\tau}\right)$, where the derivatives are understood in local coordinates. Since Ω is convex, we have $\tau \cdot \dfrac{\partial\eta}{\partial\tau}(x) \ge 0$ for all $x \in \partial\Omega$ and all vector τ tangent to $\partial\Omega$ at x (see [25, section V.B]). By assumption, $Du(x)\cdot\eta(x) = 0$ for all $x \in \partial\Omega$ and then differentiating we get

$$\frac{\partial}{\partial\tau}(Du(x)\cdot\eta(x)) = D^2u(x)\tau\cdot\eta(x) + Du(x)\cdot\frac{\partial\eta}{\partial\tau}(x) = 0, \qquad x \in \partial\Lambda,$$

for every vector τ tangent to $\partial\Omega$. For $\tau = Du(x)$ we have

$$\frac{\partial v}{\partial\eta}(x) = 2D^2u(x)\tau\cdot\eta(x) = -2\tau\cdot\frac{\partial\eta}{\partial\tau}(x) \le 0, \qquad x \in \partial\Omega.$$

□

Now we recall some known results about Neumann problems in bounded domains. Let Λ be a bounded open set in $\mathbb{R}^N$ with $C^{2+\alpha}$ boundary. Consider the realization of the operator $\mathcal{A}$ in $C(\overline{\Lambda})$ with homogeneous Neumann boundary condition

$$D_\eta(A) = \left\{ u \in W^{2,p}(\Lambda) \text{ for all } p < +\infty \ :\ \mathcal{A}u \in C(\overline{\Lambda}),\ \frac{\partial u}{\partial\eta}(x) = 0,\ x \in \partial\Lambda \right\}, \tag{2.1.7}$$

and $Au = \mathcal{A}u$ for all $u \in D_\eta(A)$.

It is well known that $(A, D_\eta(A))$ generates a strongly continuous analytic positive semigroup $(S(t))$ of contractions in the space $C(\overline{\Lambda})$ (see e.g. [32, Section 3.1.5]). It follows that for all $f \in C(\overline{\Lambda})$ the function $u(t,x) = (S(t)f)(x)$ has the following properties

(i) $u \in C([0,+\infty[;C(\overline{\Lambda})) \cap C^1(]0,+\infty[;C(\overline{\Lambda}))$,

(ii) $u(t,\cdot) \in D_\eta(A)$, for all $t > 0$,

(iii) u is the unique solution of the Neumann problem

$$
\begin{cases}
D_t u(t,x) - Au(t,x) = 0 & t > 0,\ x \in \Lambda, \\
\dfrac{\partial u}{\partial \eta}(t,x) = 0 & t > 0,\ x \in \partial\Lambda, \\
u(0,x) = f(x) & x \in \overline{\Lambda}.
\end{cases}
\tag{2.1.8}
$$

satisfying (i) and (ii).

Actually the function u enjoys further regularity, as it is shown below.

Lemma 2.1.4 *The following properties hold*

(a) $u \in C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda})$ *for all* $0 < \varepsilon < T < +\infty$ *and*

$$
\|u\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda})} \le C\,\|u\|_{C([0,T]\times\overline{\Lambda})}
\tag{2.1.9}
$$

for a suitable constant $C = C(\varepsilon,T,\Lambda) > 0$.

(b) $D_i u \in C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda}')$, *for all* $i = 1,...,N$, $0 < \varepsilon < T < +\infty$ *and* Λ' *open set with* $\overline{\Lambda}' \subset \Lambda$. *In particular* $u \in C^{1,3}(]0,+\infty[\times\Lambda)$.

Proof. (a) Assume first that $f \in C^{2+\alpha}(\overline{\Lambda})$ and $\partial f/\partial\eta = 0$ on $\partial\Lambda$. Then there exists a function $v \in C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Lambda})$, for all $T > 0$, which solves (2.1.8) (see [30, Theorem IV.5.3]). By uniqueness $v(t,x) = u(t,x)$.

Now take $f \in C(\overline{\Lambda})$ and consider a sequence $(f_n) \subseteq C^{2+\alpha}(\overline{\Lambda})$ with $\partial f_n/\partial\eta = 0$ on $\partial\Lambda$, which converges to f in $C(\overline{\Lambda})$. Let $v_n \in C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Lambda})$, for all $T > 0$, be the solution of problem (2.1.8) with initial datum f_n. Fix $0 < \varepsilon' < \varepsilon < T$, then the following Schauder estimate holds

$$
\|v_n\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda})} \le C\|v_n\|_{C([\varepsilon',T]\times\overline{\Lambda})}, \qquad n \in \mathbb{N}
\tag{2.1.10}
$$

where $C = C(\varepsilon,\varepsilon',T,\Lambda) > 0$ (see Theorem C.1.1).

On the other hand, the maximum principle implies that if $z \in C([0,T]\times\overline{\Lambda}) \cap C^1(]0,T]\times\overline{\Lambda}) \cap C^{1,2}(]0,T]\times\Lambda)$ solves problem (2.1.8) then

$$
\|z\|_{C([0,T]\times\overline{\Lambda})} \le \|f\|_{C(\overline{\Lambda})}.
$$

Applying this estimate and (2.1.10) to the difference $v_n - v_m$ we get

$$
\begin{aligned}
\|v_n - v_m\|_{C([0,T]\times\overline{\Lambda})} &\le \|f_n - f_m\|_{C(\overline{\Lambda})}, \qquad n,m \in \mathbb{N}, \\
\|v_n - v_m\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda})} &\le C\,\|f_n - f_m\|_{C(\overline{\Lambda})}, \qquad n,m \in \mathbb{N}.
\end{aligned}
$$

It follows that (v_n) is a Cauchy sequence in $C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda})$ and in $C([0,T]\times\overline{\Lambda})$, consequently it converges to a function $\overline{v} \in C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Lambda}) \cap C([0,T]\times\overline{\Lambda})$. Iterating the same argument we find a function $v \in C^{1+\alpha/2,2+\alpha}_{\text{loc}}(]0,+\infty[\times\overline{\Lambda}) \cap C([0,+\infty[\times\overline{\Lambda})$ which solves problem (2.1.8) with datum f. Again, by uniqueness, $v(t,x) = u(t,x)$. Estimate (2.1.9) is clear from (2.1.10) $n \to \infty$.

(b) The statement follows from [29, Theorem 8.12.1] since the coefficients of A belong to $C^{1+\alpha}(\overline{\Lambda})$. □

Next we prove a gradient estimate for $S(t)f$, using Bernstein's method (see [34, Theorem 2.4]). It is worth observing that, since Λ is bounded, this result is well-known. Actually, our interest is not in the estimate itself but rather in the fact that the constant C_T in (2.1.11) does not depend on the domain Λ, when it is convex. This will be an important step in the study of problem (2.0.1).

Proposition 2.1.5 *Let Λ be a bounded convex open set with $C^{2+\alpha}$ boundary. For all fixed $T > 0$ there exists a constant $C_T > 0$ independent of Λ such that*

$$|DS(t)f(x)| \le \frac{C_T}{\sqrt{t}} \|f\|_\infty \qquad 0 < t \le T,\ x \in \overline{\Lambda} \tag{2.1.11}$$

for every $f \in C(\overline{\Lambda})$.

Proof. We may suppose that $V \ge 1$; the general case follows considering the operator $A' = A - I$. Assume first that $f \in D_\eta(A)$; set $u(t,x) = (S(t)f)(x)$ and define the function

$$v(t,x) = u^2(t,x) + a\,t\,|Du(t,x)|^2\ , \quad t \ge 0,\ x \in \Lambda,$$

where $a > 0$ is a parameter that will be chosen later. Then we have $v \in C^{1,2}(]0,T] \times \Lambda) \cap C^{0,1}(]0,T] \times \overline{\Lambda})$; moreover, since $f \in D_\eta(A)$, we have $u \in C([0,T]; D_\eta(A))$; in particular $Du \in C([0,T] \times \overline{\Lambda})$ and then $v \in C([0,T] \times \overline{\Lambda})$.

We claim that for a suitable value of $a > 0$ independent of Λ, we have

$$v_t(t,x) - \mathcal{A}v(t,x) \le 0, \qquad 0 < t < T,\ x \in \Lambda, \tag{2.1.12}$$

$$\frac{\partial v}{\partial \eta}(t,x) \le 0 \qquad 0 < t < T,\ x \in \partial\Lambda; \tag{2.1.13}$$

then the maximum principle implies

$$v(t,x) \le \sup_{x \in \overline{\Lambda}} v(0,x) = \|f\|_\infty^2 \qquad 0 \le t \le T,\ x \in \overline{\Lambda},$$

which yields (2.1.11) with $C_T = a^{-1/2}$.

The boundary condition (2.1.13) follows from Lemma 2.1.3. For (2.1.12), a straightforward computation shows that v satisfies the equation

$$v_t(t,x) - \mathcal{A}v(t,x) = a|Du(t,x)|^2 - 2\sum_{i,j=1}^N q_{ij}(x)\,D_i u(t,x)\,D_j u(t,x) + g_1(t,x) + g_2(t,x),$$

where

$$\begin{aligned} g_1(t,x) &= 2\,a\,t\sum_{i,j=1}^N D_iF_j(x)\,D_iu(t,x)\,D_ju(t,x) - a\,t\,V(x)|Du(t,x)|^2 \\ &\quad -2\,a\,t\,u(t,x)Du(t,x)\cdot DV(x) - V(x)u^2(t,x), \\ g_2(t,x) &= 2\,a\,t\left(\sum_{i,j,k=1}^N D_kq_{ij}(x)D_ku(t,x)D_{ij}u(t,x) - \sum_{i,j,k=1}^N q_{ij}(x)D_{ik}u(t,x)D_{jk}u(t,x)\right). \end{aligned}$$

Let us estimate the function g_1. Using (2.1.3), (2.1.4) and recalling that $V \ge 1$ we get for all $\varepsilon > 0$

$$\begin{aligned} g_1 &\le 2at(\beta V + k_0)|Du|^2 - atV|Du|^2 + 2a\gamma C_\varepsilon t(1+V)|u|^2 + 2a\gamma\varepsilon t(1+V)|Du|^2 - Vu^2 \\ &\le at(2\beta - 1 + 2\gamma\varepsilon)V|Du|^2 + (4a\gamma C_\varepsilon t - 1)Vu^2 + 2at(k_0 + \gamma\varepsilon)|Du|^2, \end{aligned}$$

where $C_\varepsilon > 0$ is a constant. Since $\beta < 1/2$ we can choose $\varepsilon = \varepsilon(\beta, \gamma)$ such that $(2\beta - 1 + 2\gamma\varepsilon) < 0$ and we get

$$g_1 \leq (4\,a\,\gamma\,C_\varepsilon t - 1)Vu^2 + 2\,a\,t(k_0 + \gamma\varepsilon)|Du|^2. \tag{2.1.14}$$

Concerning g_2, from (2.1.2) we have

$$\begin{aligned}\sum_{i,j,k=1}^N D_k q_{ij} D_k u D_{ij} u &\leq M\nu(x) \sum_{k=1}^N |D_k u| \sum_{i,j=1}^N |D_{ij} u| \\ &\leq MN^{3/2}\nu(x)|Du| \left(\sum_{i,j=1}^N (D_{ij}u)^2 \right)^{1/2} \\ &\leq \nu(x) \sum_{i,j=1}^N (D_{ij}u)^2 + \frac{1}{4} M^2 N^3 \nu(x)|Du|^2,\end{aligned}$$

and therefore

$$\begin{aligned}g_2(t,x) &\leq 2\,a\,t\left(\nu(x) \sum_{i,j=1}^N (D_{ij}u)^2 + \frac{1}{4} M^2 N^3 \nu(x)|Du|^2 - \nu(x) \sum_{i,j=1}^N (D_{ij}u)^2 \right) \\ &= \frac{1}{2}\,a\,t\,M^2 N^3\,\nu(x)|Du|^2. \end{aligned} \tag{2.1.15}$$

Estimates (2.1.14) and (2.1.15) imply that

$$\begin{aligned} v_t(t,x) - \mathcal{A}v(t,x) &\leq \left\{ a + 2\,at(k_0 + \gamma\varepsilon)) + \left(\frac{1}{2} a\,t\,M^2 N^3 - 2 \right) \nu(x) \right\} |Du(t,x)|^2 \\ &\quad + (4\,a\,\gamma\,C_\varepsilon t - 1)V(x)u^2(t,x) \\ &\leq \left\{ a + 2\,aT(k_0^+ + \gamma\varepsilon)) + \left(\frac{1}{2} a\,T\,M^2 N^3 - 2 \right) \nu(x) \right\} |Du(t,x)|^2 \\ &\quad + (4\,a\,\gamma\,C_\varepsilon T - 1)V(x)u^2(t,x), \end{aligned}$$

for all $t \in]0, T]$ and $x \in \Lambda$. It is clear now that there exists a sufficiently small value $a > 0$ which depends on $\nu_0, M, k_0, \beta, \gamma, N, T$ but not on Λ such that (2.1.12) holds.

If $f \in C(\overline{\Lambda})$ the statement follows easily using the semigroup law, since $S(t)$ is analytic:

$$|DS(t)f(x)| = |DS(t/2)S(t/2)f(x)| \leq \frac{\sqrt{2}\,C_T}{\sqrt{t}} \|S(t/2)f\|_\infty \leq \frac{\sqrt{2}\,C_T}{\sqrt{t}} \|f\|_\infty.$$

□

2.2 Construction of the associated semigroup

In this section we prove that there exist bounded solutions to problems (2.0.1) and (2.0.2), we show that there exists a semigroup $(P_t)_{t \geq 0}$ in $C_b(\overline{\Omega})$ which yields the solution of (2.0.1) and we study the main properties of P_t.

We consider a nested sequence $\{\Omega_n\}_{n \in \mathbb{N}}$ of convex bounded open sets with $C^{2+\alpha}$ boundary such that

$$\bigcup_{n \in \mathbb{N}} \Omega_n = \Omega, \qquad \partial\Omega \subset \bigcup_{n \in \mathbb{N}} \partial\Omega_n.$$

We denote the domain of the realization of $\mathcal{A}$ in Ω_n with

$$D_n(\mathcal{A}) = \left\{ u \in W^{2,p}(\Omega_n) \text{ for all } p < \infty : \ \mathcal{A}u \in C(\overline{\Omega}_n), \ \frac{\partial u}{\partial \eta}(x) = 0, \ x \in \partial\Omega_n \right\}. \tag{2.2.1}$$

and we denote the associated semigroup with $(T_n(t))_{t\geq 0}$. Here is the existence theorem for problem (2.0.1).

Theorem 2.2.1 *For every $f \in C_b(\overline{\Omega})$ there exists a unique bounded solution $u(t,x)$ of problem (2.0.1) belonging to $C([0,+\infty[\times\overline{\Omega}) \cap C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}(]0,+\infty[\times\overline{\Omega})$. Moreover*

$$u(t,x) = \lim_{n\to\infty} (T_n(t))f(x), \qquad t \geq 0, \ x \in \overline{\Omega}. \tag{2.2.2}$$

Setting $P_t f = u(t,\cdot)$, then $(P_t)_{t\geq 0}$ is a positive contraction semigroup in $C_b(\overline{\Omega})$. Moreover

$$\|DP_t f\|_\infty \leq \frac{C_T}{\sqrt{t}} \|f\|_\infty \qquad 0 < t \leq T, \tag{2.2.3}$$

where C_T is the same as in (2.1.11).

Proof. Set $u_n(t,x) = (T_n(t)f)(x)$. Let $\Omega' \subset \Omega$ be a bounded open set and $0 < \varepsilon < T$. From [30, Theorem IV.10.1] it follows that if $\Omega'' \subset \Omega$ is a bounded open set such that $\Omega' \subset \Omega''$ and $\mathrm{dist}\,(\Omega', \Omega \setminus \Omega'') > 0$, then there exists a constant $C = C(\varepsilon, T, \Omega', \Omega'') > 0$ such that

$$\|u_n\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Omega}')} \leq C\|u_n\|_{C([0,T]\times\overline{\Omega}'')}. \tag{2.2.4}$$

Hence

$$\|u_n\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Omega}')} \leq C\|f\|_\infty,$$

for all $n \in \mathbb{N}$ such that $\Omega'' \subset \Omega_n$, and therefore the sequence $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $C^{1,2}([\varepsilon,T] \times \overline{\Omega}')$. Considering an increasing sequence of domains $[\varepsilon_n, T_n] \times \overline{\Omega}'_n$ whose union is $]0,+\infty[\times\overline{\Omega}$ and using a diagonal procedure we can conclude that there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ (possibly dependent on f) such that

$$\exists \lim_{k\to\infty} u_{n_k}(t,x) = u(t,x), \qquad t > 0, \ x \in \overline{\Omega},$$

where $u \in C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}(]0,+\infty[\times\overline{\Omega})$. Moreover $(u_{n_k})_{k\in\mathbb{N}}$ converges to u in $C^{1,2}([\varepsilon,T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and for all bounded open set $\Omega' \subset \Omega$.

We prove that u is a bounded classical solution of problem (2.0.1). The function u is a solution of the equation $u_t - \mathcal{A}u = 0$ in $]0,+\infty[\times\Omega$. This follows letting $k \to \infty$ in the equation satisfied by u_{n_k}. Moreover since

$$|u(t,x)| \leq \|f\|_\infty, \qquad t > 0, \ x \in \overline{\Omega},$$

then u is bounded in $]0,+\infty[\times\overline{\Omega}$. The boundary condition

$$\frac{\partial u}{\partial \eta}(t,x) = 0, \quad t > 0, \ x \in \partial\Omega.$$

follows immediately since u_{n_k} converges to u in $C^{1,2}([\varepsilon,T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and $\Omega' \subset \Omega$ bounded open set. Finally we prove that u is continuous at $(0,x_0)$ with value $f(x_0)$ for all $x_0 \in \overline{\Omega}$. Consider two neighborhoods $U_1 \subset U_0$ of x_0. Set $\Omega_0 = U_0 \cap \Omega$ and $\Omega_1 = U_1 \cap \Omega$ and suppose that Ω_0 is convex and has $C^{2+\alpha}$ boundary. Let $\theta \in C^2(\overline{\Omega}_0)$ be such that $\theta = 0$ in a neighborhood of $\Omega \cap \partial U_0$, $\theta = 1$ in $\overline{\Omega}_1$ and $\partial\theta/\partial\eta = 0$ in $U_0 \cap \partial\Omega$. Define

$$v_n(t,x) = \theta(x)u_n(t,x), \qquad t > 0, \ x \in \Omega_0.$$

Then v_n satisfies the boundary condition

$$\frac{\partial v_n}{\partial \eta}(t,x) = \theta(x)\frac{\partial u_n}{\partial \eta}(t,x) + u_n(t,x)\frac{\partial \theta}{\partial \eta}(x) = 0, \tag{2.2.5}$$

for all $t > 0$ and $x \in \partial\Omega_0$ and for all n such that $\Omega_0 \subset \Omega_n$. Moreover v_n satisfies the equation

$$D_t v_n(t,x) - \mathcal{A} v_n(t,x) = \psi_n(t,x), \qquad t > 0,\ x \in \Omega_0,$$

where

$$\psi_n(t,x) = -u_n(t,x)(\mathcal{A} + V(x))\theta(x) - 2\sum_{i,j=1}^{N} q_{ij}(x) D_i u_n(t,x) D_j \theta(x).$$

Since $T_n(t)$ satisfies the gradient estimate (2.1.11), it follows that there exists a constant $C > 0$ such that

$$\|\psi_n(t)\|_\infty \le \frac{C}{\sqrt{t}} \qquad 0 < t \le T, \tag{2.2.6}$$

for all $n \in \mathbb{N}$. Let $T(t)$ be the strongly continuous analytic semigroup generated by the realization of $\mathcal{A}$ in $C(\overline{\Omega}_0)$ with Neumann boundary conditions. From [32, Proposition 4.1.2] it follows that $v_n(t)$ can be written as

$$v_n(t) = T(t)(\theta f) + \int_0^t T(t-s)\psi_n(s)ds.$$

Since $v_n = u_n$ in $\overline{\Omega}_1$, if $(t,x) \in]0,T[\times\overline{\Omega}_1$ we have

$$|u_{n_k}(t,x) - f(x_0)| \le |T(t)(\theta f)(x) - f(x_0)| + \int_0^t \|T(t-s)\psi_{n_k}(s)\|_\infty ds.$$

Using (2.2.6) and letting $k \to \infty$ we get

$$|u(t,x) - f(x_0)| \le |T(t)(\theta f)(x) - f(x_0)| + \int_0^t \frac{C}{\sqrt{s}} ds$$

which shows that u is continuous at $(0,x_0)$. Since $x_0 \in \overline{\Omega}$ is arbitrary, we conclude that u is continuous in $[0,T]\times\overline{\Omega}$. Thus we have proved that u is a bounded classical solution of problem (2.0.1).

We claim that the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $C^{1,2}([\varepsilon,T]\times\overline{\Omega}')$ for all $0 < \varepsilon < T$, $\Omega' \subset \Omega$ bounded open set. Indeed consider any subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$. The previous argument can be applied to $(u_{n_k})_{k\in\mathbb{N}}$ and it follows that there is a subsequence $(u_{n_{k_j}})_{j\in\mathbb{N}}$ and a function v such that v is a classical bounded solution of problem (2.0.1) and $(u_{n_{k_j}})_{j\in\mathbb{N}}$ converges to v. But from Proposition 2.1.1 it follows that $u = v$. This show that the whole sequence converges to u.

Writing $(P_t f)(x) = u(t,x)$, we get the positivity of P_t directly from the positivity of $T_n(t)$. The semigroup law for the linear operators P_t follows in a standard way from uniqueness.

Finally, according to Proposition 2.1.5, for all $T > 0$ there exists a constant $C_T > 0$ such that

$$|DT_n(t)f(x)| \le \frac{C_T}{\sqrt{t}}\|f\|_\infty, \quad 0 < t \le T,\ x \in \overline{\Omega}_n,$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ we get (2.2.3). □

The next proposition shows some continuity properties of P_t that will be useful in the sequel.

Proposition 2.2.2 *If $(f_n)_{n\in\mathbb{N}} \subset C_b(\overline{\Omega})$ is a bounded sequence which converges pointwise in Ω to a function $f \in C_b(\overline{\Omega})$, then $(P_t f_n)(x)$ converges to $(P_t f)(x)$ in $C^{1,2}([\varepsilon, T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and all bounded sets $\Omega' \subset \Omega$. If (f_n) converges to f uniformly on compact subsets of $\overline{\Omega}$, then $(P_t f_n)(x)$ converges to $(P_t f)(x)$ uniformly in $[0,T] \times \overline{\Omega}'$ for all $T > 0$ and all bounded sets $\Omega' \subset \Omega$.*

Finally P_t can be represented in the form

$$(P_t f)(x) = \int_\Omega f(y)p(t,x;dy), \quad t > 0,\ x \in \overline{\Omega}, \tag{2.2.7}$$

where $p(t,x;dy)$ is a positive finite Borel measure on Ω.

Proof. We may assume that $f = 0$. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C_b(\overline{\Omega})$ that converges pointwise to zero in Ω, and set $u_n(t,x) = P_t f_n(x)$. Using the local Schauder estimate (2.2.4) and the maximum principle it follows that the sequence (u_n) is bounded in $C^{1+\alpha/2,2+\alpha}([\varepsilon,T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and all bounded $\Omega' \subset \Omega$. Therefore there exist a subsequence u_{n_k}, and a function $u \in C^{1,2}(]0,+\infty[\times\overline{\Omega})$ such that u_{n_k} converges to u in $C^{1,2}([\varepsilon,T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and for all bounded $\Omega' \subset \Omega$. The function u is a bounded solution of the equation

$$u_t - \mathcal{A}u = 0 \quad \text{in } (0,+\infty) \times \Omega,$$

and it satisfies the boundary condition

$$\frac{\partial u}{\partial \eta} = 0 \quad \text{in } (0,+\infty) \times \partial\Omega.$$

Now we show that u is continuous up to $t = 0$ and that $u(0,x) = 0$ in order to conclude that $u \equiv 0$, by Proposition 2.1.1. Let Ω_0, Ω_1 and θ be as in the proof of Theorem 2.2.1 and set $v_n(t,x) = \theta(x)u_n(t,x)$. Then we can write

$$v_n(t) = T(t)(\theta f_n) + \int_0^t T(t-s)\psi_n(s)ds,$$

where $T(t)$ is the semigroup generated by the realization of $\mathcal{A}$ in $C(\overline{\Omega}_0)$ with Neumann boundary condition and

$$\psi_n(t,x) = -u_n(t,x)(\mathcal{A} + V(x))\theta(x) - 2\sum_{i,j=1}^N q_{ij}(x)D_i u_n(t,x)D_j\theta(x).$$

Using the gradient estimate (2.2.3) and the boundedness of $(f_{n_k})_{k\in\mathbb{N}}$ it follows that

$$|v_{n_k}(t,x)| \le |(T(t)(\theta f_{n_k}))(x)| + C\sqrt{t}, \quad x \in \overline{\Omega}_0,\ 0 \le t \le T,\ k \in \mathbb{N}, \tag{2.2.8}$$

where $C > 0$ is a constant independent of $k \in \mathbb{N}$. For all $1 < p < +\infty$ the semigroup $(T(t))$ extends to an analytic semigroup in $L^p(\Omega_0)$ (see [32, Section 3.1.1]), and for $p > N$ the domain of the generator of $T(t)$ in $L^p(\Omega_0)$ is embedded in $C(\overline{\Omega}_0)$; since θf_{n_k} converges to zero in $L^p(\Omega_0)$ it follows that $T(t)(\theta f_{n_k})$ converges to zero uniformly in $\overline{\Omega}_0$. Thus letting $k \to \infty$ in (2.2.8) we get

$$|u(t,x)| \le C\sqrt{t}, \quad 0 < t < T,\ x \in \overline{\Omega}_1,$$

which implies that u is continuous at $(0,x_0)$ for all $x_0 \in \overline{\Omega}_1$. Since $\Omega_1 \subset \Omega$ is arbitrary, we obtain that u is continuous at $t = 0$ with $u(0,x) = 0$.

Therefore $u \equiv 0$ and the subsequence u_{n_k} converges to zero in $C^{1,2}([\varepsilon,T]\times\overline{\Omega}')$ for all $0 < \varepsilon < T$ and bounded $\Omega' \subset \Omega$. As in the proof of Theorem 2.2.1 one can prove that the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges to zero in $C^{1,2}([\varepsilon,T] \times \overline{\Omega}')$ for all $0 < \varepsilon < T$ and bounded $\Omega' \subset \Omega$, as stated.

Suppose now that $(f_n)_{n\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\overline{\Omega}$. By (2.2.8) we have

$$|u_n(t,x)| \le \|T(t)(\theta f_n)\|_\infty + C\sqrt{t} \le \|\theta f_n\|_\infty + C\sqrt{t}, \qquad x\in\overline{\Omega}_1,\ 0\le t\le T,$$

where $C>0$ does not depend on $n\in\mathbb{N}$. Therefore for all $\varepsilon>0$ we have

$$\|u_n\|_{C([0,T]\times\overline{\Omega}_1)} \le \|\theta f_n\|_\infty + C\sqrt{\varepsilon} + \|u_n\|_{C([\varepsilon,T]\times\overline{\Omega}_1)}.$$

Taking into account the first step of the proof this yields

$$\limsup_{n\to\infty} \|u_n\|_{C([0,T]\times\overline{\Omega}_1)} \le C\sqrt{\varepsilon},$$

that is u_n converges to zero uniformly in $[0,T]\times\overline{\Omega}_1$. Since Ω_1 is arbitrary, the conclusion follows.

We can prove now (2.2.7). By the Riesz representation theorem, for every $x\in\overline{\Omega}$ there exists a positive finite Borel measure $p(t,x;dy)$ in Ω such that

$$(P_tf)(x) = \int_\Omega f(y)p(t,x;dy), \qquad f\in C_0(\Omega). \tag{2.2.9}$$

If $f\in C_b(\overline{\Omega})$, we consider a bounded sequence $(f_n)_{n\in\mathbb{N}}\subset C_0(\Omega)$ which converges to f uniformly on compact sets of Ω. Writing (2.2.9) for f_n and letting $n\to+\infty$ we obtain the statement for $f\in C_b(\overline{\Omega})$, by dominated convergence. □

Using the semigroup law we extend estimate (2.2.3) to the whole half-line $[0,+\infty[$.

Corollary 2.2.3 *For all $\omega>0$ there exists $C_\omega>0$ such that*

$$\|DP_tf\|_\infty \le C_\omega \frac{e^{\omega t}}{\sqrt{t}}\|f\|_\infty, \qquad t>0,\ f\in C_b(\overline{\Omega}). \tag{2.2.10}$$

Proof. Fix $\omega>0$ and let $T=T(\omega)>0$ such that $e^{\omega t}t^{-1/2}\ge 1$, for all $t>T(\omega)$. By (2.2.3) for all $t\in]0,T]$ we have

$$\|DP_tf\|_\infty \le \frac{C_T}{\sqrt{t}}\|f\|_\infty \le C_T\frac{e^{\omega t}}{\sqrt{t}}\|f\|_\infty\ , \qquad 0<t\le T,$$

while for all $t>T$

$$\|DP_tf\|_\infty = \|DP_T\,P_{t-T}f\|_\infty \le \frac{C_T}{\sqrt{T}}\|P_{t-T}f\|_\infty \le \frac{C_T}{\sqrt{T}}\|f\|_\infty \le \frac{C_T}{\sqrt{T}}\frac{e^{\omega t}}{\sqrt{t}}\|f\|_\infty, \quad t>T.$$

So the statement follows with $C_\omega = \max\left\{C_T, \frac{C_T}{\sqrt{T}}\right\}$. □

We remark that the semigroup $(P_t)_{t\ge0}$ is not strongly continuous in $C_b(\overline{\Omega})$ in general: this is shown by the example $\Omega=\mathbb{R}^N$ and $\mathcal{A}=\Delta$. As in the case $\Omega=\mathbb{R}^N$ (see Section 5.2), we can introduce the weak generator $(\widehat{A},D(\widehat{A}))$ defined by

$$\begin{aligned} D(\widehat{A}) &= \Big\{f\in C_b(\overline{\Omega}) : \sup_{t\in(0,1)}\frac{\|P_tf-f\|}{t}<\infty \text{ and } \exists g\in C_b(\overline{\Omega}) \text{ such that} \\ &\qquad \lim_{t\to0}\frac{(P_tf)(x)-f(x)}{t} = g(x),\ \ \forall\, x\in\overline{\Omega}\Big\} \\ \widehat{A}f(x) &= \lim_{t\to0}\frac{(P_tf)(x)-f(x)}{t}, \quad f\in D(\widehat{A}),\ x\in\overline{\Omega}. \end{aligned}$$

The following results are proved in [48]: if $f \in D(\widehat{A})$, then $P_t f \in D(\widehat{A})$ and $\widehat{A} P_t f = P_t \widehat{A} f$, for all $t \geq 0$. Moreover one has $(0, +\infty) \subset \rho(\widehat{A})$, $\|R(\lambda, \widehat{A})\| \leq 1/\lambda$ and

$$(R(\lambda, \widehat{A})f)(x) = \int_0^{+\infty} e^{-\lambda t}(P_t f)(x)\, dt, \qquad x \in \overline{\Omega}, \tag{2.2.11}$$

and $R(\lambda, \widehat{A})$ is surjective from $C_b(\overline{\Omega})$ onto $D(\widehat{A})$ for all $\lambda > 0$.

Our aim now is to show that in fact $\widehat{A}$ coincides with the operator $\mathcal{A}$. This result is well known in the case where $\Omega = \mathbb{R}^N$. More precisely, one can prove that $\widehat{A} \subset \mathcal{A}$. If it is assumed that a Liapunov function exists, then one can check that also the other inclusion holds. We refer to Section 5.2, where the main properties concerning Feller semigroups in $\mathbb{R}^N$ are collected. If Ω is not the whole space, then the same result holds, but in proving it we have to pay attention to the boundary. Indeed, the main point in the proof below consists in applying suitable interior L^p estimates which involve also a part of $\partial\Omega$ (see (2.2.13)).

Proposition 2.2.4 *For all $f \in C_b(\overline{\Omega})$ and $\lambda > 0$, the function $u = R(\lambda, \widehat{A})f$ belongs to $D(\mathcal{A})$ and solves problem (2.0.2). Moreover $D(\widehat{A}) = D(\mathcal{A})$ and $\widehat{A}v = \mathcal{A}v$ for all $v \in D(\mathcal{A})$.*

Proof. Let $f \in C_b(\overline{\Omega})$ and let $u = R(\lambda, \widehat{A})f$. For all $n \in \mathbb{N}$, let $u_n = R_n(\lambda, \mathcal{A})f \in D_n(\mathcal{A})$, where $R_n(\lambda, \mathcal{A})$ is the resolvent of the operator $(\mathcal{A}, D_n(\mathcal{A}))$, that is

$$u_n(x) = \int_0^{+\infty} e^{-\lambda t}(T_n(t)f)(x)dt, \quad x \in \overline{\Omega}_n.$$

Taking into account the contractivity of $T_n(t)$, we have

$$\|u_n\|_\infty \leq \frac{1}{\lambda}\|f\|_\infty, \qquad \|\mathcal{A}u_n\|_\infty \leq 2\,\|f\|_\infty \tag{2.2.12}$$

for all $n \in \mathbb{N}$, and then from Theorem 2.2.1 and by dominated convergence it follows that

$$\lim_{n\to\infty} u_n = u,$$

pointwise in $\overline{\Omega}$ and in $L^p(\Omega_k)$, for all $k \in \mathbb{N}$. Furthermore, by Theorem C.2.1 we have

$$\|u_n - u_m\|_{W^{2,p}(\Omega_k)} \leq c(p,k)\Big(\|u_n - u_m\|_{L^p(\Omega_{k+1})}\Big), \qquad n, m > k, \tag{2.2.13}$$

for all $p \in (1, +\infty)$, where $c(p, k) > 0$ is a constant. Consequently u_n converges to u in $W^{2,p}(\Omega_k)$, for all $k \in \mathbb{N}$. Hence $u \in W^{2,p}(\Omega \cap B_R)$, for all $R < \infty$. Moreover by Sobolev embedding u_n converges to u in $C^1(\overline{\Omega}_k)$ for all $k \in \mathbb{N}$, and then we deduce that $\partial u/\partial\eta = 0$ in $\partial\Omega$. Finally, letting $n \to \infty$ in the equation $\lambda u_n - \mathcal{A}u_n = f$, it follows that $\lambda u - \mathcal{A}u = f$ in Ω. Therefore u belongs to $D(\mathcal{A})$ and it is a solution of problem (2.0.2).

In particular, since $R(\lambda, \widehat{A})$ is surjective from $C_b(\overline{\Omega})$ onto $D(\widehat{A})$, it follows that $D(\widehat{A}) \subset D(\mathcal{A})$. Conversely, let $u \in D(\mathcal{A})$ and define $f = \lambda u - \mathcal{A}u \in C_b(\overline{\Omega})$, where $\lambda \geq \lambda_0$ (see (2.1.5)). Then the function $v = R(\lambda, \widehat{A})f$ is a bounded solution of problem (2.0.2). By Proposition 2.1.2 we have $u = v$, and in particular $u \in D(\widehat{A})$. □

A consequence of the gradient estimate (2.2.10) is that $D(\mathcal{A})$ is continuously embedded in $C_b^1(\overline{\Omega})$.

Proposition 2.2.5 $D(\mathcal{A}) \subseteq C_b^1(\overline{\Omega})$. *Moreover for all $\omega > 0$ there exists a constant $M_\omega > 0$ such that:*

$$\|Du\|_\infty \leq M_\omega\, \|u\|_\infty^{\frac{1}{2}}\, \|(\mathcal{A} - \omega)\, u\|_\infty^{\frac{1}{2}} \tag{2.2.14}$$

for all $u \in D(\mathcal{A})$.

PROOF. Let $u \in D(\mathcal{A})$, $\omega > 0$ and $\lambda > 0$. Then the function $f = (\lambda + \omega)\, u - \mathcal{A}u$ belongs to $C_b(\overline{\Omega})$ and

$$u(x) = (R(\lambda+\omega, \widehat{A})f)(x) = \int_0^{+\infty} e^{-(\lambda+\omega)\,t}(P_t f)(x)\,dt\ , \qquad x \in \overline{\Omega}.$$

By using estimate (2.2.10), we may differentiate under the integral sign obtaining

$$Du(x) = \int_0^{+\infty} e^{-(\lambda+\omega)\,t}(DP_t f)(x)\,dt, \qquad x \in \Omega$$

and

$$|Du(x)| \le C_\omega \int_0^{+\infty} \frac{e^{-\lambda\, t}}{\sqrt{t}}\,dt\, \|f\|_\infty = \frac{M_\omega}{\sqrt{\lambda}}\|f\|_\infty\ , \qquad x \in \Omega,$$

where $M_\omega > 0$ is a constant. Therefore

$$\|Du\|_\infty \le M_\omega\left(\sqrt{\lambda}\|u\|_\infty + \frac{\|(\mathcal{A}-\omega)\,u\|_\infty}{\sqrt{\lambda}}\right),$$

and, taking the minimum over λ, (2.2.14) follows. □

With the same technique as in Proposition 2.1.5 we get the following gradient estimate.

Proposition 2.2.6 *For every $T > 0$ there exists $C_T > 0$ such that*

$$\|DP_t f\|_\infty \le C_T\left(\|f\|_\infty + \|Df\|_\infty\right)\ , \qquad 0 \le t \le T, \tag{2.2.15}$$

for every $f \in C^1_\eta(\overline{\Omega})$ (see (2.0.5)).

PROOF. We may suppose that $V \ge 1$; the general case follows considering the operator $\mathcal{A}' = \mathcal{A} - I$. We give the proof by steps; first we prove that there exists a constant $C_T > 0$ such that

$$|DT_n(t)f(x)| \le C_T(\|f\|_\infty + \|Df\|_\infty)\ , \qquad 0 \le t \le T,\ x \in \overline{\Omega}_n, \tag{2.2.16}$$

for every $n \in \mathbb{N}$ and $f \in C^1_\eta(\overline{\Omega}_n)$. Since $D_n(\mathcal{A})$ (see (2.2.1)) is dense in $C^1_\eta(\overline{\Omega}_n)$, it is enough to prove (2.2.16) for $f \in D_n(\mathcal{A})$.

Let $f \in D_n(\mathcal{A})$ and define

$$w(t,x) = u^2(t,x) + a\,|Du(t,x)|^2, \qquad t > 0,\ x \in \Omega_n\ ,$$

where $u(t,x) = (T_n(t)f)(x)$ and $a > 0$ is a constant. Then $w \in C([0,T] \times \overline{\Omega}_n) \cap C^{0,1}(]0,T] \times \overline{\Omega}_n) \cap C^{1,2}(]0,T] \times \Omega_n)$ and from Lemma 2.1.3 it follows that

$$\frac{\partial w}{\partial \eta}(t,x) \le 0, \qquad t > 0,\ x \in \partial\Omega_n.$$

Moreover w satisfies the equation

$$w_t(t,x) - \mathcal{A}w(t,x) = -2\sum_{i,j=1}^N q_{ij}(x)\, D_i u(t,x)\, D_j u(t,x) + h_1(t,x) + h_2(t,x),$$

where

$$\begin{aligned}
h_1(t,x) &= 2\,a \sum_{i,j=1}^N D_i F_j(x)\, D_i u(t,x)\, D_j u(t,x) - aV(x)|Du(t,x)|^2 \\
&\quad -2\,au(t,x)Du(t,x)\cdot DV(x) - V(x)u^2(t,x), \\
h_2(t,x) &= 2\,a\left(\sum_{i,j,k=1}^N D_k q_{ij}(x) D_k u(t,x) D_{ij}u(t,x) - \sum_{i,j,k=1}^N q_{ij}(x) D_{ik}u(t,x) D_{jk}u(t,x)\right).
\end{aligned}$$

The same estimates of the proof of Proposition 2.1.5 show that there exists a value of $a > 0$ independent of n such that

$$w_t(t,x) - \mathcal{A}w(t,x) \le 0, \qquad 0 \le t \le T,\ x \in \Omega_n\ .$$

Therefore the classical maximum principle yields

$$w(t,x) \le \sup_{x\in\overline{\Omega}_n} w(0,x) \le (\|f\|_\infty^2 + a\,\|Df\|_\infty^2), \qquad 0 \le t \le T,\ x \in \overline{\Omega}_n,$$

which implies (2.2.16) with $C_T = a^{-1/2} \vee 1$.

Let now $f \in C^1_\eta(\overline{\Omega})$. For all $k \in \mathbb{N}$, let $\theta_k \in C^1_b(\overline{\Omega})$ be a function with bounded support such that

$$0 \le \theta_k \le 1, \quad \|D\theta_k\|_\infty \le L,$$
$$\theta_k = 1 \ \text{ in } \Omega_k, \quad \frac{\partial \theta_k}{\partial \eta} = 0 \ \text{ in } \partial\Omega,$$

where $L > 0$ is a constant independent of $k \in \mathbb{N}$, and set $f_k = \theta_k f$. Then for all $n \in \mathbb{N}$ such that $\operatorname{supp}(\theta_k) \subset \Omega_n$ we have

$$\frac{\partial f_k}{\partial \eta}(x) = \frac{\partial \theta_k}{\partial \eta}(x) f(x) + \theta_k(x)\frac{\partial f}{\partial \eta}(x) = 0, \qquad x \in \partial\Omega_n,$$

that is $f_k \in C^1_\eta(\overline{\Omega}_n)$. Then $T_n(t)f_k$ satisfies estimate (2.2.16), and letting $n \to +\infty$ we get

$$|DP_t f_k(x)| \le C_T(\|f_k\|_\infty + \|Df_k\|_\infty) \le C_T((1+L)\|f\|_\infty + \|Df\|_\infty)\,, \qquad 0 \le t \le T,\ x \in \overline{\Omega}.$$

Taking into account Proposition 2.2.2 and letting $k \to \infty$ the statement follows. □

As a consequence we get the following result which will be used in the sequel.

Proposition 2.2.7 *If $f \in C^1_\eta(\overline{\Omega})$ then the function $DP_t f$ is continuous in $[0,+\infty) \times \overline{\Omega}$.*

Proof. Let $f \in C^1_\eta(\overline{\Omega})$. Taking account of Theorem 2.2.1 we have only to prove that $DP_t f$ is continuous at $t = 0$. Let $x_0 \in \overline{\Omega}$ be fixed and Ω_0, Ω_1, θ and $T(t)$ as in the proof of Theorem 2.2.1. We set

$$v(t,x) = \theta(x)(P_t f)(x), \qquad t \ge 0,\ x \in \overline{\Omega}_0,$$

and we prove that Dv is continuous at $(0,x_0)$; since $v(t,x) = (P_t f)(x)$ for all $x \in \Omega_1$ then the conclusion follows. We can write

$$v(t) = T(t)(\theta f) + \int_0^t T(t-s)\psi(s)ds,$$

where

$$\psi(t,x) = -P_t f(x)(\mathcal{A} + V(x))\theta(x) - 2\sum_{i,j=1}^N q_{ij}(x) D_i P_t f(x) D_j\theta(x).$$

From Proposition 2.2.6 it follows that

$$\|\psi(t)\|_\infty \le C_T(\|f\|_\infty + \|Df\|_\infty), \qquad 0 \le t \le T,$$

for some $C_T > 0$, where T is fixed, and then by (2.1.11) we have

$$\|DT(t-s)\psi(s)\|_\infty \le \frac{C}{\sqrt{t-s}}(\|f\|_\infty + \|Df\|_\infty), \qquad 0 < s < t \le T.$$

for some $C > 0$. Therefore

$$|Dv(t,x) - Df(x_0)| \le |DT(t)(\theta f)(x) - Df(x_0)| + 2C\sqrt{t}(\|f\|_\infty + \|Df\|_\infty),$$

for all $0 < t \le T$, $x \in \overline{\Omega}_0$. Taking account of

$$\lim_{(t,x)\to(0,x_0)} |DT(t)(\theta f)(x) - Df(x_0)| = 0, \tag{2.2.17}$$

we conclude that Dv is continuous at $(0, x_0)$. Relation (2.2.17) is immediate if $\theta f \in D_\eta(\mathcal{A})$, where $D_\eta(\mathcal{A})$ is the domain of the generator of $T(t)$, as in (2.1.7). Indeed in this case $T(t)(\theta f)$ belongs to $C([0,\infty); D_\eta(\mathcal{A}))$ and $D_\eta(\mathcal{A}) \subset C^1_\eta(\overline{\Omega}_0)$. In general we have $\theta f \in C^1_\eta(\overline{\Omega}_0)$ (see (2.2.5)), and (2.2.17) follows by approximation, since $D_\eta(\mathcal{A})$ is dense in $C^1_\eta(\overline{\Omega}_0)$. □

Remark 2.2.8 In the case $\Omega = \mathbb{R}^N$ the compactness of P_t in $C_b(\mathbb{R}^N)$ has been studied in [39]. The results extend to the case $\Omega \ne \mathbb{R}^N$, with the same proofs adapted to the Neumann problem. Assume that $V \equiv 0$, *i. e.* consider the conservative case where $P_t\mathbb{1} = \mathbb{1}$. First, P_t is compact in $C_b(\overline{\Omega})$ for all $t > 0$ if and only if for all $t, \varepsilon > 0$ there exists a bounded set $\Omega' \subset \Omega$ such that $p(t, x, \Omega') \ge 1 - \varepsilon$ for all $x \in \overline{\Omega}$. Secondly, if there exists a positive function $\psi \in C^2$ such that

$$\lim_{|x|\to+\infty} \psi(x) = +\infty, \qquad \frac{\partial\psi}{\partial\eta}(x) = 0, \ \ x \in \partial\Omega, \qquad \mathcal{A}\psi(x) \le -g(\psi(x)), \ \ x \in \Omega,$$

where $g : [0, +\infty[\to \mathbb{R}$ is a convex function such that $\lim_{x\to+\infty} g(x) = +\infty$ and $1/g$ is integrable at $+\infty$, then P_t is compact in $C_b(\overline{\Omega})$ for all $t > 0$.

2.3 Pointwise gradient estimates

In the whole section we assume that $V \equiv 0$ which implies that $P_t\mathbb{1} = \mathbb{1}$ for all $t > 0$, by uniqueness. Actually this is a necessary condition for the estimates that we are going to prove. Indeed, taking $f = \mathbb{1}$ in (2.3.1) it follows that $P_t\mathbb{1} = \mathbb{1}$.

Proposition 2.3.1 *Suppose $q_{ij}(x) \equiv \delta_{ij}$ for all $i, j = 1, ..., N$. Then for every $p \ge 1$ and $f \in C^1_\eta(\overline{\Omega})$ we have*

$$|DP_tf(x)|^p \le e^{pk_0t} P_t(|Df|^p)(x), \quad t \ge 0, \ x \in \overline{\Omega}. \tag{2.3.1}$$

PROOF: It is sufficient to prove the case $p = 1$. For $p > 1$, we observe that since $P_t\mathbb{1} = \mathbb{1}$ the measures $p(t, x; dy)$ given by Proposition 2.2.2 are probability measures, and then Jensen's inequality yields

$$|DP_tf(x)|^p \le \left(e^{k_0t} P_t(|Df|)(x)\right)^p \le e^{k_0pt} P_t(|Df|^p)(x).$$

Let $f \in C^1_\eta(\overline{\Omega})$ and let $\varepsilon > 0$ be fixed. Set $u(t,x) = P_tf(x)$ and define the function

$$w(t,x) = \left(|Du(t,x)|^2 + \varepsilon\right)^{\frac{1}{2}}, \quad t > 0, \ x \in \Omega.$$

From Proposition 2.2.6 and Proposition 2.2.7 it follows that w is bounded and continuous in $[0, +\infty[\times\overline{\Omega}$. Since $u \in C^{1+\alpha/2, 2+\alpha}_{\text{loc}}(]0, +\infty[\times\overline{\Omega})$ (see Theorem 2.2.1), we have that $w \in C^{0,1}(]0, +\infty[\times\overline{\Omega})$. Finally, from [29, Theorem 8.12.1] we deduce that $w \in C^{1,2}(]0, +\infty[\times\Omega)$. From Lemma 2.1.3 it follows that

$$\frac{\partial w}{\partial\eta}(t,x) = \frac{1}{2}\left(|Du(t,x)|^2 + \varepsilon\right)^{-\frac{1}{2}} \frac{\partial}{\partial\eta}|Du|^2(t,x) \le 0, \quad t > 0, \ x \in \partial\Omega.$$

A straightforward computation shows that w satisfies the equation

$$w_t(t,x) - \mathcal{A}w(t,x) = g_1(t,x) + g_2(t,x)$$

where

$$\begin{aligned} g_1 &= \left(|Du|^2+\varepsilon\right)^{-\frac{1}{2}} \sum_{i,j=1}^N (D_iF_j)(D_iu)(D_ju) \\ g_2 &= \left(|Du|^2+\varepsilon\right)^{-\frac{3}{2}} \sum_{i=1}^N \left(\sum_{j=1}^N (D_ju)(D_{ij}u)\right)^2 - \left(|Du|^2+\varepsilon\right)^{-\frac{1}{2}} \sum_{i,j=1}^N (D_{ij}u)^2 \,. \end{aligned}$$

We estimate now the functions g_1 and g_2. Since

$$\begin{aligned} \left(|Du|^2+\varepsilon\right)^{-\frac{3}{2}} \sum_{i=1}^N \left(\sum_{j=1}^N D_juD_{ij}u\right)^2 &\le \left(|Du|^2+\varepsilon\right)^{-\frac{3}{2}} |Du|^2 \sum_{i,j=1}^N (D_{ij}u)^2 \\ &\le \left(|Du|^2+\varepsilon\right)^{-\frac{1}{2}} \sum_{i,j=1}^N (D_{ij}u)^2. \end{aligned}$$

it follows that $g_2 \le 0$. On the other hand using (2.1.3) we obtain

$$g_1(t,x) \le k_0\left(|Du(t,x)|^2+\varepsilon\right)^{-\frac{1}{2}} |Du(t,x)|^2 = k_0w - k_0\varepsilon\left(|Du(t,x)|^2+\varepsilon\right)^{-\frac{1}{2}} \,.$$

If $k_0 \ge 0$ we have immediately

$$g_1(t,x) \le k_0 w\,,$$

whereas if $k_0 < 0$, we have

$$g_1(t,x) \le k_0 w - k_0\sqrt{\varepsilon}\,.$$

In any case we obtain

$$w_t - \mathcal{A}w \le k_0\left(w - \delta_\varepsilon\right)$$

where

$$\delta_\varepsilon = \begin{cases} 0 & k_0 \ge 0, \\ \sqrt{\varepsilon} & k_0 < 0. \end{cases}$$

Therefore the function $v = w - \delta_\varepsilon$ satisfies

$$\begin{cases} v_t(t,x) - \mathcal{A}v(t,x) \le k_0\, v(t,x) & t>0,\ x\in\Omega, \\ \dfrac{\partial v}{\partial \eta}(t,x) \le 0 & t>0,\ x\in\partial\Omega, \\ v(0,x) = \left(|Df(x)|^2+\varepsilon\right)^{\frac{1}{2}} - \delta_\varepsilon & x\in\overline{\Omega}. \end{cases}$$

On the other hand, the function

$$z(t,x) = e^{k_0t}P_t\left((|Df|^2+\varepsilon)^{\frac{1}{2}}\right)(x), \qquad t>0,\ x\in\Omega,$$

solves the problem

$$\begin{cases} z_t(t,x) - \mathcal{A}z(t,x) = k_0z(t,x) & t>0,\ x\in\Omega, \\ \dfrac{\partial z}{\partial \eta}(t,x) = 0 & t>0,\ x\in\partial\Omega, \\ z(0,x) = (|Df(x)|^2+\varepsilon)^{\frac{1}{2}} & x\in\overline{\Omega}. \end{cases}$$

Therefore Proposition 2.1.1 applied to $v-z$ and to the operator $\mathcal{A}+k_0I$ yields $v \le z$, that is

$$\left(|Du(t,x)|^2+\varepsilon\right)^{\frac{1}{2}} - \delta_\varepsilon \le e^{k_0t}P_t\left((|Df|^2+\varepsilon)^{\frac{1}{2}}\right)(x) \quad t\ge 0,\ x\in\overline{\Omega}.$$

Letting $\varepsilon \to 0$ estimate (2.3.1) with $p=1$ follows. □

We now consider the case of variable second order coefficients. Under the assumption

$$\sum_{i,j=1}^{N}(Dq_{ij}(x)\cdot\xi)^2 \le q_0\nu(x)|\xi|^2, \quad x\in\Omega,\ \xi\in\mathbb{R}^N, \tag{2.3.2}$$

which is slightly stronger than (2.1.2), we generalize the previous result when $p>1$.

Proposition 2.3.2 *Suppose that (2.3.2) holds. Then*

$$|DP_tf(x)|^p \le e^{\sigma_p t}P_t(|Df|^p)(x), \quad t\ge 0,\ x\in\overline{\Omega}, \tag{2.3.3}$$

for all $p>1$ and $f\in C^1_\eta(\overline{\Omega})$, where $\sigma_p = pk_0+\frac{p}{4}q_0$ if $p\ge 2$ and $\sigma_p = pk_0+\frac{p}{4(p-1)}q_0$ if $1<p<2$.

Proof. Let $f\in C^1_\eta(\overline{\Omega})$ be fixed. We first prove the statement for $p=2$. Consider the function

$$w(t,x)=|Du(t,x)|^2, \quad t>0,\ x\in\Omega,$$

where $u(t,x)=(P_tf)(x)$; then $w\in C([0,+\infty[\times\overline{\Omega})\cap C^{0,1}(]0,+\infty[\times\overline{\Omega})\cap C^{1,2}(]0,+\infty[\times\Omega)$, and from Lemma 2.1.3 we have

$$\frac{\partial w}{\partial\eta}(t,x)\le 0, \quad t>0,\ x\in\partial\Omega.$$

Moreover it is readily seen that

$$w_t(t,x)-\mathcal{A}w(t,x)=f_0(t,x),$$

where

$$f_0 = 2\left(\sum_{i,j,k}D_kq_{ij}D_kuD_{ij}u+\sum_{j,k}D_kF_jD_kuD_ju-\sum_{i,j,k}q_{ij}D_{ik}uD_{jk}u\right).$$

From (2.3.2) it follows that

$$\begin{aligned}\sum_{i,j,k=1}^{N}D_kq_{ij}(x)D_kuD_{ij}u &\le \left(\sum_{i,j=1}^{N}(D_{ij}u)^2\right)^{1/2}\left(\sum_{i,j=1}^{N}(Dq_{ij}\cdot Du)^2\right)^{1/2}\\ &\le \left(\sum_{i,j=1}^{N}(D_{ij}u)^2\right)^{1/2}\left(q_0\nu(x)|Du|^2\right)^{1/2}\\ &\le \nu(x)\sum_{i,j=1}^{N}(D_{ij}u)^2+\frac14 q_0|Du|^2, \end{aligned}\tag{2.3.4}$$

and then using (2.1.3) we get

$$\begin{aligned}f_0(t,x) &\le 2\left(\nu(x)\sum_{i,j=1}^{N}(D_{ij}u)^2+\frac14 q_0|Du|^2+k_0|Du|^2-\nu(x)\sum_{i,j=1}^{N}(D_{ij}u)^2\right)\\ &= \left(2k_0+\frac{q_0}{2}\right)|Du|^2=\sigma_2|Du|^2\end{aligned}$$

On the other hand the function

$$z(t,x)=e^{\sigma_2 t}P_t(|Df|^2)(x), \qquad t>0,\ x\in\Omega,$$

is the solution of the problem

$$\begin{cases} z_t(t,x) - \mathcal{A}z(t,x) = \sigma_2 z(t,x) & t>0,\ x\in\Omega,\\ \dfrac{\partial z}{\partial\eta}(t,x) = 0 & t>0,\ x\in\partial\Omega,\\ z(0,x) = |Df(x)|^2 & x\in\overline{\Omega}. \end{cases}$$

Using Proposition 2.1.1 we can conclude that $w \le z$, that is (2.3.3) with $p = 2$.

Now the case $p > 2$ follows easily applying Jensen's inequality:

$$|DP_tf(x)|^p \le \left(e^{\sigma_2 t}P_t(|Df|^2)(x)\right)^{\frac{p}{2}} \le e^{\sigma_p t}P_t(|Df|^p)(x), \qquad t>0,\ x\in\Omega.$$

Assume $1 < p < 2$. Fix $\varepsilon > 0$ and define the function

$$w(t,x) = \left(|Du(t,x)|^2 + \varepsilon\right)^{\frac{p}{2}},$$

where $u(t,x) = (P_tf)(x)$. Then $w \in C([0,+\infty[\times\overline{\Omega}) \cap C^{0,1}(]0,+\infty[\times\overline{\Omega}) \cap C^{1,2}(]0,+\infty[\times\Omega)$, and from Lemma 2.1.3 we have

$$\frac{\partial w}{\partial\eta}(t,x) = \frac{p}{2}\left(|Du(t,x)|^2+\varepsilon\right)^{\frac{p}{2}-1}\frac{\partial}{\partial\eta}|Du(t,x)|^2 \le 0, \qquad t>0,\ x\in\partial\Omega.$$

Moreover it turns out that

$$w_t(t,x) - \mathcal{A}w(t,x) = f_1(t,x) + f_2(t,x),$$

where

$$f_1 = p\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}} f_0$$

$$f_2 = p\,(2-p)\left(|Du|^2+\varepsilon\right)^{\frac{p-4}{2}} \sum_{i,j,k,h} q_{ij}D_kuD_{jk}uD_huD_{ih}u$$

Taking into account (2.3.4) for all $\delta > 0$ we have

$$f_1 \le p\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}}\left(\delta\nu(x)\sum_{i,j=1}^N (D_{ij}u)^2 + \frac{1}{4\delta}q_0|Du|^2 + k_0|Du|^2 - \sum_{i,j,k=1}^N q_{ij}D_{jk}uD_{ik}u\right).$$

As far as f_2 is concerned, we set $A_{kh} = \sum_{i,j=1}^N q_{ij}D_{jk}uD_{ih}u$ and we observe that, since the matrix $A = (A_{kh})$ is symmetric and nonnegative definite, we have $\sum_{k,h=1}^N A_{kh}D_huD_ku \le \mathrm{Tr}(A)|Du|^2$, where $\mathrm{Tr}(A)$ denotes the trace of A. Therefore

$$\begin{aligned} f_2 &= p(2-p)\left(|Du|^2+\varepsilon\right)^{\frac{p-4}{2}}\sum_{k,h=1}^N A_{kh}D_kuD_hu\\ &\le p(2-p)\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}}\sum_{i,j,k=1}^N q_{ij}D_{jk}uD_{ik}u. \end{aligned}$$

Choosing $\delta = p-1$ we get

$$\begin{aligned} f_1+f_2 &\le p\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}}\Bigg((p-1)\nu(x)\sum_{i,j=1}^N (D_{ij}u)^2 + \Big(\frac{q_0}{4(p-1)}+k_0\Big)|Du|^2\\ &\quad +(1-p)\sum_{i,j,k=1}^N q_{ij}D_{jk}uD_{ik}u\Bigg)\\ &\le \left(pk_0 + \frac{p}{4(p-1)}q_0\right)\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}}|Du|^2 = \sigma_p w - \varepsilon\sigma_p\left(|Du|^2+\varepsilon\right)^{\frac{p-2}{2}}, \end{aligned}$$

which implies

$$w_t - \mathcal{A}w \le \sigma_p(w - \delta_\varepsilon),$$

where

$$\delta_\varepsilon = \begin{cases} 0 & \text{if } \sigma_p \ge 0, \\ \varepsilon^{\frac{p}{2}} & \text{if } \sigma_p < 0. \end{cases}$$

Now the conclusion of the proof is the same as in Proposition 2.3.1: applying Proposition 2.1.1 to compare with $z(t,x) = e^{\sigma_p t} P_t((|Df|^2 + \varepsilon)^{\frac{p}{2}})$ we deduce

$$\left(|Du(t,x)|^2 + \varepsilon\right)^{\frac{p}{2}} - \delta_\varepsilon \le e^{\sigma_p t} P_t\left((|Df|^2 + \varepsilon)^{\frac{p}{2}}\right)(x), \qquad t \ge 0,\ x \in \overline{\Omega},$$

and then (2.3.3) follows letting $\varepsilon \to 0$. □

In the following proposition we deduce from (2.3.3) another type of pointwise gradient estimate. The basic idea of the proof is taken from [7] where it is considered the case $p = 2$.

Proposition 2.3.3 *Assume that (2.3.2) holds. Then for all $f \in C_b(\overline{\Omega})$ we have*

$$|DP_t f(x)|^p \le \left(\frac{\sigma_2 \nu_0^{-1}}{2(1 - e^{-\sigma_2 t})}\right)^{\frac{p}{2}} P_t(|f|^p)(x), \qquad t > 0,\ x \in \overline{\Omega}, \tag{2.3.5}$$

for all $p \ge 2$, and

$$|DP_t f(x)|^p \le \frac{c_p \nu_0^{-1} \sigma_p}{t^{p/2-1}(1 - e^{-\sigma_p t})} P_t(|f|^p)(x), \qquad t > 0,\ x \in \overline{\Omega}, \tag{2.3.6}$$

for all $1 < p < 2$, where $c_p = 2^p/(p(p-1))^{p/2}$ and σ_p is given by Proposition 2.3.2. When $\sigma_p = 0$ in (2.3.5) and (2.3.6) we replace $\sigma_p/(1 - e^{-\sigma_p t})$ by $1/t$.

PROOF. We prove that $T_n(t)f$ satisfies estimates (2.3.5) and (2.3.6) for $x \in \overline{\Omega}_n$, for all $n \in \mathbb{N}$; then the conclusion follows letting $n \to \infty$. Fix $n \in \mathbb{N}$ and set $T_t = T_n(t)$, for simplicity. Note that T_t satisfies estimate (2.3.3) for all the functions in $C^1_\eta(\overline{\Omega}_n)$.

First we consider the case $p = 2$. Let $f \in C_b(\overline{\Omega})$, fix $t > 0$ and set

$$\Phi(s) = T_s\left((T_{t-s}f)^2\right), \quad 0 \le s \le t - \varepsilon\ ,$$

where $\varepsilon > 0$. From the analiticity of T_t it follows that $g = T_{t-s}f \in D_n(\mathcal{A})$, for all $0 \le s \le t - \varepsilon$ (we recall that $D_n(\mathcal{A})$ is the domain of the generator of T_t, defined in (2.2.1)). Moreover from a direct calculation it is readily seen that $g^2 \in D_n(\mathcal{A})$ and

$$\Phi'(s) = \mathcal{A}T_s(g^2) - 2\,T_s(g\mathcal{A}g) = T_s(\mathcal{A}(g^2) - 2g\mathcal{A}g) = 2T_s(\langle qDg, Dg\rangle).$$

Thus

$$\Phi(t-\varepsilon) - \Phi(0) = T_{t-\varepsilon}((T_\varepsilon f)^2) - (T_t f)^2 = 2\int_0^{t-\varepsilon} T_s(\langle qDT_{t-s}f, DT_{t-s}f\rangle)\, ds.$$

Now, applying Proposition 2.3.2 to $T_{t-s}f$ we obtain

$$T_s(\langle qDT_{t-s}f, DT_{t-s}f\rangle) \ge \nu_0 T_s(|DT_{t-s}f|^2) \ge \nu_0 e^{-\sigma_2 s}|DT_t f|^2,$$

so that

$$T_{t-\varepsilon}((T_\varepsilon f)^2) - (T_t f)^2 \ge 2\,\nu_0 |DT_t f|^2 \int_0^{t-\varepsilon} e^{-\sigma_2 s}\, ds = \frac{2\nu_0(1 - e^{-\sigma_2(t-\varepsilon)})}{\sigma_2}|DT_t f|^2\ ,$$

and then

$$|DT_tf|^2 \le \frac{\sigma_2\,\nu_0^{-1}}{2(1-e^{-\sigma_2(t-\varepsilon)})}\Big(T_{t-\varepsilon}((T_\varepsilon f)^2)-(T_tf)^2\Big) \le \frac{\sigma_2\,\nu_0^{-1}}{2(1-e^{-\sigma_2(t-\varepsilon)})}\,T_{t-\varepsilon}((T_\varepsilon f)^2).$$

Letting $\varepsilon \to 0$ we obtain our claim.

If $p > 2$, using Jensen's inequality we get

$$|DT_tf|^p \le \left(\frac{\sigma_2\,\nu_0^{-1}}{2(1-e^{-\sigma_2 t})}T_t(f^2)\right)^{\frac{p}{2}} \le \left(\frac{\sigma_2\,\nu_0^{-1}}{2(1-e^{-\sigma_2 t})}\right)^{\frac{p}{2}} T_t(|f|^p).$$

Now assume $1 < p < 2$. Let first $f \in C_b(\overline{\Omega})$ with $f \ge \delta$ for some $\delta > 0$. Fix $t, \varepsilon > 0$ and define the function

$$\Psi(s) = T_s\left((T_{t-s}f)^p\right) \qquad 0 \le s \le t-\varepsilon.$$

Then $g = T_{t-s}f \ge \delta > 0$ and a straightforward computation shows that

$$\mathcal{A}(g^p) = pg^{p-1}\mathcal{A}g + p(p-1)g^{p-2}\langle qDg, Dg\rangle, \qquad \frac{\partial g^p}{\partial\eta} = pg^{p-1}\frac{\partial g}{\partial \eta}$$

which imply that $g^p \in D_n(\mathcal{A})$, since $g \in D_n(\mathcal{A})$. Moreover

$$\Psi'(s) = T_s\Big(\mathcal{A}(g^p) - pg^{p-1}\mathcal{A}g\Big) = p(p-1)\,T_s\Big((T_{t-s}f)^{p-2}\langle qDT_{t-s}f, DT_{t-s}f\rangle\Big),$$

and hence

$$T_{t-\varepsilon}((T_\varepsilon f)^p) - (T_tf)^p = p(p-1)\int_0^{t-\varepsilon} T_s\Big((T_{t-s}f)^{p-2}\langle qDT_{t-s}f, DT_{t-s}f\rangle\Big)\,ds. \tag{2.3.7}$$

Applying Proposition 2.3.2 and Hölder's inequality we get for all $\beta \in \mathbb{R}$

$$\begin{aligned}
|DT_tf|^p &= |DT_sT_{t-s}f|^p \le e^{\sigma_p s}T_s(|DT_{t-s}f|^p)\\
&= e^{\sigma_p s}T_s\Big(|DT_{t-s}f|^p\,(T_{t-s}f)^{-\beta}\,(T_{t-s}f)^{\beta}\Big)\\
&\le e^{\sigma_p s}\left\{T_s\left(|DT_{t-s}f|^2\,(T_{t-s}f)^{-\frac{2\beta}{p}}\right)\right\}^{p/2}\left\{T_s\,(T_{t-s}f)^{\frac{2\beta}{2-p}}\right\}^{1-p/2}\\
&\le e^{\sigma_p s}\nu_0^{-1}\left\{T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{-\frac{2\beta}{p}}\right)\right\}^{p/2}\left\{T_s\,(T_{t-s}f)^{\frac{2\beta}{2-p}}\right\}^{1-p/2}.
\end{aligned}$$

Choosing $\beta = p(2-p)/2$ and using Jensen's and Young's inequalities we get for all $\delta > 0$

$$\begin{aligned}
|DT_tf|^p &\le \nu_0^{-1}e^{\sigma_p s}\left\{T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{p-2}\right)\right\}^{p/2}\left\{T_s\,(T_{t-s}f)^p\right\}^{1-p/2}\\
&\le \nu_0^{-1}e^{\sigma_p s}\left\{T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{p-2}\right)\right\}^{p/2}\left\{T_t(f^p)\right\}^{1-p/2}\\
&\le \nu_0^{-1}e^{\sigma_p s}\left\{\frac{p}{2}\delta^{\frac{2}{p}}T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{p-2}\right) + \Big(1-\frac{p}{2}\Big)\,\delta^{\frac{2}{p-2}}T_t(f^p)\right\},
\end{aligned}$$

so that

$$\nu_0 e^{-\sigma_p s}|DT_tf|^p \le \frac{p}{2}\delta^{\frac{2}{p}}T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{p-2}\right) + \Big(1-\frac{p}{2}\Big)\,\delta^{\frac{2}{p-2}}T_t(f^p).$$

Integrating from 0 to $t-\varepsilon$ and using (2.3.7) we get

$$\begin{aligned}
\frac{\nu_0(1-e^{-\sigma_p(t-\varepsilon)})}{\sigma_p}|DT_tf|^p &\le \frac{p}{2}\delta^{\frac{2}{p}}\int_0^{t-\varepsilon} T_s\left(\langle qDT_{t-s}f, DT_{t-s}f\rangle\,(T_{t-s}f)^{p-2}\right)ds\\
&\quad + \Big(1-\frac{p}{2}\Big)\,\delta^{\frac{2}{p-2}}\,(t-\varepsilon)\,T_t(f^p)\\
&= \frac{p}{2}\delta^{\frac{2}{p}}\,\frac{T_{t-\varepsilon}((T_\varepsilon f)^p)-(T_tf)^p}{p(p-1)} + \Big(1-\frac{p}{2}\Big)\,\delta^{\frac{2}{p-2}}\,(t-\varepsilon)\,T_t(f^p)
\end{aligned}$$

and then letting $\varepsilon \to 0$

$$|DT_tf|^p \le \frac{\nu_0^{-1}\sigma_p}{1-e^{-\sigma_p t}} T_t(f^p)\left(\frac{p}{2}\delta^{\frac{2}{p}}\frac{1}{p(p-1)} + \left(1-\frac{p}{2}\right)\delta^{\frac{2}{p-2}}\, t\right).$$

Taking the optimal choice $\delta = \{p(p-1)t\}^{\frac{p(2-p)}{4}}$ we finally obtain

$$|DT_tf|^p \le \frac{\nu_0^{-1}\sigma_p}{[p(p-1)]^{p/2}t^{p/2-1}(1-e^{-\sigma_p t})}\, T_t(f^p). \tag{2.3.8}$$

If $f \in C_b(\overline{\Omega})$ and $f \ge 0$ then (2.3.8) follows by approximating f with $f + \frac{1}{n}$ and using Proposition 2.2.2. If $f \in C_b(\overline{\Omega})$ then

$$\begin{aligned} |DT_tf|^p &= |DT_t(f^+ - f^-)|^p \le 2^{p-1}(|DT_t(f^+)|^p + |DT_t(f^-)|^p) \\ &\le \frac{2^{p-1}\nu_0^{-1}\sigma_p}{[p(p-1)]^{p/2}t^{p/2-1}(1-e^{-\sigma_p t})}\,(T_t((f^+)^p) + T_t((f^-)^p)) \\ &\le \frac{2^{p}\nu_0^{-1}\sigma_p}{[p(p-1)]^{p/2}t^{p/2-1}(1-e^{-\sigma_p t})}\, T_t(|f|^p), \end{aligned}$$

which concludes the proof. □

Remark 2.3.4 If $\Omega = \mathbb{R}^N$, we can consider the case of operators with locally Hölder continuous but not differentiable coefficients. In the case of differentiable coefficients, (2.1.2) and (2.1.3) are consequences of

$$|q_{ij}(x) - q_{ij}(y)| \le M\nu(x)|x-y|, \qquad x, y \in \Omega, \tag{2.3.9}$$

$$(F(x) - F(y))\cdot(x-y) \le (\beta V(x) + k_0)|x-y|^2, \qquad x, y \in \Omega. \tag{2.3.10}$$

Assume that the coefficients q_{ij} and F_i belong to $C^{\alpha}_{\rm loc}(\mathbb{R}^N)$ and satisfy (2.3.9) and (2.3.10), and assume that $V \in C^{1+\alpha}_{\rm loc}(\mathbb{R}^N)$ and it satisfies (2.1.4). If one considers a standard family of mollifiers $(\zeta_\varepsilon)_{\varepsilon>0}$ and define $q^\varepsilon_{ij} = q_{ij} * \zeta_\varepsilon$ and $F^\varepsilon_i = F_i * \zeta_\varepsilon$, then the functions q^ε_{ij} and F^ε_i are regular and satisfy (2.3.9) and (2.3.10) with the same constants q_0, β, k_0 for all $\varepsilon > 0$. Therefore q^ε_{ij} and F^ε_i satisfy (2.1.2) and (2.1.3); if $\mathcal{A}^\varepsilon$ denotes the operator with coefficients q^ε_{ij}, F^ε_i and V, and if P^ε_t denotes the associated semigroup, then P^ε_t satisfies all the gradient estimates that we have proved, with the same constants for all $\varepsilon > 0$. As $\varepsilon \to 0$ we get the gradient estimates for the semigroup P_t associated with the operator with coefficients q_{ij}, F_i and V. Indeed from the interior estimates [30, Theorem IV.10.1] it follows that $P^\varepsilon_t f \to P_t f$ in $C^{1,2}_{\rm loc}((0,\infty)\times\mathbb{R}^N)$.

2.4 Consequences and counterexamples

The aim of this section is to show on one hand some consequences of the gradient estimates proved so far and on the other two counterexamples to some of them.

We start by giving a new formulation of the uniform gradient estimate (2.2.3): now we precise how the constant C_T depends on the operator $\mathcal{A}$. This allows us to deduce a Liouville type theorem.

Corollary 2.4.1 *Suppose that $V \equiv 0$ and (2.3.2) holds. Then for every $f \in C_b(\overline{\Omega})$*

$$\|DP_tf\|_\infty \le \left(\frac{\nu_0^{-1}\sigma_2}{2(1-e^{-\sigma_2 t})}\right)^{\frac{1}{2}} \|f\|_\infty\ , \qquad t > 0\ ,$$

if $\sigma_2 \neq 0$ *and*

$$\|DP_t f\|_\infty \le \left(\frac{1}{2\,\nu_0\,t}\right)^{\frac{1}{2}} \|f\|_\infty\ , \qquad t>0\ ,$$

if $\sigma_2 = 0$.

The proof is an easy consequence of Proposition (2.3.3) with $p = 2$.

Proposition 2.4.2 *Suppose that* $V \equiv 0$*, (2.3.2) holds and* $\sigma_2 = 2k_0 + \frac{1}{2}q_0 \le 0$*. If* $f \in D(\mathcal{A})$ *is such that* $\mathcal{A}f = 0$ *then* f *is constant.*

Proof. Let $f \in D(\mathcal{A})$ and $\mathcal{A}f = 0$. Then $P_t f = f$, for all $t \ge 0$. Applying Corollary 2.4.1 and letting $t \to +\infty$ it turns out that $Df \equiv 0$ and consequently f is constant. □

Now we assume that $(P_t)_{t\ge0}$ extends to a contractive semigroup in $L^1_\mu(\Omega) = L^1(\Omega,\mu)$, for some measure μ. Then, by interpolation, P_t extends to a contractive semigroup in $L^p_\mu(\Omega)$ for all $1 \le p < \infty$.

In this situation, the pointwise gradient estimates of Section 2.3 imply global gradient estimates with respect to the L^p-norm. Moreover, if $(A_p, D(A_p))$ denotes the generator of P_t in $L^p_\mu(\Omega)$, we deduce that $D(A_p)$ embeds continuously in $W^{1,p}_\mu(\Omega)$.

Proposition 2.4.3 *Suppose that* $V \equiv 0$ *and (2.3.2) holds. For all* $f \in L^p_\mu(\Omega)$*, we have* $P_t f \in W^{1,p}_\mu(\Omega)$ *and*

$$\|DP_t f\|_p \le \left(\frac{\nu_0^{-1}\sigma_2}{2(1-e^{-\sigma_2 t})}\right)^{\frac{1}{2}} \|f\|_p\ , \qquad t>0,\ p\ge2 \tag{2.4.1}$$

$$\|DP_t f\|_p \le t^{\frac{1}{p}-\frac{1}{2}} \left(\frac{c_p\,\nu_0^{-1}\sigma_p}{1-e^{-\sigma_p t}}\right)^{\frac{1}{p}} \|f\|_p\ , \qquad t>0,\ 1<p<2. \tag{2.4.2}$$

In the case where $\sigma_p = 0$*,* $\sigma_p/(1-e^{-\sigma_p t})$ *is replaced by* $1/t$*.*

Proof. Fix $p \ge 2$. If $f \in C_b(\overline{\Omega}) \cap L^p_\mu(\Omega)$ integrating (2.3.5) it follows that $P_t f \in W^{1,p}_\mu(\Omega)$ and it satisfies (2.4.1). If $f \in L^p_\mu(\Omega)$, take a sequence $(f_n) \subset C_b(\overline{\Omega}) \cap L^p_\mu(\Omega)$ that converges to f in $L^p_\mu(\Omega)$. Writing (2.4.1) for $f_n - f_m$ it follows that $P_t f_n$ is a Cauchy sequence in $W^{1,p}_\mu(\Omega)$. Therefore $P_t f \in W^{1,p}_\mu(\Omega)$ and it satisfies (2.4.1). The case $1 < p < 2$ follows similarly from (2.3.6). □

Corollary 2.4.4 *Suppose that* $V \equiv 0$*. For all* $p > 1$ *and* $\omega > 0$ *there exists* $C = C(p,\omega) > 0$ *such that*

$$\|DP_t f\|_p \le C\,\frac{e^{\omega t}}{\sqrt{t}}\ \|f\|_p\ , \qquad t>0, \tag{2.4.3}$$

for every $f \in L^p_\mu$*. Consequently,* $D(A_p) \subset W^{1,p}_\mu(\Omega)$ *and for all* $\omega > 0$ *there exists* $M_\omega > 0$ *such that*

$$\|Du\|_p \le M_\omega \|u\|_p^{\frac{1}{2}}\ \|(A_p-\omega)u\|_p^{\frac{1}{2}} \tag{2.4.4}$$

for all $u \in D(A_p)$*.*

Proof. Fix $T > 0$. From Proposition 2.4.3 it follows that $\|DP_t f\|_p \le C_T t^{-1/2}\|f\|_p$ for every $t \in]0, T[$ and $f \in L^p_\mu(\Omega)$ for some constant $C_T > 0$. Therefore arguing as in Corollary 2.2.3 we get (2.4.3).

For the second statement, fix $\omega, \lambda > 0$. Let $f \in C_b(\overline{\Omega}) \cap L^p_\mu(\Omega)$ and set $u = R(\lambda + \omega, \mathcal{A})f$. Then

$$Du(x) = \int_0^{+\infty} e^{-(\lambda+\omega)t}(DP_t f)(x)dt, \qquad x \in \overline{\Omega}.$$

As in Proposition 2.2.5, with estimate (2.2.10) replaced by (2.4.3), we deduce that

$$\|Du\|_p \le M_\omega \|u\|_p^{\frac{1}{2}} \, \|(A_p - \omega)u\|_p^{\frac{1}{2}}.$$

Since $C_b(\overline{\Omega}) \cap L^p_\mu(\Omega)$ is dense in $L^p_\mu(\Omega)$, $R(\lambda, \mathcal{A})(C_b(\overline{\Omega}) \cap L^p_\mu(\Omega))$ is a core for $(A_p, D(A_p))$. Thus, the general case $u \in D(A_p)$ easily follows from the previous step by approximation. □

Remark 2.4.5 We note that, in particular, one may take as μ the invariant measure of P_t (when it exists), which is, by definition, a Borel probability measure such that

$$\int_\Omega P_t f d\mu = \int_\Omega f d\mu,$$

for all $t \ge 0$ and $f \in C_b(\overline{\Omega})$ (we refer to Chapter 5 for more details concerning invariant measures). In this case estimate (2.0.6) and (2.0.8) have interesting consequences. (2.0.6) with $p = 1$ and $k_0 < 0$ yields the hypercontractivity of (P_t) in $L^2(\Omega, \mu)$, which means that for every $f \in L^2(\Omega, \mu)$ one has

$$\|P_t f\|_{L^{q(t)}(\Omega,\mu)} \le \|f\|_{L^2(\Omega,\mu)}, \tag{2.4.5}$$

where $q(t) = 1 + e^{\lambda t}$ for a suitable $\lambda > 0$. One can check that (2.4.5) is equivalent to the logarithmic Sobolev inequality

$$\int_\Omega |f|^2 \log |f| d\mu \le \|f\|^2_{L^2(\Omega,\mu)} \log \|f\|_{L^2(\Omega,\mu)} + \frac{2}{\lambda}\int_\Omega |Df|^2 d\mu,$$

for every $f \in W^{1,2}(\Omega, \mu)$.

(2.0.8) with $p = 2$ and $\sigma_2 < 0$ yields the Poincaré inequality in $W^{1,2}(\Omega, \mu)$

$$\int_\Omega |f - \overline{f}\,|^2 d\mu \le C \int_\Omega |Df|^2 d\mu, \tag{2.4.6}$$

where $\overline{f} = \int_\Omega f d\mu$. As a consequence, one obtains the spectral gap for the generator A_2 of (P_t) in $L^2(\Omega, \mu)$, which means that

$$\sigma(A_2) \setminus \{0\} \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \le -1/C\}$$

where C is determined by (2.4.6).

We do not enter in the details of such consequences, but we limit ourselves to mention them. We refer to [20, Section 10.5].

Example 2.4.6 This example shows that Proposition 2.3.3 fails in general for $p = 1$. Consider the heat semigroup in $\mathbb{R}$

$$P_t f(x) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)dy, \qquad t > 0, \ x \in \mathbb{R}$$

generated by the operator $\mathcal{A}u(x) = u''(x)$. The derivative is given by

$$DP_t f(x) = \frac{1}{2t(4\pi t)^{1/2}} \int_{\mathbb{R}} (y-x) e^{-\frac{(x-y)^2}{4t}} f(y)dy, \qquad t>0,\ x\in\mathbb{R}.$$

Fix $R > 0$. Let $f \in C_b(\mathbb{R})$ be such that $0 \le f \le 1$, $f(x) = 0$ for $x < R - R^{-1}$ and $f(x) = 1$ for $x > R$. Then

$$P_t f(0) \le \frac{1}{(4\pi t)^{1/2}} \int_{R-R^{-1}}^{\infty} e^{-\frac{|y|^2}{4t}} dy, \qquad DP_t f(0) \ge \frac{1}{2t(4\pi t)^{1/2}} \int_{R}^{\infty} y e^{-\frac{|y|^2}{4t}} dy.$$

Therefore

$$DP_t f(0) \ge c_R P_t f(0), \qquad c_R = \frac{1}{2t} \int_R^\infty y e^{-\frac{|y|^2}{4t}} dy \left(\int_{R-R^{-1}}^\infty e^{-\frac{|y|^2}{4t}} dy \right)^{-1}.$$

Using the De L'Hôspital rule, it is readily seen that $c_R \to +\infty$ as $R \to +\infty$. This means that no pointwise estimate similar to (2.3.5) can hold for $p = 1$.

With the next counterexample we show that gradient estimate (2.2.3) is not true in general without assuming the dissipativity condition (2.1.3). In particular we show an example in which $D(\mathcal{A})$ is not contained in $C^1_\eta(\overline{\Omega})$.

Example 2.4.7 Consider in $\Omega = \mathbb{R}$ the operator

$$\mathcal{A}u(x) = u''(x) + B'(x)u'(x) = e^{-B(x)} \left(e^{B(x)} u'(x) \right)', \qquad x \in \mathbb{R},$$

where $B \in C^2(\mathbb{R})$ is such that $Q(x) = e^{B(x)} \int_0^x e^{-B(t)}dt \in L^1(\mathbb{R})$. Then, in particular $e^B \in L^1(\mathbb{R})$. Let $D(\mathcal{A}) = \{u \in C^2(\mathbb{R}) \cap C_b(\mathbb{R}) \,:\, \mathcal{A}u \in C_b(\mathbb{R})\}$. It follows from [55, page 242] (see also [40, Proposition 2.1]) that $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a semigroup in $C_b(\mathbb{R})$ having $e^{B(x)}dx$ as its invariant measure.

If $f \in C_b(\mathbb{R})$, then the function

$$u(x) = C_1 + \int_0^x e^{-B(t)} \left(C_2 + \int_0^t f(s) e^{B(s)} ds \right) dt, \tag{2.4.7}$$

for arbitrary $C_1, C_2 \in \mathbb{R}$, is the general solution of the equation $\mathcal{A}u = f$. Assuming that

$$\int_{-\infty}^{+\infty} f(t) e^{B(t)} dt = 0, \tag{2.4.8}$$

and setting

$$C_2 = -\int_0^{+\infty} f(t)e^{B(t)}dt = \int_{-\infty}^0 f(t) e^{B(t)} dt,$$

for $x > 0$ (2.4.7) gives

$$\begin{aligned} u(x) &= C_1 - \int_0^x e^{-B(t)} \int_t^{+\infty} f(s) e^{B(s)} ds\, dt \\ &= C_1 - \int_0^{+\infty} e^{B(s)} f(s) \int_0^{s\wedge x} e^{-B(t)} dt\, ds. \end{aligned}$$

It follows that

$$|u(x)| \le |C_1| + \|f\|_\infty \int_0^{+\infty} Q(s) ds, \qquad x > 0,$$

which implies that u is bounded at $+\infty$. Similarly, since $Q \in L^1(]-\infty,0[)$, u is bounded at $-\infty$. Since $\mathcal{A}u = f$, we conclude that $u \in D(\mathcal{A})$. The derivative of u is given by

$$u'(x) = -e^{-B(x)} \int_x^{+\infty} f(s)e^{B(s)}ds, \qquad x \in \mathbb{R}.$$

We claim that we can choose the functions B and f so that $Q \in L^1(\mathbb{R})$, (2.4.8) holds but u' is not bounded. To this aim, take

$$B(x) = -x^4 + \log h(x),$$

where $h \in C^2(\mathbb{R})$ satisfies

$$\begin{cases} h(x) = \varepsilon_n & \text{if } \ x = n - \dfrac{\delta_n}{2}, \ n \in \mathbb{N}, \\ \varepsilon_n \le h(x) \le 1 & \text{if } \ n - \delta_n < x < n, \ n \in \mathbb{N}, \\ h(x) = 1 & \text{otherwise}, \end{cases}$$

with

$$\varepsilon_n = \frac{1}{n}\, e^{\left(n - \frac{1}{2}\right)^4 - \left(n + \frac{1}{2}\right)^4}, \qquad \delta_n = \frac{e^{-n^4}}{n^2}\varepsilon_n\,.$$

As a consequence of this choice

$$Q(x) = e^{-x^4}\int_0^x e^{t^4}dt, \quad x<0, \qquad Q(x) = h(x)e^{-x^4}\int_0^x \frac{e^{t^4}}{h(t)}dt, \quad x>0.$$

Using the De L'Hôspital rule one sees that $\lim_{x\to-\infty} x^3 Q(x) = 1/4$ and hence that $Q \in L^1(]-\infty,0[)$. If $x>0$ then

$$\begin{aligned} Q(x) \quad &\le \quad e^{-x^4}\int_0^x \frac{e^{t^4}}{h(t)}dt \le e^{-x^4}\int_0^x e^{t^4}dt + e^{-x^4}\sum_{n=1}^{[x]+1}\int_{n-\delta_n}^n \frac{e^{n^4}}{\varepsilon_n}dt \\ &\le \quad e^{-x^4}\int_0^x e^{t^4}dt + e^{-x^4}\sum_{n=1}^{\infty}\frac{\delta_n e^{n^4}}{\varepsilon_n} = e^{-x^4}\int_0^x e^{t^4}dt + e^{-x^4}\sum_{n=1}^{\infty}\frac{1}{n^2}, \end{aligned}$$

which shows that $Q \in L^1(]0,+\infty)$. Let $f \in C_b(\mathbb{R})$ be such that $f(x)=1$ for all $x>0$ and (2.4.8) holds. Then

$$u'(x) = -\frac{e^{x^4}}{h(x)}\int_x^\infty h(t)e^{-t^4}dt, \qquad x>0$$

and in particular, at $x_n = n - \delta_n/2$

$$\begin{aligned} |u'(x_n)| \quad &= \quad \frac{e^{{x_n}^4}}{\varepsilon_n}\int_{x_n}^{+\infty} h(t)e^{-t^4}dt \ge \frac{e^{\left(n-\frac{1}{2}\right)^4}}{\varepsilon_n}\int_n^{n+\frac{1}{2}} e^{-t^4}dt \\ &\ge \quad \frac{e^{\left(n-\frac{1}{2}\right)^4}}{2\varepsilon_n}\, e^{-\left(n+\frac{1}{2}\right)^4} = \frac{n}{2}, \end{aligned}$$

which implies that $u'(x)$ is unbounded at $+\infty$.

Therefore we have shown that the function u belongs to $D(\mathcal{A})$ but not to $C_b^1(\mathbb{R})$. This means that the gradient estimate (2.2.3) cannot be true. We note that in this situation the dissipativity assumption (2.1.3) fails since B'' is unbounded from above.

Example 2.4.8 We see now an example of a Neumann problem in a domain Ω with Lipschitz continuous boundary. In spite of the lower regularity of $\partial\Omega$, the associated semigroup satisfies the gradient estimate (2.3.1). Consider the Ornstein-Uhlenbeck operator

$$\mathcal{A}u(x) = \frac{1}{2}\Delta u(x) - x \cdot Du(x), \qquad x \in \mathbb{R}^N.$$

If we set

$$\begin{aligned} N(m,\sigma^2)(y) &= \frac{1}{\left(\sqrt{2\pi}\,\sigma\right)^N} e^{-\frac{|y-m|^2}{2\sigma^2}}, \qquad \sigma > 0,\ m, y \in \mathbb{R}^N, \\ \Gamma(t,x,y) &= N(e^{-t}x, 1-e^{-2t})(y), \quad t > 0,\ x, y \in \mathbb{R}^N, \end{aligned}$$

then the Ornstein-Uhlenbeck semigroup in $C_b(\mathbb{R}^N)$ is given by the formula

$$(U_t f)(x) = \int_{\mathbb{R}^N} f(y)\Gamma(t,x,y)dy, \quad t > 0,\ x \in \mathbb{R}^N.$$

We fix $k \in \mathbb{N}$, $0 \le k < N$ and we consider the domain $\Omega = \{x \in \mathbb{R}^N : x_{k+1}, ..., x_N > 0\}$. We define now the Ornstein-Uhlenbeck operator in Ω with Neumann boundary conditions. For $k+1 \le j \le N$ consider the reflections

$$\theta_j : \mathbb{R}^N \to \mathbb{R}^N, \qquad \theta_j x = (x_1, ..., x_{j-1}, -x_j, x_{j+1}, ..., x_N), \qquad x \in \mathbb{R}^N,$$

and the family

$$\Lambda = \{\theta = \theta_{i_1} \circ \cdots \circ \theta_{i_n},\ k+1 \le i_j \le N,\ i_j < i_h \text{ if } j < h,\ 1 \le n \le N-k\}.$$

Moreover if $f \in C_b(\overline{\Omega})$ we define the extension $Ef \in C_b(\mathbb{R}^N)$ by

$$(Ef)(x) = f(x_1, ..., x_k, |x_{k+1}|, ..., |x_N|), \quad x \in \mathbb{R}^N.$$

The Ornstein-Uhlenbeck semigroup in Ω is given by the formula

$$(P_t f)(x) = (U_t Ef\,)(x) = \int_{\mathbb{R}^N} (Ef)(y)\Gamma(t,x,y)dy, \qquad t > 0,\ x \in \Omega.$$

With the changes of variable $y' = \theta y$ and using the identity $\Gamma(t,x,\theta y) = \Gamma(t,\theta x,y)$ for all $\theta \in \Lambda$, we get

$$\begin{aligned} (P_t f)(x) &= \int_\Omega f(y)\Big\{\Gamma(t,x,y) + \sum_{\theta\in\Lambda}\Gamma(t,x,\theta y)\Big\}dy \\ &= \int_\Omega f(y)\Big\{\Gamma(t,x,y) + \sum_{\theta\in\Lambda}\Gamma(t,\theta x,y)\Big\}dy \qquad (2.4.9) \end{aligned}$$

The Neumann boundary condition can be verified in the following way. Let $x \in \partial\Omega$ be such that $x_j = 0$ for some $j \in \{k+1, ..., N\}$ and $x_i \ne 0$ for all $i \in \{k+1, ..., N\}$, $i \ne j$. Then the outward unit normal vector is $\eta(x) = -e_j$. For all $\theta \in \Lambda$ the normal derivative of the function $\Gamma(t,\theta x,y)$ is

$$\frac{\partial}{\partial x_j}\Gamma(t,\theta x,y) = \frac{(\pm y_j - e^{-t}x_j)e^{-t}}{(1-e^{-2t})}\Gamma(t,\theta x,y), \quad t > 0,\ x, y \in \Omega,$$

where in the right hand side we have the sign $+$ if θ does not contain the reflection θ_j and the sign $-$ otherwise. Let now $\theta \in \Lambda$ such that it does not contain the reflection θ_j and let $\theta' = \theta_j \circ \theta \in \Lambda$; then if $x_j = 0$ we have $\theta x = \theta' x$ and

$$\frac{\partial}{\partial x_j}\Gamma(t,\theta x,y) + \frac{\partial}{\partial x_j}\Gamma(t,\theta' x,y) = \frac{y_j}{(1-e^{-2t})}\Gamma(t,\theta x,y) - \frac{y_j}{(1-e^{-2t})}\Gamma(t,\theta' x,y) = 0,$$

for all $t > 0$ and $y \in \Omega$. Thus the Neumann boundary condition for $P_t f$ follows coupling in the sum in formula (2.4.9) all the maps $\theta \in \Lambda$ that does not contain the reflection θ_j with the respective maps $\theta' = \theta_j \circ \theta$. In this way all the terms of the sum are considered and the normal derivative turns out to be zero.

Since $DU_t Ef(x) = e^{-t} U_t(DEf)(x)$ for all $x \in \mathbb{R}^N$, we have

$$|DP_t f(x)| \leq e^{-t} U_t(|DEf|)(x) = e^{-t} P_t(|Df|)(x), \qquad t \geq 0, \; x \in \overline{\Omega},$$

that is P_t satisfies the gradient estimate (2.3.1) for $p = 1$ and hence for all $p \geq 1$.

Chapter 3

Gradient estimates in Dirichlet parabolic problems in regular domains

The aim of the present chapter is to prove global gradient estimates for the bounded classical solution u to the following Dirichlet parabolic problem

$$
\begin{cases} u_t(t,x) - Au(t,x) = 0 & t \in (0,T), \ x \in \Omega, \\ u(t,\xi) = 0 & t \in (0,T), \ \xi \in \partial\Omega, \\ u(0,x) = f(x) & x \in \Omega, \end{cases} \tag{3.0.1}
$$

where Ω is an unbounded smooth connected open set in $\mathbb{R}^N$, f a continuous and bounded function in Ω and A a second order elliptic operator, with (possibly) unbounded regular coefficients, i.e.,

$$
A = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i,j=1}^{N} F_i D_i - V = \mathrm{Tr}(qD^2) + \langle F, D \rangle - V. \tag{3.0.2}
$$

More precisely, we determine conditions on the coefficients of A yielding the following estimate

$$
\|Du(t,\cdot)\|_\infty \leq \frac{C}{\sqrt{t}} \|f\|_\infty, \ \ t \in (0,T). \tag{3.0.3}
$$

In Chapter 2 we have already studied gradient estimates for parabolic problems with Neumann boundary conditions. The main tool was Bernstein's method, which consists in applying the maximum principle to the function $u_n^2 + at|Du_n|^2$, where (u_n) approximates the solution. The crucial point was that the convexity assumption on Ω ensured the boundary condition $\frac{\partial |Du_n|^2}{\partial \eta} \leq 0$. Here, we cannot proceed exactly in the same way, since for a given function v satisfying $v = 0$ on $\partial\Omega$, it is not possible to establish *a priori* the sign of $|Dv|^2$ on $\partial\Omega$. Hence, after having proved existence and uniqueness of bounded classical solutions u to (3.0.1) (Section 3.2), our first aim is to obtain boundary estimates for Du. This is done by comparison with certain one dimensional operators, which arise by introducing the distance function from the boundary. Then, using Bernstein's method, one shows that the boundary estimates can be extended to the whole Ω (Section 3.3). However, the method works (and gives (3.0.3) with the right dependence of all constants involved), if one already knows that Du is bounded up to the boundary of Ω for positive t, see Proposition 3.3.3. To circumvent this difficulty, we subtract to the operator A a potential εW, where W is big enough to dominate the growth of F and, following ideas in [11], [12], [41], we show that the perturbed operator $A_\varepsilon = A - \varepsilon W$ generates an analytic semigroup in $L^p(\Omega)$ and characterize its domain. Choosing a large p and using Sobolev embedding, it follows that the bounded classical solution u_ε of problem (3.0.1) with A_ε instead of A and a smooth f has a bounded gradient in $[0,T) \times \Omega$. Therefore Proposition 3.3.3 applies and gives (3.0.3) for u_ε with a constant C independent of ε. An approximation argument then completes the proof. This program is carried out in Sections 3.4 and 3.5. In Section 3.6 we present a counterexample.

3.1 Assumptions and main result

Let us collect our hypotheses on Ω and the coefficients of A.

Hypothesis 1.1

(i) Ω is a connected open subset of $\mathbb{R}^N$ with uniformly $C^{2+\alpha}$-boundary for some $0 < \alpha < 1$, see Appendix B.

(ii) $q_{ij}, F_i, V \in C^{1+\alpha}(\Omega \cap B_R)$ for every $i, j = 1, \ldots, N$ and $R > 0$; moreover $V \geq 0$ in Ω.

(iii) $q_{ij} = q_{ji} \in C_b^1(\Omega)$, and there exists $\nu_0 > 0$ such that $\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \geq \nu_0|\xi|^2$, for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.

(iv) There exist a positive function $\varphi \in C^2(\overline{\Omega})$ and $\lambda_0 > 0$ such that

$$\lim_{|x|\to+\infty,\, x\in\overline{\Omega}} \varphi(x) = +\infty, \qquad A\varphi - \lambda_0\varphi \leq 0.$$

The *Lyapunov* map φ introduced in assumption (iv) ensures that maximum principles hold, see Appendix A. Moreover condition (i) ensures that the *distance function*

$$r(x) = \mathrm{dist}(x, \partial\Omega), \quad x \in \overline{\Omega} \tag{3.1.1}$$

is a C^2-function with bounded second order derivatives in Ω_δ, for some $\delta > 0$, where we set

$$\Omega_\delta = \{x \in \overline{\Omega} : \mathrm{dist}(x, \partial\Omega) < \delta\},$$

see [26, Lemma 14.16] and also Appendix B (note that (i) implies that the principal curvatures of $\partial\Omega$, when $\partial\Omega$ is considered as an hypersurface, are bounded). Our main result will be proved assuming also the conditions listed below.

$$\sum_{i,j=1}^N D_iF_j(x)\,\xi_i\,\xi_j \leq (sV(x) + k)\,|\xi|^2, \quad x \in \Omega,\ \xi \in \mathbb{R}^N, \tag{3.1.2}$$

$$\sum_{i,j=1}^N q_{ij}(x)D_{ij}r(x) + \sum_{i=1}^N F_i(x)\,D_ir(x) \ \leq M, \quad x \in \Omega_\delta \ \text{(for some } \delta > 0), \tag{3.1.3}$$

$$|DV(x)| \leq \beta(1 + V(x)), \quad x \in \Omega, \tag{3.1.4}$$

$$|F(x)| \leq c_1e^{c_2|x|}, \quad x \in \Omega, \tag{3.1.5}$$

for some constants $k, M, \beta, c_1, c_2 \in \mathbb{R}$, $s < 1/2$.

Observe that, since $q_{ij} \in C_b^1(\Omega)$ and Ω is uniformly C^2, (3.1.3) is only a condition on the component of F along the inner normal to $\partial\Omega$ in a neighborhood of $\partial\Omega$.

Let us explain our main assumptions in the particular case where $A = \Delta + \langle F, D\rangle$. The dissipativity condition on F (3.1.2) is quite natural since a one-dimensional counterexample to gradient estimates has been constructed in Example 2.4.7 when it fails. Observe also that, if $F = D\Phi$, then (3.1.2) is a concavity assumption on Φ.

Condition (3.1.3) means that the component of the drift F along the inner normal is bounded from above in a neighborhood of $\partial\Omega$. Even though its connection with gradient estimates is not evident from an analytic point of view, its necessity is clear if one considers the Markov process governed by the operator A under Dirichlet boundary conditions. In fact the solution $u(t,x)$ to (3.0.1) corresponding to $f = \mathbf{1}$ represents the probability that the process starting from $x \in \Omega$ at time $t = 0$ is not absorbed by the boundary up to time t. If the (inner) normal component of F is unbounded from above in a neighborhood of $\partial\Omega$, one expects that $u(t,x) \to 1$ as $|x| \to \infty$ along the boundary. Since $u(t,\xi) = 0$ for $\xi \in \partial\Omega$, it follows that $u(t,\cdot)$ is even not uniformly continuous, see Example 3.6.1 where this heuristic argument is made rigorous.

Finally, we point out that the growth assumption (3.1.5), even though not very restrictive, seems to be a technical one in order to use our methods, see the proof of Theorem 3.1.2.

We stress the fact that we use mainly analytic tools and we do not need any convexity assumption on Ω. Moreover we note that our operator A may contain a potential term V which is difficult to treat by probabilistic methods.

Remark 3.1.1 Observe that assumption (iv) of Hypothesis 1.1 follows from the positivity of V and the boundedness of q_{ij}, when condition (3.1.2) holds with $s = 0$. In fact (3.1.2) implies, by differentiating the function $t \to \langle F(tx), x\rangle$, that $\langle F(x), x\rangle \le \langle F(0), x\rangle + k|x|^2$, hence the function $\varphi(x) = 1 + |x|^2$ satisfies (iv), for a suitable λ_0.

To specify the dependence of some constants we also introduce the quantity

$$h = \sup_{x\in\Omega}\Big(\sum_{i,j=1}^{N} |Dq_{ij}(x)|^2\Big)^{1/2}$$

which is finite, since $q_{ij} \in C_b^1(\Omega)$.

We will prove the following theorem.

Theorem 3.1.2 *There exists a constant C depending on $\nu_0, k, s, h, N, M, \beta, \delta, T$ such that the bounded classical solution u of (3.0.1) satisfies*

$$\|Du(t,\cdot)\|_\infty \le \frac{C}{\sqrt{t}}\|f\|_\infty, \quad t \in (0,T), \quad f \in C_b(\Omega).$$

3.2 Existence and uniqueness

In this section we show that (3.0.1) has a unique bounded classical solution, where by *bounded classical solution* of (3.0.1) we mean a function $u \in C^{1,2}(Q)$, such that u is continuous in $\overline{Q}\setminus\partial_{tx}Q$, bounded in Q and solves (3.0.1). To this purpose we use both classical Schauder estimates and a nonstandard maximum principle for discontinuous solutions to (3.0.1), see Theorem A.0.13.

Proposition 3.2.1 *Assume Hypothesis 1.1. If $f \in C^{2+\alpha}(\Omega)$ has compact support in Ω, then problem (3.0.1) has a unique bounded solution u which belongs to $C^{1+\alpha/2,2+\alpha}((0,T)\times(\Omega\cap B_R))$ for every $R > 0$. Moreover, $\|u\|_\infty \le \|f\|_\infty$ and $u \ge 0$ if $f \ge 0$. Finally, Du belongs to $C^{1+\alpha/2,2+\alpha}((\varepsilon,T)\times\Omega'))$ for every $\varepsilon > 0$ and Ω' open bounded set with* $\mathrm{dist}\,(\Omega', \mathbb{R}^N \setminus \Omega) > 0$. *In particular, $Du \in C^{1,2}(Q)$.*

Proof. Uniqueness is immediate consequence of a classical maximum principle, see Proposition A.0.12.

To prove the existence part, we consider a sequence of uniformly elliptic operators with coefficients in $C^\alpha(\Omega)$,

$$A^n = \sum_{i,j=1}^N q_{ij} D_{ij} + \sum_{i=1}^N F_i^n D_i - V^n u,$$

such that $F_i^n = F_i$, $V^n = V$ in $\Omega \cap B_n$, $V^n \geq 0$ and let $u_n \in C^{1+\alpha/2,2+\alpha}(Q)$ be the solution of (3.0.1), with A^n instead of A (see e.g. [30, Theorem IV.5.2]). The classical maximum principle yields $\|u_n\|_\infty \leq \|f\|_\infty$. Let us fix $R > 0$ and observe that, since Ω is unbounded and connected, $\mathrm{dist}(\Omega \setminus B_{R+1}, \Omega \cap B_R) > 0$. Since $A^n = A^m = A$ in $\Omega \cap B_{R+1}$ for $n, m > R+1$, by the local Schauder estimates [30, Theorem IV.10.1], there exists a constant C such that

$$\|u_n - u_m\|_{C^{1+\alpha/2,2+\alpha}((0,T)\times(\Omega\cap B_R))} \leq C\|u_n - u_m\|_{C((0,T)\times(\Omega\cap B_{R+1}))} \leq 2C\|f\|_\infty.$$

Therefore (u_n) is relatively compact in $C^{1,2}([0,T] \times (\overline{\Omega \cap B_R}))$. Considering an increasing sequence of balls and using a diagonal procedure we can extract a subsequence (u_{n_k}) convergent to a function $u \in C^{1+\alpha/2,2+\alpha}((0,T) \times (\Omega \cap B_R))$ for every $R > 0$ which solves (3.0.1) and satisfies $\|u\|_\infty \leq \|f\|_\infty$. By the maximum principle, $u \geq 0$, whenever $f \geq 0$.

In order to prove the last part of the statement it is sufficient to apply [29, Theorem 8.12.1] directly to the operator $D_t - A$. □

We now introduce linear operators $(P_t)_{t\geq 0}$ via the formula $(P_t f)(x) = u(t,x)$ for $f \in C^{2+\alpha}(\Omega)$, with compact support in Ω, where u is the solution of (3.0.1) given by the above proposition. Each operator P_t is positive and contractive with respect to the sup-norm, by the above proposition.

Now we consider the case where f is only continuous and bounded in Ω and extend the above maps $(P_t)_{t\geq 0}$ to a semigroup in $C_b(\Omega)$.

Proposition 3.2.2 *Assume Hypothesis 1.1. If f belongs to $C_b(\Omega)$, then problem (3.0.1) has a unique bounded classical solution u. Moreover, $u(t,x) \to f(x)$ as $t \to 0$, uniformly on compact sets of Ω.*

Proof. Uniqueness is an immediate consequence of a nonstandard maximum principle, see Theorem A.0.13. To show existence, we consider a sequence $(f_n) \in C_0^\infty(\Omega)$ convergent to f uniformly on compact subsets of Ω and such that $\|f_n\|_\infty \leq \|f\|_\infty$. Let $u_n \in C^{1+\alpha/2,2+\alpha}((0,T) \times (\Omega \cap B_R))$, for every $R > 0$, be the solution of (3.0.1) with f_n instead of f, given by the previous proposition. Let us fix $\varepsilon > 0$. By the Schauder estimates [30, Theorem IV.10.1], as in the proof of Proposition 3.2.1, we get a constant C such that

$$\|u_n - u_m\|_{C^{1+\alpha/2,2+\alpha}((\varepsilon,T)\times(\Omega\cap B_R))} \leq C\|u_n - u_m\|_{C((0,T)\times(\Omega\cap B_{R+1}))} \leq 2C\|f\|_\infty$$

and then, by a compactness argument, we can extract a subsequence (u_{n_k}) convergent to a function $u \in C^{1+\alpha/2,2+\alpha}((\varepsilon,T)\times(\Omega\cap B_R))$ for every $\varepsilon, R > 0$ which solves the equation $u_t - Au = 0$ in Q and such that $u(t,x) = 0$ for $t \in (0,T), x \in \partial\Omega$. In the following, we write $u = P_t f$, for $f \in C_b(\Omega)$.

It remains to show that $u(t,x) \to f(x)$ as $t \to 0$, uniformly on compact sets of Ω.

Assume first that $f \in C_0(\Omega)$, i.e. f vanishes on $\partial\Omega$ and at infinity. Then we can choose (f_n) as above in such a way that $\|f_n - f\|_\infty \to 0$. The maximum principle implies that (u_n) is a Cauchy sequence in $C([0,T] \times \overline{\Omega})$, hence $u_n \to u$ uniformly in $\overline{Q}$ and $u(0,x) = f(x)$ for every $x \in \overline{\Omega}$.

Let $K \subset \Omega$ be a compact set and $\eta \in C_0(\Omega)$, $0 \leq \eta \leq 1$, be such that $\eta = 1$ in K. Then $P_t\eta \to \eta$ as $t \to 0$, uniformly in Ω, hence $P_t\eta \to 1$ uniformly in K and, since $0 \leq P_t(1-\eta) \leq 1 - P_t\eta$, we get $P_t(1-\eta) \to 0$ uniformly in K. For $f \in C_b(\Omega)$, writing $P_t f = P_t(\eta f) + P_t((1-\eta)f)$ and observing that $P_t(\eta f) \to \eta f$ uniformly in Ω and that $P_t((1-\eta)f) \to 0$ uniformly in K we obtain that $P_t f \to f$, uniformly in K. □

Corollary 3.2.3 *The family $(P_t)_{t\geq 0}$ is a semigroup in $C_b(\Omega)$.*

PROOF. The semigroup law $P_{t+s} = P_t P_s$ is immediate consequence of the uniqueness statement in Proposition 3.2.2. □

Observe that the semigroup $(P_t)_{t\geq 0}$ is not strongly continuous. In fact $P_t f \to f$ as $t \to 0$, only uniformly on compact subsets of Ω. However, $P_t f \to f$ uniformly in Ω for every $f \in C_0(\Omega)$.

3.3 Some a-priori estimates

In the following proposition we prove a preliminary boundary gradient estimate for bounded solutions of problem (3.0.1). We need the following lemma on gradient estimates for certain one-dimensional operators.

Lemma 3.3.1 *Let $\delta > 0$ and $g : [0, +\infty) \times [0, \delta] \to \mathbb{R}$ be the solution to*

$$\begin{cases} g_t(t,r) = \nu_0 g_{rr}(t,r) + M g_r(t,r), & t > 0,\ r \in (0,\delta), \\ g(t,0) = 0,\ \ g(t,\delta) = 1 & t > 0, \\ g(0,r) = 1 & r \in (0,\delta). \end{cases} \tag{3.3.1}$$

Then $g_r \geq 0$, $g_{rr} \leq 0$ and for any $T > 0$ there exists $c_T > 0$ such that

$$0 \leq g(t,r) \leq \frac{c_T}{\sqrt{t}} r,\ \ 0 < t \leq T,\ r \in (0,\delta).$$

PROOF. We define the operator $(B, D(B))$ in $C([0,\delta])$ by

$$Bu = \nu_0 u'' + M u' \qquad D(B) = \{u \in C^2([0,\delta]) : u(0) = 0, (Bu)(\delta) = 0\}.$$

Let us show that $(B, D(B))$ generates an analytic semigroup S_t of positive contractions in $C([0,\delta])$ (note that S_t is not strongly continuous since the domain $D(B)$ is not dense in $C([0,\delta])$.

Let $D = \{u \in C^2([0,\delta]) : u(0) = u(\delta) = 0\}$. Then (B, D) generates an analytic semigroup $(T_t)_{t\geq 0}$ in $C([0,\delta])$. Set $\psi(r) = a\int_0^r e^{-M\,s/\nu_0} ds$. Then $B\psi = 0$, $\psi(0) = 0$ and $\psi(\delta) = 1$, if a is suitably chosen. It is easily seen that $S_t f = T_t(f - f(\delta)\psi) + f(\delta)\psi$ is the analytic semigroup generated by $(B, D(B))$ in $C([0,\delta])$. Since the regularity properties of $S_t f$ coincide with those of $T_t f$, it follows that $u(t,r) = S_t f(r)$ is a C^∞ function for $t > 0$, continuous at the points $(0,r)$, with $0 < r < \delta$. The maximum principle, see Theorem A.0.13, now yields positivity and contractivity of S_t.

We can prove the stated properties of g. Since $g = S_t 1$ we have $0 \leq g \leq 1$. Moreover $g(t+s,\cdot) = S_{t+s}1 = S_t S_s 1 \leq S_t 1 = g(t,\cdot)$, hence g is decreasing with respect to t and $g_t \leq 0$. To prove that $g_r \geq 0$ we write

$$g_t = \nu_0\Big(g_{rr} + \frac{M}{\nu_0} g_r\Big) = \nu_0 e^{-\frac{M}{\nu_0}r} \frac{d}{dr}\Big(e^{\frac{M}{\nu_0}r} g_r\Big) \leq 0,$$

$r \in (0,\delta)$. Then $e^{\frac{M}{\nu_0}r} g_r$ is decreasing. Since $g(t,\delta) = 1$ and $0 \leq g \leq 1$, we have $g_r(t,\delta) \geq 0$, hence $g_r \geq 0$. Now the identity $g_t = \nu_0 g_{rr} + M g_r$ yields $g_{rr} \leq 0$.

Since $(S_t)_{t\geq 0}$ is analytic, for $0 < t \leq T$ we have $\|D^2 g(t,\cdot)\| \leq c_T t^{-1}$, hence $\|Dg(t,\cdot)\| \leq c_T t^{-1/2}$ and the inequality $g(t,r) \leq c_T t^{-1/2} r$ follows, since $g(t,0) = 0$. □

Proposition 3.3.2 *Assume Hypothesis 1.1 and (3.1.3). Then there exists γ depending on ν_0, M, δ, T such that every bounded classical solution u of (3.0.1), differentiable with respect to the space variables on $]0,T[\times\overline{\Omega}$, satisfies the estimate*

$$|Du(t,\xi)| \leq \frac{\gamma}{\sqrt{t}}\|f\|_\infty,\ t \in (0,T),\ \xi \in \partial\Omega. \tag{3.3.2}$$

PROOF. For each $x \in \Omega_\delta$ let $\xi(x)$ be the unique point in $\partial\Omega$ satisfying $|x-\xi| = r(x)$. Note that

$$x = \xi(x) + \eta(\xi(x))r(x),$$

where $\eta(\xi)$ is the unit inner normal to $\partial\Omega$ at $\xi \in \partial\Omega$. Recall also that $Dr(x) = \eta(\xi(x))$, $x \in \Omega_\delta$. See Appendix B for these properties of the distance function r. To proceed we remark that, since $u = 0$ on $\partial\Omega$,

$$Du(t,\xi) = \partial_\eta u(t,\xi), \;\; \xi \in \partial\Omega, \;\; t > 0.$$

In order to prove the claim it is enough to show that

$$|w(t,x)| = w(t,x) \le \frac{\gamma}{\sqrt{t}}\, r(x), \quad t \in (0,T), \;\; x \in \Omega_\delta, \tag{3.3.3}$$

where w is the solution to (3.0.1), corresponding to $f = 1$, and γ depends only on the stated parameters. Indeed, in the general case it is sufficient to observe that, for $x = \xi + r(x)\eta(\xi)$, $\xi \in \partial\Omega$ fixed,

$$|P_t f(x) - P_t f(\xi)| = |P_t f(x)| \le P_t|f|(x) \le \|f\|_\infty P_t 1(x) = \|f\|_\infty w(t,x) \le \frac{\gamma}{\sqrt{t}} r(x)\|f\|_\infty,$$

and (3.3.2) follows easily dividing by r and letting $r \to 0$. To prove (3.3.3) we compare w with an auxiliary function z, using Theorem A.0.13. Let

$$z(t,x) = g(t,r(x)), \;\; x \in \Omega_\delta,$$

where $g : [0,+\infty) \times [0,\delta] \to \mathbb{R}$ is the solution to (3.3.1). Now Lemma 3.3.1 yields

$$|z(t,x)| = g(t,r(x)) \le \frac{\gamma}{\sqrt{t}} r(x), \;\; 0 < t < T, \; x \in \Omega_\delta.$$

Thus we have only to prove that

$$w(t,x) \le z(t,x), \;\; x \in \Omega_\delta, \; t \in (0,T). \tag{3.3.4}$$

To verify (3.3.4), we consider $v = z - w$ in the cylinder $Q_\delta = (0,T)\times\Omega_\delta$. It is clear that v belongs to $C^{1,2}(Q_\delta)$, is continuous in $\overline{Q}_\delta \setminus \partial_{tx}Q_\delta$, bounded on Q_δ and nonnegative on $\partial' Q_\delta \setminus \partial_{tx} Q_\delta$. Moreover

$$\begin{aligned} v_t - Av = z_t - Az = g_t - \nu_0 g_{rr} - M g_r \\ +\Big(\nu_0 g_{rr} + M g_r - g_{rr}\sum_{i,j=1}^N q_{ij} D_i r D_j r - g_r\langle F, Dr\rangle - g_r \sum_{i,j=1}^N q_{ij} D_{ij} r + Vz\Big) \\ = g_{rr}\Big(\nu_0 - \sum_{i,j=1}^N q_{ij} D_i r D_j r\Big) + g_r\Big(M - \sum_{i,j=1}^N q_{ij} D_{ij} r - \langle F, Dr\rangle\Big) + Vz \ge 0, \end{aligned}$$

since $z, g_r \ge 0$, $g_{rr} \le 0$. The maximum principle Theorem A.0.13 now implies (3.3.4) and concludes the proof. □

The following proposition is an a-priori estimate on Du, where u is the bounded classical solution of (3.0.1). Its importance relies on pointing out the dependence of the constant C below.

Proposition 3.3.3 *Assume Hypothesis 1.1, (3.1.2) and (3.1.4). Then there exists a constant C depending on ν_0, h, k, s, β, T with the following property. Every bounded classical solution u of (3.0.1) such that*

(i) Du belongs to $C^{1,2}(Q)$,

(ii) $\sqrt{t}|Du|$ *is continuous in* $\overline{Q}\setminus\partial_{tx}Q$*, bounded in* Q *and verifies* $\lim_{t\to 0}\sqrt{t}|Du(t,x)|=0$*,* $x\in\Omega$*,*

(iii) u *satisfies (3.3.2)*

fulfills the estimate

$$\|Du(t,\cdot)\|_\infty \le \frac{C}{\sqrt{t}}\|f\|_\infty,\ \ t\in(0,T). \tag{3.3.5}$$

PROOF. Changing V to $V+1$ (hence u to $e^{-t}u$) we may assume that $|DV|\le\beta V$. We use Bernstein's method and define the function

$$v(t,x)=u^2(t,x)+a\,t\,|Du(t,x)|^2\ ,\quad t\in(0,T),\ x\in\Omega,$$

where $a>0$ is a parameter to be chosen later. Then we have $v\in C^{1,2}(Q)$, v is continuous in $\overline{Q}\setminus\partial_{tx}Q$, bounded in Q and $v(0,x)=f^2(x)$. We claim that for a suitable value of $a>0$, depending on ν_0,h,k,s,β,T we have

$$v_t(t,x)-Av(t,x)\le 0,\qquad 0<t<T,\ x\in\Omega. \tag{3.3.6}$$

This, by Theorem A.0.13, implies that

$$v(t,x)\le\sup_{x\in\Omega}|v(0,x)|+\sup_{\xi\in\partial\Omega,\ t\in(0,T)} at\,|Du(t,\xi)|^2\le(1+a\gamma^2)\|f\|_\infty^2,$$

$0<t\le T$, $x\in\Omega$, and (3.3.5) follows with $C=(a^{-1}+\gamma^2)^{1/2}$.

To verify inequality (3.3.6), note that, by a straightforward computation, v satisfies the equation

$$v_t-Av=a|Du|^2-2\sum_{i,j=1}^N q_{ij}\,D_iu\,D_ju+g_1+g_2,$$

where

$$\begin{aligned} g_1 &= a\,t\left(2\sum_{i,j=1}^N D_iF_j\,D_iu\,D_ju-2u\langle Du,DV\rangle-V|Du|^2\right)-Vu^2,\\ g_2 &= 2\,a\,t\left(\sum_{i,j,k=1}^N D_kq_{ij}D_kuD_{ij}u-\sum_{i,j,k=1}^N q_{ij}D_{ik}uD_{jk}u\right). \end{aligned}$$

Using the assumptions one has, for all $\varepsilon>0$, $x\in\Omega$, $t\in(0,T)$,

$$\begin{aligned} v_t-Av &\le \Big(a-2\nu_0+2akt+at(2s-1)V\Big)|Du|^2\\ &\quad+2at\Big(h|Du||D^2u|+\beta V|u||Du|-\nu_0|D^2u|^2\Big)-Vu^2\\ &\le \Big(a-2\nu_0+2akt+at(2s-1)V\Big)|Du|^2\\ &\quad+at\Big(h\varepsilon^{-1}|Du|^2+h\varepsilon|D^2u|^2+\beta\varepsilon^{-1}Vu^2+\beta\varepsilon V|Du|^2-2\nu_0|D^2u|^2\Big)-Vu^2, \end{aligned}$$

where $|D^2u|^2=\sum_{i,j=1}^N|D_{ij}u|^2$. Since $2s<1$, choosing ε and a small enough we get immediately (3.3.6). □

3.4 An auxiliary problem

In this section we keep Hypothesis 1.1 and condition (3.1.4) and write our operator in divergence form

$$A = A_0 + \sum_{i=1}^N G_i D_i - V,$$

where $A_0 = \sum_{i,j=1}^N D_i(q_{ij} D_j)$ and $G_i = F_i - \sum_{j=1}^N D_j q_{ij}$.

Moreover, we assume that the potential V and the drift G satisfy the inequality

$$|G(x)| \le \sigma V(x)^{1/2} + c_\sigma, \qquad x \in \Omega, \tag{3.4.1}$$

for some $\sigma > 0$ and show generation of an analytic semigroup in $L^p(\Omega)$, for $\sigma < \min\{2\nu_0(p-1), 2\}$. We follow the ideas of [11], [12] and [41] where the situation $\Omega = \mathbb{R}^N$ is considered.

For simplicity, we assume throughout this section that $2 \le p < \infty$. Observe that, since $q_{ij} \in C_b^1(\Omega)$, condition (3.4.1) holds equivalently for F or G with the same constant σ, possibly with a different choice of c_σ.

We endow A with the domain

$$D_p = \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : Vu \in L^p(\Omega)\}$$

which is a Banach space when endowed with the norm

$$\|u\|_{D_p} = \|u\|_{W^{2,p}(\Omega)} + \|Vu\|_{L^p(\Omega)},$$

and remark that the set

$$D = \{u \in C^\infty(\overline{\Omega}) : u_{|\partial\Omega} = 0, \operatorname{supp} u \text{ compact in } \overline{\Omega}\}$$

is dense in D_p.

We need the following interpolative lemma which is analogous to [41, Proposition 2.3].

Lemma 3.4.1 *Assume Hypothesis 1.1 and that condition (3.1.4) hold. Then there exists C depending on N, p, β and the coefficients (q_{ij}) such that for every $0 < \varepsilon < 1$ and $u \in D_p$, $2 \le p < \infty$, the following inequality holds:*

$$\|V^{1/2} Du\|_p \le \varepsilon \|A_0 u\|_p + C\varepsilon^{-1}(\|u\|_p + \|Vu\|_p).$$

Proof. It suffices to establish the inequality above for functions $u \in D$. Moreover, changing V with $V+1$, we may assume that $|DV| \le \beta V \le \beta V^{3/2}$.

Integrating by parts and using the fact that $u = 0$ on $\partial\Omega$ and $p \ge 2$ we have

$$\begin{aligned}
\int_\Omega V^{\frac{p}{2}} |D_k u|^p &= \int_\Omega V^{\frac{p}{2}} |D_k u|^{p-2} D_k u D_k u \\
&= -\frac{p}{2}\int_\Omega V^{\frac{p}{2}-1} D_k V u |D_k u|^{p-2} D_k u - (p-1)\int_\Omega V^{\frac{p}{2}} u |D_k u|^{p-2} D_{kk} u \\
&\le \frac{\beta p}{2}\int_\Omega |u||D_k u|^{p-1} V^{\frac{p-1}{2}} V + (p-1)\int_\Omega V^{\frac{p-2}{2}} |D_k u|^{p-2} V |u||D_{kk} u| \\
&\le \frac{\beta p}{2}\Big(\int_\Omega V^{\frac{p}{2}} |D_k u|^p\Big)^{1-1/p}\Big(\int_\Omega V^p |u|^p\Big)^{1/p} \\
&+ (p-1)\Big(\int_\Omega V^{\frac{p}{2}} |D_k u|^p\Big)^{1-2/p}\Big(\int_\Omega V^p |u|^p\Big)^{1/p}\Big(\int_\Omega |D_{kk} u|^p\Big)^{1/p}.
\end{aligned}$$

Setting $x = \|V^{1/2}D_k u\|_p$, $y = \|Vu\|_p$, $z = \|D_{kk}u\|_p$ we have obtained $x^2 \le (\beta p)/2xy + (p-1)yz$, hence

$$x \le \frac{\beta p}{2} y + \sqrt{(p-1)yz} \le C\varepsilon^{-1} y + \varepsilon z$$

for $\varepsilon < 1$, with C depending on β, p and the statement follows with $\|D^2 u\|_p$ instead of $\|A_0 u\|_p$. To complete the proof it suffices to use the closedness of A_0 on $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. □

Proposition 3.4.2 *Assume Hypothesis 1.1, condition (3.1.4) and suppose that (3.4.1) holds with σ satisfying $\sigma < \min\{2\nu_0(p-1), 2\}$. Then (A, D_p) is closed in $L^p(\Omega)$, $2 \le p < \infty$. Moreover, there is a constant λ_0 depending on c_σ with the following property: for every $\lambda > \lambda_0$ there exist C_1, C_2 depending only on $\lambda, N, p, \beta, \sigma, c_\sigma$ and the coefficients (q_{ij}), such that for every $u \in D_p$*

$$\|u\|_{D_p} \le C_1 \|\lambda u - Au\|_p \le C_2 \|u\|_{D_p}.$$

Finally, if $c_\sigma = 0$, then $\lambda_0 = 0$ and the inequality $\lambda\|u\|_p \le \|(\lambda - A)u\|_p$ holds.

Proof. By density we may assume that $u \in D$. The right hand side of the above inequality follows immediately from Lemma 3.4.1, since $|G| \le \sigma V^{1/2} + c_\sigma$.

Changing V with $V+\omega$ for a suitable large ω, we may assume that $c_\sigma = 0$ and that $|DV| \le \beta V$.

Let us multiply the identity $f = \lambda u - Au$ by $u|u|^{p-2}$. Integrating over Ω we get, since $u = 0$ on $\partial\Omega$,

$$\int_\Omega (\lambda + V)|u|^p + (p-1)\int_\Omega q_{ij}|u|^{p-2} D_i u D_j u \le \|f\|_p \|u\|_p^{p-1} + \sigma \int_\Omega V^{1/2}|Du||u|^{p-1}.$$

The last term can be estimated with

$$\sigma\Big(\int_\Omega V|u|^p\Big)^{1/2}\Big(\int_\Omega |u|^{p-2}|Du|^2\Big)^{1/2} \le \frac{\sigma}{2}\Big(\int_\Omega V|u|^p + |u|^{p-2}|Du|^2\Big).$$

Since $\sigma < \min\{2\nu_0(p-1), 2\}$ we easily obtain, for $\lambda > 0$, $\lambda\|u\|_p \le \|f\|_p$. To estimate Vu we observe that

$$\begin{aligned}
\int_\Omega (A_0 u)V^{p-1}u|u|^{p-2} &= -\sum_{i,j=1}^N \int_\Omega q_{ij} D_i u D_j \big(V^{p-1}u|u|^{p-2}\big) \\
&= -(p-1)\int_\Omega \sum_{i,j=1}^N q_{ij} V^{p-1}|u|^{p-2} D_i u D_j u \\
&\quad -(p-1)\int_\Omega \sum_{i,j=1}^N q_{ij} V^{p-2} u|u|^{p-2} D_i u D_j V.
\end{aligned}$$

Multiplying the identity $\lambda u - Au = f$ by $V^{p-1}u|u|^{p-2}$ and integrating over Ω we obtain

$$\begin{aligned}
&\int_\Omega (\lambda V^{p-1} + V^p)|u|^p + \nu_0(p-1)\int_\Omega V^{p-1}|u|^{p-2}|Du|^2 \\
&\quad \le \int_\Omega (\lambda V^{p-1} + V^p)|u|^p + (p-1)\int_\Omega V^{p-1}|u|^{p-2} q(Du, Du) \\
&\quad = -(p-1)\int_\Omega V^{p-2}u|u|^{p-2} q(Du, DV) + \int_\Omega V^{p-1}u|u|^{p-2}\langle G, Du\rangle + \int_\Omega fV^{p-1}u|u|^{p-2},
\end{aligned}$$

where $q(Du, DV) = \sum_{i,j=1}^N q_{ij} D_i u D_j V$ and similarly for $q(Du, Du)$. Next, observe that

$$\begin{aligned}
\Big|\int_\Omega V^{p-1}u|u|^{p-2}\langle G, Du\rangle\Big| &\le \sigma \int_\Omega V^{p-1/2}|u|^{p-1}|Du| \\
&\le \sigma\Big(\int_\Omega V^{p-1}|u|^{p-2}|Du|^2\Big)^{1/2}\Big(\int_\Omega V^p|u|^p\Big)^{1/2} \\
&\le \frac{\sigma}{2}\Big(\int_\Omega V^{p-1}|u|^{p-2}|Du|^2 + \int_\Omega V^p|u|^p\Big)
\end{aligned}$$

and that, for a suitable K depending only on $\|q_{ij}\|_\infty$,

$$\begin{aligned}\int_\Omega |u|^{p-1}V^{p-2}|q(Du,DV)| &\le K\int_\Omega |u|^{p-1}V^{p-2}|Du||DV|\\ &\le K\beta\Big(\int_\Omega V^{p-1}|u|^{p-2}|Du|^2\Big)^{1/2}\Big(\int_\Omega |u|^pV^{p-1}\Big)^{1/2}\\ &\le K\beta\varepsilon\Big(\int_\Omega V^{p-1}|u|^{p-2}|Du|^2+\int_\Omega V^p|u|^p\Big)+C_\varepsilon\int_\Omega |u|^p.\end{aligned}$$

In the last inequality we have used the inequality $t^{p-1}\le \varepsilon t^p+C_\varepsilon$.

Since $\sigma<\min\{2\nu_0(p-1),2\}$, taking a small ε one concludes that $\|Vu\|_p\le C\|f\|_p$, with C as in the statement.

We now use Lemma 3.4.1 to estimate the second order derivatives of u. We have

$$\begin{aligned}\|\langle G,Du\rangle\|_p &\le \sigma\|V^{1/2}Du\|_p\le\sigma(\varepsilon\|A_0u\|_p+C\varepsilon^{-1}\|u\|_p+C\varepsilon^{-1}\|Vu\|_p)\\ &\le \sigma(\varepsilon\|f\|_p+\varepsilon\|\langle G,Du\rangle\|_p+\varepsilon\|Vu\|_p+\varepsilon\lambda\|u\|_p+C\varepsilon^{-1}\|u\|_p+C\varepsilon^{-1}\|Vu\|_p)\end{aligned}$$

hence, taking a small ε, $\|\langle G,Du\rangle\|_p\le C\|f\|_p$ and $\|A_0u\|_p\le C\|f\|_p$, by difference. Using the closedness of A_0 on $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ given by the Calderon-Zygmund estimates, we get $\|D^2u\|_p\le C\|f\|_p$, with C as in the statement. □

Proposition 3.4.3 *Assume Hypothesis 1.1, condition (3.1.4) and suppose that (3.4.1) holds with σ satisfying $\sigma<\min\{2\nu_0(p-1),2\}$. Then (A,D_p) generates a semigroup in $L^p(\Omega)$, $2\le p<\infty$.*

Proof. As in the proof of Proposition 3.4.2, we may assume that $c_\sigma=0$, $|DV|\le\beta V$, so that $\lambda\|u\|_p\le\|\lambda u-Au\|_p$ for $\lambda>0$. By the Lumer-Phillips Theorem it suffices to show $\lambda-A$ is surjective for $\lambda>0$.

Setting for $\varepsilon>0$

$$V_\varepsilon=\frac{V}{1+\varepsilon V}\qquad G_\varepsilon=\frac{G}{\sqrt{1+\varepsilon V}},$$

it is immediate to check that $V_\varepsilon, G_\varepsilon$ satisfy

$$|DV_\varepsilon|\le\beta V_\varepsilon\qquad |G_\varepsilon|\le\sigma V_\varepsilon^{1/2}.$$

Since $V_\varepsilon,G_\varepsilon$ are bounded, the operator $A_\varepsilon=A_0+\langle G_\varepsilon,D\rangle-V_\varepsilon$ with domain $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ generates an analytic semigroup in $L^p(\Omega)$ see [32, Theorem 3.1.3], which is contractive by Proposition 3.4.2.

Given $f\in L^p(\Omega)$, let $u_\varepsilon\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ such that $(\lambda-A_\varepsilon)u_\varepsilon=f$. By Proposition 3.4.2, $\|u_\varepsilon\|_{2,p},\|V_\varepsilon u_\varepsilon\|_p\le C\|f\|_p$ with C independent of ε. By weak compactness we find $\varepsilon_n\to0$ such that (u_{ε_n}) converges weakly to a function u in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ and strongly in $W^{1,p}_{loc}(\Omega)$. Moreover we may assume that $(u_{\varepsilon_n})\to u$ a.e. in Ω. By Fatou's lemma $\|Vu\|_p\le C\|f\|_p$, hence $u\in D_p$ and it is easy to check that $(\lambda-A)u=f$. □

Let us show that the above semigroup is analytic.

Theorem 3.4.4 *Assume Hypothesis 1.1, condition (3.1.4) and suppose that (3.4.1) holds with σ satisfying $\sigma<\min\{2\nu_0(p-1),2\}$. Then (A,D_p) generates an analytic semigroup in $L^p(\Omega)$, $2\le p<\infty$.*

Proof. We keep the same notation of the proof of Proposition 3.4.2. We may assume that $c_\sigma=0$. Let $u\in D$ and set $u^*:=\overline{u}|u|^{p-2}$. Integrating by parts, since $u=0$ on $\partial\Omega$, a lengthy but straightforward computation yields

$$-Re\left(\int_\Omega (Au)u^*\right) = (p-1)\int_\Omega |u|^{p-4}q(Re(\overline{u}Du), Re(\overline{u}Du))$$
$$+\int_\Omega |u|^{p-4}q(Im(\overline{u}Du), Im(\overline{u}Du)) - \int_\Omega \langle G, Re(\overline{u}Du)|u|^{p-2}\rangle + \int_\Omega V|u|^p$$

and

$$\left|Im\int_\Omega (Au)u^*\right| \le (p-2)\int_\Omega |u|^{p-4}q(Re(\overline{u}Du), Im(\overline{u}Du)) + \int_\Omega |G||u|^{p-2}|Im(\overline{u}Du)|.$$

Condition (3.4.1) implies

$$\begin{aligned}\int_\Omega |G||u|^{p-2}|Im(\overline{u}Du)| &\le \sigma\int_\Omega V^{\frac12}|Im(\overline{u}Du)||u|^{\frac{p}{2}}|u|^{\frac{p-4}{2}}\\ &\le \sigma\left(\int_\Omega V|u|^p\right)^{\frac12}\left(\int_\Omega |u|^{p-4}|Im(\overline{u}Du)|^2\right)^{\frac12}\\ &\le \frac{\sigma}{\sqrt{\nu_0}}\left(\int_\Omega V|u|^p\right)^{\frac12}\left(\int_\Omega |u|^{p-4}q(Im(\overline{u}Du), Im(\overline{u}Du))\right)^{\frac12}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |G||u|^{p-2}|Re(\overline{u}Du)| &\le \sigma\int_\Omega V^{\frac12}|Re(\overline{u}Du)||u|^{\frac{p}{2}}|u|^{\frac{p-4}{2}}\\ &\le \sigma\left(\int_\Omega V|u|^p\right)^{\frac12}\left(\int_\Omega |u|^{p-4}|Re(\overline{u}Du)|^2\right)^{\frac12}\\ &\le \frac{\sigma}{\sqrt{\nu_0}}\left(\int_\Omega V|u|^p\right)^{\frac12}\left(\int_\Omega |u|^{p-4}q(Re(\overline{u}Du), Re(\overline{u}Du))\right)^{\frac12}.\end{aligned}$$

If we put $B^2 := \int_\Omega |u|^{p-4}q(Re(\overline{u}Du), Re(\overline{u}Du))$, $C^2 := \int_\Omega |u|^{p-4}q(Im(\overline{u}Du), Im(\overline{u}Du))$, and $D^2 := \int_\Omega V|u|^p$, then we deduce from the previous estimates

$$-Re\left(\int_\Omega (Au)u^*\right) \ge \left(p-1-\frac{\sigma}{2\nu_0}\right)B^2 + C^2 + \left(1-\frac{\sigma}{2}\right)D^2.$$

Therefore,

$$\left|Im\left(\int_\Omega (Au)u^*\right)\right| \le (p-2)BC + \frac{\sigma}{\sqrt{\nu_0}}CD$$

and one can find $\kappa > 0$ such that

$$\left|Im\left(\int_\Omega (Au)u^*\right)\right| \le \kappa\left[-Re\left(\int_\Omega (Au)u^*\right)\right]$$

for every $u \in D$ and, by density, for every $u \in D_p$. Since we already know that (A, D_p) generates a semigroup, by [44, Theorem 3.9, Chapter I] the proof is complete. □

Remark 3.4.5 Observe that all the results proved until now, in this section (but not the next lemma), hold assuming less local regularity on the coefficients. For example $q_{ij} \in C_b^1(\Omega), F \in L^\infty_{loc}(\Omega), V \in C^1(\Omega)$ suffice. Moreover, the existence of the Lyapunov function φ is not necessary.

We call $(T_t)_{t\ge0}$ the semigroup generated by A in $L^p(\Omega)$. For the proof of our main result we need some regularity results of the function $u(t,x) = (T_tf)(x)$.

Lemma 3.4.6 *Assume that the conditions of Theorem 3.4.4 hold for a fixed $p > N+1$ and let $f \in C_0^\infty(\Omega)$. Then the function $u(t,x) = (T_t f)(x)$ is the bounded classical solution of problem (3.0.1) and therefore has the regularity properties stated in Proposition 3.2.1. Moreover, Du is continuous and bounded in $\overline{Q}$.*

Proof. Since $f \in D_p$, the function $t \to T_t f$ is continuous from $[0,T]$ to $W^{2,p}(\Omega)$ and Sobolev embedding implies that u, Du are bounded and continuous in $\overline{Q}$. To complete the proof, we have to show that $u \in C^{1,2}(Q)$.

Let us fix $\varepsilon > 0$ and open bounded sets Ω_1, Ω_2 such that $\overline{\Omega}_1 \subset \Omega_2$ and $\overline{\Omega}_2 \subset \Omega$. Since $(T_t)_{t\geq 0}$ is analytic, u is continuously differentiable from $[\varepsilon, T]$ to $W^{2,p}(\Omega)$ and Sobolev embedding yields $u_t \in C(Q)$. Set

$$\kappa = \sup_{\varepsilon \leq t \leq T} \Big(\|u(t,\cdot)\|_{W^{2,p}(\Omega)} + \|u_t(t,\cdot)\|_{W^{2,p}(\Omega)} \Big).$$

For every fixed $t \in [\varepsilon, T]$ the function $u(t,\cdot)$ belongs to $W^{2,p}(\Omega)$ and solves the equation

$$\sum_{i,j=1}^N q_{ij} D_{ij} u = -\langle F, Du \rangle + Vu - u_t$$

in Ω. Since the right hand side belongs to $W^{1,p}_{loc}(\Omega)$ it follows that $u(t,\cdot) \in W^{3,p}_{loc}(\Omega)$ and that, for a suitable c depending on Ω_1, Ω_2 and the coefficients of A,

$$\sup_{\varepsilon \leq t \leq T} \|u(t,\cdot)\|_{W^{3,p}(\Omega_1)} \leq c\kappa,$$

see [26, Theorem 9.19]. We have proved that for every $i,j = 1,\ldots,N$, $D_t D_{ij} u, DD_{ij} u \in L^p([\varepsilon,T] \times \Omega_1)$. By Sobolev embedding, since $p > N+1$, $D_{ij} u \in C(Q)$ and the proof is complete. □

3.5 Proof of Theorem 3.1.2

For $\varepsilon > 0$ let $V_\varepsilon(x) = \varepsilon \exp\{4c_2\sqrt{1+|x|^2}\}$. Then $|DV_\varepsilon| \leq 4c_2 V_\varepsilon$ and for every $\sigma > 0$ there exists $c_\sigma > 0$ (depending on ε) such that $|F| \leq \sigma(V+V_\varepsilon)^{1/2} + c_\sigma$. Define $A_\varepsilon = A - V_\varepsilon$ and note that the hypotheses of Theorem 3.4.4 are satisfied.

Fix $p > N+1$, $f \in C_0^\infty(\Omega)$ and let u_ε be the semigroup solution of (3.0.1) with A_ε instead of A, given by Theorem 3.4.4. By Lemma 3.4.6 the function u_ε is the bounded solution of the above problem and Du_ε is continuous and bounded in $\overline{Q}$. By Proposition 3.3.2 we deduce that $|Du_\varepsilon(t,\xi)| \leq (\gamma/\sqrt{t})\|f\|_\infty$, $\xi \in \partial\Omega$, with γ depending on ν_0, M, δ, T and independent of ε.

Since u_ε satisfies the hypotheses of Proposition 3.3.3, we deduce that

$$\|Du_\varepsilon(t,\cdot)\|_\infty \leq (C/\sqrt{t})\|f\|_\infty,$$

with C as in the statement.

Observe that $\|u_\varepsilon\|_\infty \leq \|f\|_\infty$. Let us fix $R > 0$ and note that the C^α-norm of the coefficients of A_ε is bounded, uniformly with respect to $\varepsilon < 1$, in $\Omega \cap B_{R+1}$. By the local Schauder estimates [30, Theorem IV.10.1] applied to the operator $D_t - A_\varepsilon$, there exists a constant C, independent of $\varepsilon < 1$, such that

$$\begin{aligned}
\|u_\varepsilon\|_{C^{1+\alpha/2, 2+\alpha}((0,T)\times(\Omega\cap B_R))} &\leq C\Big(\|u_\varepsilon\|_{C((0,T)\times(\Omega\cap B_{R+1})}\Big) + \|f\|_{C^{2+\alpha}(\Omega\cap B_{R+1})} \\
&\leq 2C\|f\|_{C^{2+\alpha}(\Omega)}.
\end{aligned}$$

By a standard compactness argument we conclude that a subsequence (u_{ε_n}) converges in $C^{1,2}([0,T]\times\overline{(\Omega\cap B_R)})$ for every R to a function u which is the bounded classical solution of (3.0.1) and satisfies $\|Du(t,\cdot)\|_\infty \le (C/\sqrt{t})\|f\|_\infty$.

Finally, to treat the general case of $f\in C_b(\Omega)$ we consider a sequence $(f_n)\in C_0^\infty(\Omega)$ convergent to f uniformly on compact subsets of Ω and such that $\|f_n\|_\infty \le \|f\|_\infty$. Let u_n be the bounded classical solution of (3.0.1) relative to f_n. Then $\|Du_n(t,\cdot)\|_\infty \le (C/\sqrt{t})\|f\|_\infty$, by the previous step. Since $(u_n)\to u$ in $C^{1,2}(Q)$, see the proof of Proposition 3.2.2, the estimate for Du follows. □

3.6 Examples and applications

We first show that gradient estimates fail, in general, if condition (3.1.3) is not satisfied. We refer the reader to [8, Example 5.6] for an operator defined on the whole space, for which condition (3.1.2) is violated and gradient estimates fail. The following result refines and generalizes an example in [57].

Example 3.6.1 We consider the following Dirichlet problem in $\Omega=\mathbb{R}^2_+=\{(x,y)\in\mathbb{R}^2,\ x>0\}$

$$\begin{cases} u_t(t,x,y)=u_{xx}(t,x,y)+u_{yy}(t,x,y)+g(y)u_x(t,x,y) & t>0,\ x>0,\\ u(t,0,y)=0 & t>0,\ y\in\mathbb{R},\\ u(0,x,y)=1 & (x,y)\in\Omega,\end{cases}$$

where $g\in C^2(\mathbb{R})$ and

$$\lim_{y\to+\infty} g(y)=+\infty.$$

Observe that (3.1.3) fails. However, Proposition 3.2.2 yields existence and uniqueness of a bounded solution u. Let us show that, for $t>0$, $u(t,\cdot)$ is *not* uniformly continuous in Ω. To this end, it is enough to show that, for every $t,x>0$,

$$\sup_{y>0} u(t,x,y)=1. \tag{3.6.1}$$

Fix $n>0$ and take c_n such that $g(y)\ge n$ for $y\ge c_n$. Define $R_n=(0,+\infty)\times(c_n,+\infty)$ and consider $v=v_n$ which solves

$$\begin{cases} v_t(t,x,y)=v_{xx}(t,x,y)+v_{yy}(t,x,y)+nv_x(t,x,y) & t>0,\ (x,y)\in R_n,\\ v(t,z)=0 & t>0,\ z\in\partial R_n,\\ v(0,x,y)=1 & (x,y)\in R_n,\end{cases}$$

We prove that for $t,x>0$

$$(i)\ \lim_{n\to\infty}\sup_{y>c_n} v_n(t,x,y)=1;\quad (ii)\ u(t,x,y)\ge v_n(t,x,y).$$

Clearly (i) and (ii) give (3.6.1). Let us verify (i). Note that $v_n(t,x,y)=a_n(t,x)b_n(t,y)$, where $a=a_n$, $b=b_n$ solve respectively

$$\begin{cases} a_t(t,x)=a_{xx}(t,x)+na_x(t,x) & t>0,\\ a(t,0)=0 & t>0,\\ a(0,x)=1 & x>0,\end{cases}\qquad \begin{cases} b_t(t,y)=b_{yy}(t,y) & t>0,\\ b(t,c_n)=0 & t>0,\\ b(0,y)=1 & y>c_n.\end{cases}$$

To find an explicit formula for a_n, we first remark that $a_n(t,x)=a_1(n^2t,nx)$. Then, setting $v(t,x)=e^{x/2}e^{\frac{1}{4}t}a_1(t,x)$, v solves

$$\begin{cases} v_t(t,x)=v_{xx}(t,x) & t>0,\ x>0,\\ v(t,0)=0 & t>0,\\ v(0,x)=e^{x/2} & x>0;\end{cases}$$

By a reflection argument we get easily an explicit expression for v and finally we obtain for any $t > 0$, $y \geq c_n$, $x \geq 0$,

$$a_n(t,x) = \frac{e^{\frac{-n^2 t}{4}}}{n\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{|nx-z|^2}{4n^2 t}} - e^{-\frac{|nx+z|^2}{4n^2 t}} \right) e^{\frac{z-nx}{2}} dz, \quad b_n(t,y) = \int_0^{y-c_n} \frac{e^{\frac{-z^2}{4t}}}{\sqrt{\pi t}} dz.$$

To we check that (i) holds we write

$$a_n(t,x) = A_n^1(t,x) - A_n^2(t,x),$$
$$A_n^1(t,x) = \frac{e^{\frac{-n^2 t}{4}}}{n\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{|nx-z|^2}{4n^2 t}} \right) e^{\frac{z-nx}{2}} dz,$$
$$A_n^2(t,x) = \frac{e^{\frac{-n^2 t}{4}}}{n\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{|nx+z|^2}{4n^2 t}} \right) e^{\frac{z-nx}{2}} dz.$$

Let us consider A_n^1. By a change of variables we obtain

$$A_n^1(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\frac{x+n}{2\sqrt{t}}}^{+\infty} e^{-s^2}\, ds,$$

which is increasing in x and converges to 1 as $n \to +\infty$. In a similar way we get that $A_n^2(t,x)$ is decreasing in x and converges to 0 as $n \to +\infty$. Then (i) easily follows.

To prove (ii) we use Theorem A.0.13. Set $w = u - v_n$ in $(0,T) \times R_n$. We have $w(0,x,y) = 0$, $(x,y) \in R_n$. Moreover $w(t,z) \geq 0$, $z \in \partial R_n$, $t > 0$. To conclude it suffices to verify that

$$w_t(t,x,y) \geq \Delta w(t,x,y) + g(y) w_x(t,x,y), \quad t > 0,\ (x,y) \in R_n. \tag{3.6.2}$$

Since $w_t = \Delta w + g(y) w_x + [g(y) - n](v_n)_x$, $g(y) \geq n$, for $y \geq c_n$ and $(v_n)_x(t,x,y) = a_n'(t,x) b_n(t,y) \geq 0$, $t > 0$, $(x,y) \in R_n$, as verified above, (3.6.2) follows and the proof is complete.

For instance, we can take, in the above example, $g(y) = \sqrt{1+y^2}$. On the other hand, if $g(y) = -\sqrt{1+y^2}$ then all the conditions of Theorem 3.1.2 hold and gradient estimates hold.

Remark 3.6.2 We point out that our main result can be used to prove some boundary gradient estimates for solutions of Dirichlet elliptic problems, involving the operator A. Indeed if $\varphi \in C_b(\Omega) \cap C^2(\Omega)$ solves

$$\begin{cases} A\varphi(x) = 0, & x \in \Omega, \\ \varphi(\xi) = 0, & \xi \in \partial\Omega, \end{cases} \tag{3.6.3}$$

then φ is the bounded classical solution of (3.0.1) with $f = \varphi$. Thus, under the assumptions of Theorem 3.1.2, we get

$$\sup_{x \in \overline{\Omega}} |D\varphi(x)| \leq C \|\varphi\|_\infty.$$

This extends some classical boundary gradient estimates concerning linear and nonlinear second elliptic operators, involving bounded coefficients, see for instance [26, Section 14].

Remark 3.6.3 Theorem 3.1.2 has also some applications to isoperimetric inequalities, see [31] and [57].

Chapter 4

On the domain of some ordinary differential operators in spaces of continuous functions

The present chapter is devoted to the study of the following second order ordinary differential operator

$$Au = au'' + bu'$$

in spaces of continuous functions. In particular, we are interested in a precise description of the domain on which A generates a semigroup. In Chapter 1 we have computed explicitly the domain of the generator in the framework of L^p spaces, for $1 < p < \infty$, in higher dimensions. In Chapters 2 and 3 we have studied parabolic problems with Neumann or Dirichlet boundary conditions in an open set Ω of $\mathbb{R}^N$ and, by means of gradient estimates, we have obtained some information on the domains of the generators of the semigroups yielding the classical solutions to the above problems. But we did not come to a complete description of such domains. Also in the literature, one can find more results concerning L^p spaces, with $1 < p < \infty$ (see [11], [12], [37], [41]), rather than spaces of continuous functions. In [41] a complete description of the domain is given in $C_0(\mathbb{R}^N)$ when the operator contains a potential term which balances the growth of the drift coefficient. We refer to [34] for the case of Hölder spaces.

In this chapter we limit ourselves to the special case $N = 1$ and we deal with $C_b(\mathbb{R})$ and with $C(\overline{\mathbb{R}})$, the space of continuous functions having finite limits at $\pm\infty$. Here a detailed theory has been developed in the fifties by W. Feller who gave an explicit description of all the boundary conditions under which A generates a semigroup of positive contractions. An introduction to Feller's theory which is sufficient for our purposes can be found in [21, Subsection VI.4.c].

We consider A with its maximal domain in $C_b(\mathbb{R})$

$$D_{\max}(A) := \{u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R}) \mid Au \in C_b(\mathbb{R})\}$$

and we assume that

($\mathbf{H}_0$) $\lambda - A$ is injective on $D_{\max}(A)$ for some $\lambda > 0$.

This is equivalent to saying that $(A, D_{\max}(A))$ generates a semigroup of positive contractions in $C_b(\mathbb{R})$, which is not however strongly continuous (see Proposition 5.2.3).

If (H_0) holds, then $\lambda - A$ is injective on $D_{\max}(A)$ for all $\lambda > 0$. Moreover it turns out that $\lambda - A$ is injective on $D_{\max}(A)$ if and only if it is injective on $D_{\mathrm{m}}(A)$, where

$$D_{\mathrm{m}}(A) := \{u \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid Au \in C(\overline{\mathbb{R}})\}$$

is the maximal domain in $C(\overline{\mathbb{R}})$, see Proposition 4.1.1 below. Then, from [21, Theorem VI.4.15], it follows that $(A, D_{\rm m}(A))$ generates a strongly continuous semigroup of positive contractions in $C(\overline{\mathbb{R}})$.

We point out that (H_0) is equivalent to requiring that $\pm\infty$ are inaccessible boundary points according to Feller's terminology, which means that, if $W(x) := \exp\left(-\int_0^x \frac{b(t)}{a(t)}\,dt\right)$, the function

$$R(x) := W(x)\int_0^x \frac{1}{a(t)W(t)}\,dt$$

is not summable either in $(-\infty, 0)$ or in $(0, +\infty)$. In many cases verifying these integral conditions is not by any means an easy task. A sufficient condition, which has the advantage to be easy to handle, is the existence of a positive function $V \in C^2(\mathbb{R})$ such that $\lim_{|x|\to\infty} V(x) = \infty$ and $AV \le \lambda V$ for some $\lambda > 0$, see again Proposition 4.1.1.

Our main results show that, under suitable conditions,

$$D_{\max}(A) = \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}$$

and, if a is bounded,

$$D_{\rm m}(A) = \{u \in C^2(\overline{\mathbb{R}}) \mid bu' \in C(\overline{\mathbb{R}})\}.$$

In this way, requiring that $Au \in C_b(\mathbb{R})$ (resp. $C(\overline{\mathbb{R}})$) is the same to requiring that the two terms au'' and bu' separately belong to $C_b(\mathbb{R})$ (resp. $C(\overline{\mathbb{R}})$).

Let us state our main assumptions:

($\mathbf{H_1}$) $a \in C(\mathbb{R})$ and $a \ge \delta$ for some $\delta > 0$.

($\mathbf{H_2}$) $b \in C^1(\mathbb{R})$ and there exist constants $c_1 \in \mathbb{R}$ and $c_2 < 1$ such that

$$a(x)b'(x) \le c_1 + c_2 b^2(x), \quad x \in \mathbb{R}\,.$$

We shall keep hypothesis (H_1) and (H_2) throughout Sections 4.1 and 4.2 together with (H_0), but we shall need stronger assumptions in Subsection 4.2.2. In fact, to describe the domain in $C(\overline{\mathbb{R}})$ we assume that $a \in C_b(\mathbb{R})$ and that b satisfies $|b'| \le c(1 + |b|)$.

4.1 Preliminary results

In this section we collect some preliminary results which will be useful for the sequel. We start by studying the injectivity of the operator $\lambda - A$ on $D_{\max}(A)$ and $D_{\rm m}(A)$, i.e. the uniqueness of the solution in $D_{\rm m}(A)$ and $D_{\max}(A)$ of the elliptic equation $\lambda u - Au = f$.

Proposition 4.1.1 *The following assertions are equivalent:*

(i) *(H_0) holds.*

(ii) *$\lambda - A$ is injective on $D_{\max}(A)$ for all $\lambda > 0$, hence $(A, D_{\max}(A))$ generates a semigroup of positive contractions in $C_b(\mathbb{R})$.*

(iii) *$\lambda - A$ is injective on $D_{\rm m}(A)$ for all $\lambda > 0$, hence $(A, D_{\rm m}(A))$ generates a strongly continuous semigroup of positive contractions in $C(\overline{\mathbb{R}})$.*

Moreover, if there exists a positive function $V \in C^2(\mathbb{R})$ such that $\lim_{|x|\to+\infty} V(x) = +\infty$ and $AV \le \lambda V$ for some $\lambda > 0$, then the above conditions are satisfied.

Proof. For $(i) \Leftrightarrow (ii)$ see [38, Proposition 3.5]. Implication $(ii) \Rightarrow (iii)$ is obvious, see also [21, Theorem VI.4.15].

Now we prove that (iii) implies (ii). Let $u \in D_{\max}(A)$ be such that $\lambda u - Au = 0$. From [21, Theorem VI.4.14] it follows that there exist two linearly independent solutions v_1 and v_2 of $(\lambda - A)v = 0$ such that v_1 (resp. v_2) is bounded (resp. unbounded) at $+\infty$ and unbounded (resp. bounded) at $-\infty$. Then $u = k_1 v_1 + k_2 v_2$, for some constants $k_1, k_2 \in \mathbb{R}$. Since u is bounded, $k_1 = k_2 = 0$, which means $u = 0$.

Finally if there exists a function V as above then (ii) holds as a consequence of Proposition 5.2.3. □

Now we prove some estimates which will be the main tool for the description of $D_{\max}(A)$.

Proposition 4.1.2 *Assume that $a > 0$ and that (H$_2$) holds. Let $M > 0$ and v be a function in $C^1([-M, M])$ such that $v(-M) = v(M) = 0$. Then*

$$\|bv\|_{[-M,M]} \leq \frac{1}{1-c_2}\, \|av' + bv\|_{[-M,M]} + \sqrt{\frac{c_1^+}{1-c_2}}\, \|v\|_{[-M,M]}\ , \tag{4.1.1}$$

where $c_1^+ = \max\{c_1, 0\}$.

Proof. Set $f = av' + bv$. Let $x_0 \in [-M, M]$ be a maximum point of the function bv. We may suppose that $x_0 \in]-M, M[$ and $b(x_0) \neq 0$, otherwise $b(x_0)v(x_0) = 0$ and estimate (4.1.1) is trivially satisfied. Moreover, without loss of generality we assume that $\|bv\|_{[-M,M]} = b(x_0)v(x_0)$. Then $(bv)'(x_0) = 0$ and from hypothesis (H$_2$) it follows that

$$a(x_0)v'(x_0) = -a(x_0)b'(x_0)\frac{v(x_0)}{b(x_0)} \geq -c_1\frac{v(x_0)}{b(x_0)} - c_2 b(x_0)v(x_0)$$

and consequently

$$\|f\|_{[-M,M]} \geq f(x_0) = a(x_0)v'(x_0) + b(x_0)v(x_0) \geq (1-c_2)b(x_0)v(x_0) - c_1\frac{v(x_0)}{b(x_0)}\ .$$

Multiplying by $b(x_0)v(x_0) = \|bv\|_{[-M,M]}$ both sides of the previous inequality we get

$$\|bv\|_{[-M,M]}\|f\|_{[-M,M]} \geq (1-c_2)\|bv\|^2_{[-M,M]} - c_1 v^2(x_0) \geq (1-c_2)\|bv\|^2_{[-M,M]} - c_1^+\|v\|^2_{[-M,M]}\ .$$

If $x := \|bv\|_{[-M,M]}$, we have $x^2 \leq \alpha x + \beta$ with $\alpha = \dfrac{1}{1-c_2}\|f\|_{[-M,M]}$, $\beta = \dfrac{c_1^+}{1-c_2}\|v\|^2_{[-M,M]}$. It follows that $x \leq \alpha + \sqrt{\beta}$, that is

$$\|bv\|_{[-M,M]} \leq \frac{1}{1-c_2}\|f\|_{[-M,M]} + \sqrt{\frac{c_1^+}{1-c_2}}\, \|v\|_{[-M,M]}\ ,$$

which is the statement. □

Remark 4.1.3 Assume (H$_1$) and (H$_2$). If $u \in C^2([-M, M])$ is such that $u'(-M) = u'(M) = 0$ then Proposition 4.1.2 implies

$$\|bu'\|_{[-M,M]} \leq \frac{1}{1-c_2}\, \|Au\|_{[-M,M]} + \sqrt{\frac{c_1^+}{1-c_2}}\, \|u'\|_{[-M,M]}\ .$$

Now, if $\varepsilon > 0$ is sufficiently small, there exists a constant C_ε, independent of M, such that

$$\|u'\|_{[-M,M]} \leq \varepsilon\|u''\|_{[-M,M]} + C_\varepsilon\|u\|_{[-M,M]}\ .$$

Moreover we have that

$$\|u''\|_{[-M,M]} \le \frac{1}{\delta}\|au''\|_{[-M,M]} \le \frac{1}{\delta}\left(\|bu'\|_{[-M,M]} + \|Au\|_{[-M,M]}\right) .$$

Taking into account these estimates and choosing ε small enough we get

$$\|bu'\|_{[-M,M]} \le C\,\left(\|Au\|_{[-M,M]} + \|u\|_{[-M,M]}\right) \tag{4.1.2}$$

where C depends only on c_1, c_2 and δ.

Estimate (4.1.2) still holds for every function $u \in C^2(\mathbb{R})$ with compact support; indeed, it is sufficient to consider an interval containing the support of u. The next step is to show that if a is bounded then this estimate extends to every function $u \in C_b^2(\mathbb{R})$. This will be used in Subsection 4.2.2.

Proposition 4.1.4 *If $a \in C_b(\mathbb{R})$, $a \ge \delta > 0$ and (H_2) holds, then for every $u \in C_b^2(\mathbb{R})$ we have*

- *(i)* $\|bu'\|_\infty \le C(\|Au\|_\infty + \|u\|_\infty)$ *;*
- *(ii)* $\|u''\|_\infty \le C\,(\|Au\|_\infty + \|u\|_\infty)$,

where $C = C(c_1, c_2, \delta)$.

Proof. Let $u \in C_b^2(\mathbb{R})$. We prove that

$$\|bu'\|_\infty \le \frac{1}{1-c_2}\|Au\|_\infty + \sqrt{\frac{c_1^+}{1-c_2}}\,\|u'\|_\infty. \tag{4.1.3}$$

Let $v = u'$ and $\eta \in C_c^\infty(\mathbb{R})$ be such that $0 \le \eta \le 1$, $\eta \equiv 1$ in $[-1,1]$ and $\eta \equiv 0$ in $\mathbb{R} \setminus [-2,2]$. Set $\eta_n(x) = \eta(x/n)$. Then $a(v\,\eta_n)' + b(v\,\eta_n) = (av' + bv)\eta_n + a\,v\,\eta_n'$ and applying (4.1.1) to $v\,\eta_n \in C_c^1(\mathbb{R})$ we have

$$\|b\,v\,\eta_n\|_\infty \le \frac{1}{1-c_2}\,\|av' + bv\|_\infty + \frac{\|a\|_\infty}{1-c_2}\|v\,\eta_n'\|_\infty + \sqrt{\frac{c_1^+}{1-c_2}}\,\|v\|_\infty .$$

Letting $n \to \infty$ it follows that

$$\|bv\|_\infty \le \frac{1}{1-c_2}\,\|av' + bv\|_\infty + \sqrt{\frac{c_1^+}{1-c_2}}\,\|v\|_\infty ,$$

which is just estimate (4.1.3). Now, (i) follows from (4.1.3) as in Remark 4.1.3.

Estimate (ii) easily follows from (i). □

4.2 Characterization of the domain

4.2.1 The case of $C_b(\mathbb{R})$

In this subsection we show that $D_{\max}(A)$ is given by

$$D_{\max}(A) = \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}.$$

The crucial point is to prove that $\lambda - A$ is surjective from the right-hand side above onto $C_b(\mathbb{R})$. This is done through an approximation procedure by considering the solutions of the equation $\lambda u - Au = f$ in bounded intervals with Neumann boundary conditions and applying the estimates of Section 4.1.

Proposition 4.2.1 *Assume that (H_0), (H_1) and (H_2) hold. Then*

$$D_{\max}(A) = \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}.$$

PROOF. Set $D(A) := \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}$. Let $\lambda > 0$ and $f \in C_b(\mathbb{R})$ be fixed. For every $n \in \mathbb{N}$ consider the problem

$$\begin{cases} \lambda u - Au = f & \text{in } [-n, n] \\ u'(-n) = u'(n) = 0 \end{cases}$$

It is well known that there exists a unique solution $u_n \in C^2([-n, n])$ which satisfies the following estimate

$$\|u_n\|_{[-n,n]} \leq \frac{1}{\lambda}\|f\|_\infty \tag{4.2.1}$$

(see e.g. [21, Theorem VI.4.16]). The equality $\lambda u_n - Au_n = f$ implies that

$$\|Au_n\|_{[-n,n]} \leq 2\|f\|_\infty. \tag{4.2.2}$$

Taking into account estimate (4.1.2) we have

$$\|bu_n'\|_{[-n,n]} \leq C \left(\|Au_n\|_{[-n,n]} + \|u_n\|_{[-n,n]}\right) \leq \overline{C}\, \|f\|_\infty, \tag{4.2.3}$$

where $\overline{C} = \overline{C}(c_1, c_2, \delta, \lambda)$. Moreover

$$\delta\|u_n''\|_{[-n,n]} \leq \|au_n''\|_{[-n,n]} \leq \|Au_n\|_{[-n,n]} + \|bu_n'\|_{[-n,n]} \leq \overline{C}_1\, \|f\|_\infty \tag{4.2.4}$$

and, by interpolation

$$\|u_n'\|_{[-n,n]} \leq C_2(\|Au_n\|_{[-n,n]} + \|u_n\|_{[-n,n]}) \leq \overline{C}_2\, \|f\|_\infty \tag{4.2.5}$$

with $\overline{C}_1$ and $\overline{C}_2$ depending only on $c_1, c_2, \delta, \lambda$. Now fix $k \in \mathbb{N}$ and consider $n \geq k$. Then the previous estimates imply that $\|u_n\|_{C^2([-k,k])}$ is bounded by a constant independent of n and k. It follows that the sequences (u_n), (u_n') are bounded and equicontinuous, then by Ascoli-Arzelà Theorem there exists a subsequence of (u_n) which converges in $C^1([-k, k])$. Using a diagonal procedure we can construct a subsequence, still denoted by (u_n), and a function $u \in C^1(\mathbb{R})$ such that u_n converges to u together with the first derivatives uniformly on every compact subset of $\mathbb{R}$. It follows that bu_n' converges to bu' uniformly on compact sets and, using the equation $\lambda u_n - Au_n = f$, it turns out that au_n'' and consequently u_n'' converge, too. Therefore $u \in C^2(\mathbb{R})$ and $\lambda u - Au = f$. Writing estimates (4.2.3), (4.2.4) and (4.2.5) for the function u_n in $[-k, k]$ with $n \geq k$ and letting first $n \to \infty$ and then $k \to \infty$ we obtain that $u \in C_b^2(\mathbb{R})$ with $au'', bu' \in C_b(\mathbb{R})$, i.e. $u \in D(A)$.

This shows that $\lambda - A : D(A) \to C_b(\mathbb{R})$ is surjective. Since $D(A) \subset D_{\max}(A)$ and $\lambda - A : D_{\max}(A) \to C_b(\mathbb{R})$ is bijective we deduce that $D(A) = D_{\max}(A)$, as claimed. □

4.2.2 The case of $C(\overline{\mathbb{R}})$

As in the previous subsection we show that the domain $D_{\mathrm{m}}(A)$ on which A generates a strongly continuous semigroup in $C(\overline{\mathbb{R}})$ is given by

$$D_{\mathrm{m}}(A) = \{u \in C^2(\overline{\mathbb{R}}) \mid \; bu' \in C(\overline{\mathbb{R}})\}.$$

To this aim we require that

($\mathbf{H_0'}$) there exist positive constants d_1, d_2 such that

$$b(x)x \le d_1(1+x^2)\log(1+x^2) + d_2, \quad x \in \mathbb{R}.$$

($\mathbf{H_1'}$) $a \in C_b(\mathbb{R})$ and $a \ge \delta$ for some $\delta > 0$.

($\mathbf{H_2'}$) $b \in C^1(\mathbb{R})$ and $|b'(x)| \le c(1+|b(x)|)$, for some constant $c > 0$ and for all $x \in \mathbb{R}$.

Since a is bounded one easily verify that the function $V(x) = 1+\log(1+x^2)$ satisfies the hypothesis of Proposition 4.1.1. Hence $(A, D_{\rm m}(A))$ generates a semigroup in $C(\overline{\mathbb{R}})$. Clearly (H_1') and (H_2') imply (H_1) and (H_2), thus we may use the results of Subsection 4.2.1.

Proposition 4.2.2 *Assume that (H_0'), (H_1') and (H_2') hold. Then*

$$D_{\rm m}(A) = \{u \in C^2(\overline{\mathbb{R}}) \mid \ bu' \in C(\overline{\mathbb{R}})\}.$$

Proof. Set $D := \{u \in C^2(\overline{\mathbb{R}}) \mid bu' \in C(\overline{\mathbb{R}})\}$. Since $\lambda - A : D_{\rm m}(A) \to C(\overline{\mathbb{R}})$ is bijective and $D \subset D_{\rm m}(A)$, it is sufficient to prove that $\lambda - A : D \to C(\overline{\mathbb{R}})$ is surjective.

Step 1: We assume first that $a \equiv 1$. Let $\lambda > 0$ and $f \in C(\overline{\mathbb{R}})$ be fixed. From Proposition 4.2.1 we know that there exists $u \in D_{\max}(A) = \{u \in C_b^2(\mathbb{R}) \mid bu' \in C_b(\mathbb{R})\}$ such that $\lambda u - Au = f$. On the other hand, since $(A, D_{\rm m}(A))$ generates a strongly continuous semigroup of contractions, there is $w \in D_{\rm m}(A)$ which solves the same equation. By uniqueness $u = w$. This means that $u \in C_b^2(\mathbb{R}) \cap C(\overline{\mathbb{R}})$ with $Au \in C(\overline{\mathbb{R}})$, $bu' \in C_b(\mathbb{R})$ and $\lambda u - Au = f$. It remains to prove that $u', u'', bu' \in C(\overline{\mathbb{R}})$. Since u' is uniformly continuous and u admits finite limits at $\pm\infty$ we deduce that $\lim_{|x|\to\infty} u'(x) = 0$. In order to use the same argument for u'' we first assume $f \in C(\overline{\mathbb{R}}) \cap C_b^1(\mathbb{R})$. Then we may differentiate the equation

$$\lambda u - u'' - bu' = f \tag{4.2.6}$$

obtaining

$$\lambda v - v'' - bv' = f' + b'v \,,$$

where $v = u'$. (H_2') implies that $g := f' + b'v \in C_b(\mathbb{R})$. Therefore $v \in D_{\max}(A)$ and Proposition 4.2.1 implies that $v \in C_b^2(\mathbb{R})$. This means that $u \in C_b^3(\mathbb{R})$ and as before it implies that $u'' \in C(\overline{\mathbb{R}})$, with $\lim_{|x|\to\infty} u''(x) = 0$.

Now take $f \in C(\overline{\mathbb{R}})$. Set $f_\varepsilon := \Phi_\varepsilon * f \in C(\overline{\mathbb{R}}) \cap C_b^1(\mathbb{R})$ for $\varepsilon > 0$, where (Φ_ε) is a family of standard mollifiers. From the previous computations, for every $\varepsilon > 0$ the solution u_ε of the equation $\lambda u_\varepsilon - Au_\varepsilon = f_\varepsilon$ belongs to D. Let $u \in D_{\max}(A)$ be the solution of $\lambda u - Au = f$ and consider the difference $u - u_\varepsilon$. Then $u - u_\varepsilon \in C_b^2(\mathbb{R})$ with $A(u - u_\varepsilon) \in C_b(\mathbb{R})$ and $\lambda(u - u_\varepsilon) - A(u - u_\varepsilon) = f - f_\varepsilon$. Moreover

$$\|u - u_\varepsilon\|_\infty \le \frac{1}{\lambda}\|f_\varepsilon - f\|_\infty \ ,$$

thus from the equation we get

$$\|Au - Au_\varepsilon\|_\infty \le 2\|f_\varepsilon - f\|_\infty$$

and from Proposition 4.1.4(ii) it follows that

$$\|u'' - u_\varepsilon''\|_\infty \le C(\|Au - Au_\varepsilon\|_\infty + \|u - u_\varepsilon\|_\infty) \,.$$

Since f_ε converges uniformly to f as $\varepsilon \to 0$, we obtain that u_ε'' converges uniformly to u'' as $\varepsilon \to 0$. Since each u_ε'' tends to 0 as $|x| \to \infty$, we conclude that $\lim_{|x|\to\infty} u'' = 0$. Therefore $u \in C^2(\overline{\mathbb{R}})$ and $bu' \in C(\overline{\mathbb{R}})$, i.e. $u \in D$.

Step 2: Now we consider a generic function a satisfying (H_1'). We endow the domain D with the canonical norm

$$\|u\|_D = \|u\|_{C^2(\overline{\mathbb{R}})} + \|bu'\|_\infty \ ,$$

and we apply the method of continuity to the operators

$$A_t := (ta + 1 - t)\frac{d^2}{dx^2} + b\frac{d}{dx}, \qquad t \in [0,1] \ .$$

Let $u \in D \subset D_{\max}(A)$. We observe that the constants c_1, c_2 in (H_2) and δ in (H_1') are independent of $t \in [0,1]$, so, applying Proposition 4.2.1 with A_t instead of A and letting $n \to \infty$ in estimates (4.2.1), (4.2.3), (4.2.4) and (4.2.5), we obtain for $\lambda > 0$

$$\|u\|_D \le C\|(\lambda - A_t)u\|_\infty \ ,$$

where the constant C is independent of $t \in [0,1]$.

Since $\lambda - A_0 : D \to C(\overline{\mathbb{R}})$ is bijective from step 1, we conclude that $\lambda - A_1 = \lambda - A$ is bijective, too. □

4.2.3 Examples

Assume for simplicity that $a \equiv 1$. If b is given by $b(x) = -|x|^r x$, with $r \ge 0$, then it is readily seen that the function $V(x) = 1 + x^2$ satisfies $AV \le \lambda V$ for $\lambda > 0$ sufficiently large. Then Proposition 4.1.1 holds and A endowed with its maximal domain is a generator both in $C_b(\mathbb{R})$ and in $C(\overline{\mathbb{R}})$. The corresponding semigroup is differentiable for $r > 0$, but never analytic in $C_b(\mathbb{R})$ (see [40, Propositions 4.4 and 3.5]). Since (H_1') and (H_2') are satisfied, Propositions 4.2.1 and 4.2.2 hold.

Condition (H_2) is satisfied by all polynomials and functions like e^P with P a polynomial. But if b oscillates too fast then (H_2) is not true and $D_{\max}(A)$ is not contained in general in $C_b^1(\mathbb{R})$ as shown in Example 2.4.7.

As far as hypothesis (H_2') is concerned, we remark that it holds for example for e^x but not for e^{x^2}. In this last situation we do not know whether Proposition 4.2.2 still holds.

Chapter 5

Invariant measures: main properties and some applications

In this last chapter we collect some known facts concerning invariant measures, most of which have been already used. Here we provide the relative proofs and we also show some other results which complete the exposition and make it clearer. Even though the subject has a certain relevance from a probabilistic point of view and can be treated by making use of probabilistic tools, our approach is purely analytic.

We start by introducing Feller semigroups in $C_b(\mathbb{R}^N)$. These are semigroups of positive contractions that are not strongly continuous in general, but continuous only with respect to the pointwise convergence. In our framework, we also assume that each operator of a Feller semigroup admits an integral representation and that it can be extended to the bounded Borel functions in $\mathbb{R}^N$. Then we give the definition of an invariant measure μ for a Feller semigroup (P_t). If one considers the underlying stochastic process $\{\xi_t\}$, μ can be interpreted as a stationary distribution for $\{\xi_t\}$. A quite general result concerning existence of invariant measures is given by Krylov and Bogoliubov (see Theorem 5.1.6). The main tool to prove it is a weak* compactness result for probability measures, which is due to Prokhorov. As a consequence, we infer that the semigroup (P_t) extends to a strongly continuous contractions semigroup in $L^p(\mathbb{R}^N, \mu)$, for all $1 \leq p < +\infty$. In order to deal with uniqueness, we have to require some regularity properties to (P_t), namely irreducibility and strong Feller property. Under these further assumptions, if an invariant measure exists, it is unique. To prove such a result we make use of some known facts concerning ergodic means of linear operators in Hilbert spaces and in particular the Von Neumann Theorem. Ergodicity of invariant measures concludes the first section.

In the second section we show how Feller semigroups arise naturally when one deals with a second order partial differential operator in $\mathbb{R}^N$ of the form

$$A = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i=1}^{N} F_i D_i.$$

The absence of a zero order term is a necessary condition for the existence of an invariant measure for the associated semigroup $T(t)$ (see Remark 5.2.12). The construction of $T(t)$ is based on an approximation argument which consists of finding a bounded classical solution u to the Cauchy problem

$$\begin{cases} u_t - Au = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = f(x) & x \in \mathbb{R}^N \end{cases}$$

as limit of solutions of parabolic problems in cylinders $(0,\infty) \times B_\rho$. The main tools to carry out this procedure are the classical maximum principle and interior Schauder estimates. Then one sets $u(t,x) = T(t)f(x)$. It turns out that $T(t)$ is a Feller semigroup in $C_b(\mathbb{R}^N)$, which is represented by a strictly positive integral kernel. Even though $T(t)$ is not strongly continuous we can associate a "weak" generator, which enjoys several classical properties of generators of strongly continuous semigroups. We show that assuming the existence of a Liapunov function, the weak generator coincides with the operator A endowed with the maximal domain in $C_b(\mathbb{R}^N)$ (see Proposition 5.2.3). Under the same assumption the semigroup $T(t)$ yields the *unique* bounded classical solution to the problem above. Concerning invariant measures, we establish two existence criteria, whose assumptions are expressed in terms of the coefficients of the operator A. The first is due to Khas'minskii and uses the existence of suitable supersolutions of the equation $\lambda u - Au = 0$ to apply the Krylov-Bogoliubov Theorem. The second is due to Varadhan and show directly the existence of an invariant measure for an operator of the form $\Delta - \langle D\Phi + G, D\rangle$, given by $\mu(dx) = e^{-\Phi}dx$.

The last section is devoted to the characterization of the domain of a class of elliptic operators in $L^p(\mathbb{R}^N,\mu)$. The main tools are the results of Chapter 1, where the same problem has been studied for differential operators in $L^p(\mathbb{R}^N)$. In fact, we show that the given operator on $L^p(\mathbb{R}^N,\mu)$ is similar to an operator in the unweighted space $L^p(\mathbb{R}^N)$ which satisfy the generation results of Chapter 1 that provide also an explicit description of the domain.

5.1 Existence and uniqueness of invariant measures for Feller semigroups

Throughout this section $(P_t)_{t\geq 0}$ is a family of linear operators in $C_b(\mathbb{R}^N)$, the space of all continuous and bounded functions in $\mathbb{R}^N$, satisfying the following properties:

(i) $P_0 = I, P_{t+s} = P_tP_s$, for all $t,s \geq 0$;

(ii) $P_tf \geq 0$ for all $t \geq 0$ and $f \in C_b(\mathbb{R}^N)$ with $f \geq 0$;

(iii) $\lim_{t\to 0} P_tf(x) = f(x)$, for all $x \in \mathbb{R}^N$ and $f \in C_b(\mathbb{R}^N)$;

(iv) $P_t\mathbb{1} = \mathbb{1}$, for all $t \geq 0$,

where $\mathbb{1}$ denotes the function with constant value 1. From *(ii)* and *(iv)* it follows that each operator P_t is a contraction. Indeed, for all $f \in C_b(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$

$$|P_tf(x)| \leq P_t|f|(x) \leq \|f\|_\infty P_t\mathbb{1} = \|f\|_\infty,$$

hence $\|P_tf\|_\infty \leq \|f\|_\infty$. Under a probabilistic point of view (P_t) is a *Feller semigroup* and condition *(iii)* represents the *stochastic continuity* of (P_t).

It is useful to make the following additional assumption:

(I) for all $t > 0$ and $x \in \mathbb{R}^N$ there exists a positive Borel measure $p_t(x,\cdot)$ such that $p_t(x,\mathbb{R}^N) = 1$ and

(5.1.1) $$(P_tf)(x) = \int_{\mathbb{R}^N} f(y)p_t(x,dy),$$

for all $f \in C_b(\mathbb{R}^N)$.

We set $p_0(x,\cdot) = \delta_x$, the Dirac measure concentrated at x.

We note that (5.1.1) makes sense also for bounded Borel functions. In particular, if Γ is a Borel set of $\mathbb{R}^N$ and χ_Γ is the corresponding characteristic function, then

(5.1.2) $$(P_t\chi_\Gamma)(x) = p_t(x,\Gamma), \qquad x \in \mathbb{R}^N,\ t \geq 0.$$

Then we also assume that

(II) for every bounded Borel function f and for every $t \geq 0$ the function $P_t f$ is still Borel measurable.

In general such a semigroup is not strongly continuous in $C_b(\mathbb{R}^N)$, a simple counterexample being the heat semigroup.

Definition 5.1.1 *A probability Borel measure μ is said to be invariant for (P_t) if*

$$\int_{\mathbb{R}^N} (P_t f)(x)\mu(dx) = \int_{\mathbb{R}^N} f(x)\mu(dx) \tag{5.1.3}$$

for all $t \geq 0$ and for every bounded Borel function f.

It is readily seen that μ is invariant if and only if

$$\mu(\Gamma) = \int_{\mathbb{R}^N} p_t(x, \Gamma)\mu(dx) \tag{5.1.4}$$

for any borelian set Γ. Indeed, if (5.1.3) holds, then (5.1.4) easily follows by taking $f = \chi_\Gamma$. Conversely, assume that (5.1.4) is true. This means that (5.1.3) is satisfied by any characteristic function. By linearity, one has the same formula also for simple functions. If f is a bounded nonnegative Borel function, then let (s_n) be an increasing sequence of simple functions such that $s_n(x)$ converges to $f(x)$, for every $x \in \mathbb{R}^N$. Writing (5.1.3) for each s_n and letting $n \to \infty$, by monotone convergence we get the identity for f. In the general case, it is sufficient to write $f = f^+ - f^-$.

From a probabilistic point of view, let us consider the stochastic process $\{\xi_t\}$ having $p_t(x, \Gamma)$ as transition functions. This means that $p_t(x, \Gamma)$ represents the probability that the process reaches Γ at the time t starting from x at $t = 0$. In order to determine completely the process, that is the probability that the process is in Γ at the time t, for any Γ and $t > 0$, it is sufficient to know the law $p_t(x, \Gamma)$ and the initial distribution σ, since, applying the formula of total probability, it holds

$$P(\xi_t \in \Gamma) = \int_{\mathbb{R}^N} p_t(x, \Gamma)\sigma(dx).$$

In this context, an invariant measure is a stationary distribution for the process, since

$$P(\xi_t \in \Gamma) = \int_{\mathbb{R}^N} p_t(x, \Gamma)\mu(dx) = \mu(\Gamma) = P(\xi_0 \in \Gamma),$$

for all $t \geq 0$.

A first basic result is the following.

Proposition 5.1.2 *Assume that μ is an invariant measure for (P_t). Then for all $p \in [1, +\infty[$, (P_t) can be extended uniquely to a strongly continuous contractions semigroup in $L^p(\mathbb{R}^N, \mu)$, still denoted by (P_t). Moreover, if $(A_p, D(A_p))$ is the generator of such a semigroup, then (5.1.3) is equivalent to have $\int_{\mathbb{R}^N} (A_p f)(x)\mu(dx) = 0$, for all $f \in D(A_p)$.*

Proof. Let $\varphi \in C_b(\mathbb{R}^N)$. From (5.1.1) and Hölder's inequality it follows that

$$|P_t\varphi(x)|^p \leq \int_{\mathbb{R}^N} |\varphi(y)|^p p_t(x, dy) = P_t(|\varphi|^p)(x).$$

Integrating with respect to μ, we get

$$\int_{\mathbb{R}^N} |P_t\varphi(x)|^p \mu(dx) \leq \int_{\mathbb{R}^N} P_t(|\varphi|^p)(x)\mu(dx) = \int_{\mathbb{R}^N} |\varphi(x)|^p \mu(dx).$$

Since $C_b(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N,\mu)$, P_t has a unique continuous extension to $L^p(\mathbb{R}^N,\mu)$, still denoted by P_t, such that $\|P_t\| \le 1$. The strong continuity of $P_t f$ in $L^p(\mathbb{R}^N,\mu)$ for $f \in C_b(\mathbb{R}^N)$ follows easily from property *(iii)* of (P_t) and the dominated convergence theorem. The general case can be treated by a standard density argument.

Let us prove the last assertion. If $f \in D(A_p)$ then $P_t f \in D(A_p)$, the map $t \to P_t f$ is of class $C^1([0,+\infty[; L^p(\mathbb{R}^N,\mu))$ and $\frac{d}{dt}P_t f = A_p P_t f = P_t A_p f$. Differentiating with respect to t the identity (5.1.3) we have

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{\mathbb{R}^N}(P_t f)(x)\mu(dx) = \int_{\mathbb{R}^N}\frac{d}{dt}(P_t f)(x)\mu(dx) = \int_{\mathbb{R}^N} P_t(A_p f)(x)\mu(dx) \\ &= \int_{\mathbb{R}^N}(A_p f)(x)\mu(dx). \end{aligned}$$

Conversely, if $\int_{\mathbb{R}^N}(A_p f)(x)\mu(dx) = 0$, for all $f \in D(A_p)$, then

$$\frac{d}{dt}\int_{\mathbb{R}^N}(P_t f)(x)\mu(dx) = \int_{\mathbb{R}^N} A_p(P_t f)(x)\mu(dx) = 0$$

and (5.1.3) holds in $D(A_p)$. Since $D(A_p)$ is dense in $L^p(\mathbb{R}^N,\mu)$, (5.1.3) is also true for $f \in L^p(\mathbb{R}^N,\mu)$. □

Now, our aim is to prove a quite general result on existence of invariant measures due to Krylov and Bogoliubov. Before stating it, we need to introduce some basic notions from measure theory.

We denote by $\mathcal{M}(\mathbb{R}^N)$ the set of all Borel probability measures on $\mathbb{R}^N$.

Definition 5.1.3 *A subset Λ of $\mathcal{M}(\mathbb{R}^N)$ is said to be relatively weakly compact if for any sequence (μ_n) in Λ there exist a subsequence (μ_{n_k}) and $\mu \in \mathcal{M}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} f(x)\mu_{n_k}(dx) \to \int_{\mathbb{R}^N} f(x)\mu(dx)$, for all $f \in C_b(\mathbb{R}^N)$. In this case, we say that μ_{n_k} weakly converges to μ.*

The set Λ is said to be tight if for all $\varepsilon > 0$ there is a compact set K_ε such that $\mu(K_\varepsilon) \ge 1-\varepsilon$, for all $\mu \in \Lambda$.

The Prokhorov theorem, proved below, shows that in fact the previous two notions are equivalent. Even though it holds in a general separable complete metric space, we state and prove it in $\mathbb{R}^N$, since this case is closer to our interests. We first need a lemma.

Lemma 5.1.4 *Let $\mu_n, \mu \in \mathcal{M}(\mathbb{R}^N)$ be such that μ_n converges weakly to μ. Then one has $\limsup_{n\to\infty}\mu_n(F) \le \mu(F)$, for every closed set F of $\mathbb{R}^N$ or, equivalently, $\liminf_{n\to\infty}\mu_n(G) \ge \mu(G)$, for every open set G of $\mathbb{R}^N$.*

Proof. Let F be a closed set and consider $F_\delta = \{x \in \mathbb{R}^N \mid \mathrm{dist}(x,F) < \delta\}$. Since F_δ is decreasing with respect to δ and $\cap_{\delta>0}F_\delta = F$, we have that $\lim_{\delta\to 0}\mu(F_\delta) = \mu(F)$. Therefore, given a positive ε there exists $\delta > 0$ such that $\mu(F_\delta) < \mu(F) + \varepsilon$. Let

$$\varphi(t) = \begin{cases} 1 & \text{if} \quad t \le 0, \\ 1-t & \text{if} \quad 0 \le t \le 1, \\ 0 & \text{if} \quad t \ge 1, \end{cases}$$

and define $f(x) = \varphi(\delta^{-1}\mathrm{dist}(x,F))$. Since f is nonnegative and assumes the value 1 on F, we have

$$\mu_n(F) = \int_F f(x)\mu_n(dx) \le \int_{\mathbb{R}^N} f(x)\mu_n(dx).$$

Since f vanishes outside F_δ and never exceeds 1

$$\int_{\mathbb{R}^N} f(x)\mu(dx) = \int_{F_\delta} f(x)\mu(dx) \le \mu(F_\delta).$$

Finally, since μ_n converges weakly to μ and $f \in C_b(\mathbb{R}^N)$ we deduce

$$\limsup_{n\to\infty} \mu_n(F) \le \lim_{n\to\infty} \int_{\mathbb{R}^N} f(x)\mu_n(dx) = \int_{\mathbb{R}^N} f(x)\mu(dx) \le \mu(F_\delta) < \mu(F) + \varepsilon.$$

Since ε was arbitrary, the thesis follows. A simple complementation argument proves the last assertion. □

Theorem 5.1.5 (Prokhorov) *A subset Λ in $\mathcal{M}(\mathbb{R}^N)$ is relatively weakly compact if and only if it is tight.*

Proof. Let $\overline{B}_n$ be the closed ball in $\mathbb{R}^N$ with radius $n \in \mathbb{N}$ and centered at zero. Assume first that Λ is tight and consider a sequence (μ_k) in Λ. We have to show that it is possible to extract a weakly convergent subsequence. Consider the restrictions $\big(\mu_{k|\overline{B}_1}\big)$. Since $C(\overline{B}_1)$ is separable, the weak* topology of the unit ball of the dual space (of all finite Borel measures) is metrizable. Hence, there exists a subsequence of $\big(\mu_{k|\overline{B}_1}\big)$ which converges weakly in $C(\overline{B}_1)^*$. By a diagonal procedure, since $(\overline{B}_n)$ is increasing, we can construct a subsequence (μ_{n_k}) such that $\int_{\overline{B}_n} f(x)\mu_{n_k}(dx)$ converges to $\int_{\overline{B}_n} f(x)\mu(dx)$ for all $f \in C(\overline{B}_n)$, and $n \in \mathbb{N}$ and for some positive Borel measure μ with $\mu(\mathbb{R}^N) \le 1$. Now, let $\varepsilon > 0$ be fixed. Since Λ is tight, there exists $r \in \mathbb{N}$ such that $\mu_{n_k}(\mathbb{R}^N \setminus \overline{B}_r) < \varepsilon$, for all $k \in \mathbb{N}$. If $n > r$, let $g \in C(\mathbb{R}^N)$ be such that $0 \le g \le 1$, $g \equiv 1$ in $\overline{B}_n \setminus B_{r+1}$ and supp $g \subset B_{n+1} \setminus \overline{B}_r \subset B_{n+1}$. Then

$$\mu(\overline{B}_n \setminus B_{r+1}) \le \int_{\mathbb{R}^N} g(x)\mu(dx) = \lim_{k\to+\infty} \int_{\mathbb{R}^N} g(x)\mu_{n_k}(dx) \le \limsup_{k\to+\infty} \mu_{n_k}(B_{n+1} \setminus \overline{B}_r) \le \varepsilon.$$

Letting $n \to +\infty$ we find that $\mu(\mathbb{R}^N \setminus B_{r+1}) \le \varepsilon$. Now, we can conclude. Indeed, if $f \in C_b(\mathbb{R}^N)$ then

$$\begin{aligned}\left|\int_{\mathbb{R}^N} f(x)\mu(dx) - \int_{\mathbb{R}^N} f(x)\mu_{n_k}(dx)\right| \;\le\; & \left|\int_{\overline{B}_{r+1}} f(x)\mu(dx) - \int_{\overline{B}_{r+1}} f(x)\mu_{n_k}(dx)\right| \\ & + \int_{\mathbb{R}^N\setminus\overline{B}_{r+1}} |f(x)|\mu(dx) \\ & + \int_{\mathbb{R}^N\setminus\overline{B}_{r+1}} |f(x)|\mu_{n_k}(dx).\end{aligned}$$

If $\varepsilon > 0$ is given, we first choose $r \in \mathbb{N}$ sufficiently large in such a way that $\mu(\mathbb{R}^N \setminus \overline{B}_{r+1}), \mu_{n_k}(\mathbb{R}^N \setminus \overline{B}_{r+1}) \le \varepsilon$ for all $k \in \mathbb{N}$. Then we choose $k \in \mathbb{N}$ large enough to make the first term in the right hand side smaller than ε. At the end we find

$$\left|\int_{\mathbb{R}^N} f(x)\mu(dx) - \int_{\mathbb{R}^N} f(x)\mu_{n_k}(dx)\right| \;\le\; \varepsilon + 2\varepsilon\|f\|_\infty$$

for k large. Thus the statement follows. In particular, taking $f = \mathbf{1}$, we have that μ is a probability measure, i.e. $\mu \in \mathcal{M}(\mathbb{R}^N)$.

Conversely, let us show that a relatively weakly compact set Λ must be tight. Consider the open ball B_n of $\mathbb{R}^N$ centered at zero and with radius $n \in \mathbb{N}$. For each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\nu(B_n) > 1 - \varepsilon$ for all $\nu \in \Lambda$. Otherwise, for each n we have $\nu_n(B_n) \le 1 - \varepsilon$, for some $\nu_n \in \Lambda$. By weakly compactness, there exist a subsequence (ν_{n_k}) and $\nu_0 \in \mathcal{M}(\mathbb{R}^N)$ such that ν_{n_k} converges to ν_0 weakly. From Lemma 5.1.4 it follows that $\nu_0(B_n) \le \liminf_{k\to\infty} \nu_{n_k}(B_n) \le \liminf_{k\to\infty} \nu_{n_k}(B_{n_k}) \le 1 - \varepsilon$, which is impossible, since $B_n \uparrow \mathbb{R}^N$. Thus, the closure of B_n is a compact set of $\mathbb{R}^N$ such that $\nu(\overline{B}_n) > 1 - \varepsilon$, for all $\nu \in \Lambda$. □

Now, we are ready to prove the announced result of existence of an invariant measure for the semigroup (P_t).

Theorem 5.1.6 (Krylov-Bogoliubov) *Assume that for some $T_0 > 0$ and $x_0 \in \mathbb{R}^N$ the set $\{\mu_T\}_{T>T_0}$, where*

$$\mu_T = \frac{1}{T}\int_0^T p_t(x_0, \cdot)dt,$$

is tight. Then there is an invariant measure μ for (P_t).

Proof. From Theorem 5.1.5 it follows that there exist a sequence (T_n) going to $+\infty$ and a probability measure μ such that $\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x)\mu_{T_n}(dx) = \int_{\mathbb{R}^N} f(x)\mu(dx)$, for all $f \in C_b(\mathbb{R}^N)$. Taking into account (5.1.1), this is equivalent to

$$\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}(P_t f)(x_0)dt = \int_{\mathbb{R}^N} f(x)\mu(dx). \tag{5.1.5}$$

Setting $f = P_s g$ we have

$$\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}(P_{t+s}g)(x_0)dt = \int_{\mathbb{R}^N}(P_s g)(x)\mu(dx),$$

for all $g \in C_b(\mathbb{R}^N)$. Now, we show that the limit at the left hand side above is equal to $\int_{\mathbb{R}^N} g(x)\mu(dx)$. We have in fact

$$\begin{aligned}\frac{1}{T_n}\int_0^{T_n}(P_{t+s}g)(x_0)dt &= \frac{1}{T_n}\int_s^{T_n+s}(P_t g)(x_0)dt\\ &= \frac{1}{T_n}\int_0^{T_n}(P_t g)(x_0)dt + \frac{1}{T_n}\int_{T_n}^{T_n+s}(P_t g)(x_0)dt\\ &\quad -\frac{1}{T_n}\int_0^s(P_t g)(x_0)dt.\end{aligned}$$

Since the last two terms above are infinitesimal and condition (5.1.5) holds, we find that (5.1.3) holds for $g \in C_b(\mathbb{R}^N)$. If g is a bounded Borel function in $\mathbb{R}^N$, then $g \in L^1(\mathbb{R}^N, \mu)$, hence, by density, there exists a sequence (g_n) in $C_b(\mathbb{R}^N)$ converging to g in $L^1(\mathbb{R}^N, \mu)$. By continuity, $P_t g_n$ converges to $P_t g$ in $L^1(\mathbb{R}^n, \mu)$ as well. Now, the thesis follows easily writing (5.1.3) for g_n and letting $n \to \infty$. □

In the next section we will see an application of this general result in the case of semigroups associated with differential operators.

Once that an invariant measure exists, one can ask whether it is unique or not. Such a problem requires more attention and suitable regularity properties for the semigroup (P_t) that we introduce below.

Definition 5.1.7 - *(P_t) is irreducible if for any ball $B(z, \varepsilon)$ one has $P_t\chi_{B(z,\varepsilon)}(x) > 0$ or, equivalently, $p_t(x, B(z, \varepsilon)) > 0$ for all $t > 0$ and $x \in \mathbb{R}^N$.*

- *(P_t) has the strong Feller property if for any bounded Borel function f and $t > 0$ we have $P_t f \in C_b(\mathbb{R}^N)$.*

- *P_t is called regular if all the probabilities $p_t(x, \cdot)$, $t > 0, x \in \mathbb{R}^N$, are equivalent, i.e. they are mutually absolutely continuous.*

It is clear that if (P_t) is irreducible, then it is *positivity improving*, in the sense that given a bounded Borel nonnegative function φ on $\mathbb{R}^N$ such that φ is strictly positive on some ball, then $P_t\varphi(x) > 0$, for all $t > 0$ and $x \in \mathbb{R}^N$. In this way, irreducibility says that a strong maximum principle holds. From a probabilistic point of view, this means that the underlying Markov process diffuses with infinite speed.

The main result concerning uniqueness is the following.

Theorem 5.1.8 *If (P_t) is regular then it has at most one invariant measure μ. Moreover, μ is equivalent to $p_t(x,\cdot)$, for all $t > 0, x \in \mathbb{R}^N$.*

Before proving the above theorem, we show an important tool to have regularity due to Khas'minskii.

Proposition 5.1.9 *If (P_t) is strong Feller and irreducible, then it is regular.*

Proof. It is sufficient to prove that all the probabilities $p_t(x,\cdot)$, $t > 0, x \in \mathbb{R}^N$, have the same null sets. This means that if Γ is a Borel set, then

(i) either $p_t(x,\Gamma) = 0$, for all $t > 0, x \in \mathbb{R}^N$,

(ii) or $p_t(x,\Gamma) > 0$, for all $t > 0, x \in \mathbb{R}^N$.

Assume that (i) does not hold. Then, there exist $x_0 \in \mathbb{R}^N$ and $t_0 > 0$ such that $P_{t_0}\chi_\Gamma(x_0) > 0$. By the strong Feller property, $P_{t_0}\chi_\Gamma \in C_b(\mathbb{R}^N)$, hence $P_{t_0}\chi_\Gamma(x) > 0$ for $x \in B(x_0,\delta)$. From the irreducibility and the semigroup law it follows $P_t\chi_\Gamma(x) > 0$ for all $x \in \mathbb{R}^N$, $t > t_0$, respectively. We claim that this holds for $t \le t_0$, too. If $t_1 < t \le t_0$ then there exists $x_1 \in \mathbb{R}^N$ such that $P_{t_1}\chi_\Gamma(x_1) > 0$ (otherwise $P_{t_0}\chi_\Gamma$ would be identically zero). By the same argument as before, we have $P_t\chi_\Gamma(x) > 0$ for all $x \in \mathbb{R}^N$ and the proof is concluded. □

In order to prove Theorem 5.1.8, we need some results about ergodic means of linear operators, in particular the Von Neumann Theorem. Let T be a linear bounded operator on a Hilbert space H and set

$$M_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k, \quad n \in \mathbb{N}.$$

Proposition 5.1.10 *Assume that there is a positive constant K such that $\|T^n\| \le K$, for all $n \in \mathbb{N}$. Then, the limit*

$$\lim_{n\to\infty} M_n x =: M_\infty x \tag{5.1.6}$$

exists for every $x \in H$. Moreover, $M_\infty^2 = M_\infty$, $M_\infty(H) = \ker(I-T)$, that is M_∞ is a projection on $\ker(I-T)$.

Proof. The stated limit trivially exists when $x \in \ker(I-T)$ or $x \in (I-T)(H)$. Indeed, in the first case we have $T^k x = x$ for all $k \in \mathbb{N}$, hence $M_n x = x$ for all $n \in \mathbb{N}$. In the second case, if $x = (I-T)y$, for some $y \in H$, taking into account the identity

$$M_n(I-T) = (I-T)M_n = \frac{1}{n}(I-T^n), \tag{5.1.7}$$

we have

$$\|M_n x\| = \left\| \frac{1}{n}(y - T^n y) \right\| \le \frac{1}{n}(\|y\| + K\|y\|),$$

and consequently $\lim_{n\to\infty} M_n x = 0$. Since $\|M_n x\| \le K\|x\|$, it follows that

$$\lim_{n\to\infty} M_n x = 0, \quad x \in \overline{(I-T)(H)}. \tag{5.1.8}$$

Now, let $x \in H$ be fixed. Then, there exist $y \in H$ and a subsequence $M_{n_k}x$ weakly convergent to y. Since T is bounded, $TM_{n_k}x$ converges weakly to Ty. On the other hand, from (5.1.7) it follows that $TM_nx = M_nx - \frac{1}{n}x + \frac{1}{n}T^nx$, hence $TM_{n_k}x$ converges weakly also to y. By uniqueness, $Ty = y$, i.e. $y \in \ker(I-T)$. Now we claim that M_nx converges to y. Since $y \in \ker(I-T)$, we have $M_ny = y$ and consequently

$$M_nx = M_ny + M_n(x-y) = y + M_n(x-y),$$

so that it is sufficient to show that $M_n(x-y)$ converges to zero. To this aim, recalling (5.1.8), we prove that $x - y \in \overline{(I-T)(H)}$. We have in fact $x - M_{n_k}x \in (I-T)(H)$, because

$$x - M_{n_k}x = \frac{1}{n_k}\sum_{j=0}^{n_k-1}(I-T^j)x = \frac{1}{n_k}(I-T)\sum_{j=0}^{n_k-1}(I+T+\cdots+T^{j-1})x$$

and $x - M_{n_k}x$ converges weakly to $x-y$. Since $(I-T)(H)$ is convex, its strong and weak closures coincide, hence $x - y \in \overline{(I-T)(H)}$. Therefore (5.1.6) is proved. As far as the last part of the statement is concerned, since $(I-T)M_n = M_n(I-T)$ converges to zero in the strong topology, we have $M_\infty = TM_\infty$ and therefore $M_\infty = T^kM_\infty$, for every $k \in \mathbb{N}$. This implies that $M_\infty = M_nM_\infty$, which yields, as $n \to \infty$, $M_\infty = M_\infty^2$, as required. □

Now we use this general result in our framework. More precisely, let μ be an invariant measure for the semigroup (P_t) and consider the Hilbert space $L^2(\mathbb{R}^N, \mu)$. Proposition 5.1.2 ensures that each P_t extends to a linear bounded operator in $L^2(\mathbb{R}^N, \mu)$ with $\|P_t\| \le 1$. Consider the ergodic mean

(5.1.9) $$M(T)\varphi = \frac{1}{T}\int_0^T P_s\varphi ds, \quad \varphi \in L^2(\mathbb{R}^N, \mu),\ T > 0.$$

Clearly, $M(T)$ is a linear operator and, by the Minkowski inequality, it is bounded in $L^2(\mathbb{R}^N, \mu)$:

$$\|M(T)\varphi\|_{L^2(\mathbb{R}^N,\mu)} \le \frac{1}{T}\int_0^T \|P_s\varphi\|_{L^2(\mathbb{R}^N,\mu)}ds \le \|\varphi\|_{L^2(\mathbb{R}^N,\mu)}.$$

Theorem 5.1.11 (Von Neumann) *For every $\varphi \in L^2(\mathbb{R}^N, \mu)$, the limit*

$$\lim_{T\to\infty} M(T)\varphi =: M_\infty\varphi$$

exists in $L^2(\mathbb{R}^N, \mu)$. Moreover $M_\infty = M_\infty^2$ and $M_\infty(L^2(\mathbb{R}^N, \mu)) = \Sigma$, where Σ is the set of all the stationary points of (P_t), i.e.

(5.1.10) $$\Sigma = \{\varphi \in L^2(\mathbb{R}^N, \mu) \mid P_t\varphi = \varphi,\ \mu \text{ a.e.},\ \forall t \ge 0\}.$$

Finally

$$\int_{\mathbb{R}^N} M_\infty\varphi(x)\mu(dx) = \int_{\mathbb{R}^N}\varphi(x)\mu(dx).$$

Proof. For all $T > 0$, let $n_T \in \mathbb{N}\cup\{0\}$ and $r_T \in [0,1[$ be the integer and fractional part of T, respectively. If $\varphi \in L^2(\mathbb{R}^N, \mu)$, then

$$\begin{aligned} M(T)\varphi &= \frac{1}{T}\sum_{k=0}^{n_T-1}\int_k^{k+1}P_s\varphi ds + \frac{1}{T}\int_{n_T}^T P_s\varphi ds = \frac{1}{T}\sum_{k=0}^{n_T-1}\int_0^1 P_{s+k}\varphi ds \\ &\quad + \frac{1}{T}\int_0^{r_T}P_{s+n_T}\varphi ds \\ &= \frac{n_T}{T}\frac{1}{n_T}\sum_{k=0}^{n_T-1}P_1^k(M(1)\varphi) + \frac{r_T}{T}P_1^{n_T}(M(r_T)\varphi). \end{aligned}$$

Since

$$\lim_{T\to\infty}\frac{n_T}{T}=1,\quad \lim_{T\to\infty}\frac{r_T}{T}=0,$$

letting $T\to\infty$ and recalling Proposition 5.1.10, we get that $M(T)\varphi$ has limit in $L^2(\mathbb{R}^N,\mu)$, say $M_\infty\varphi$. Let us prove that

$$M_\infty P_t = P_t M_\infty = M_\infty. \tag{5.1.11}$$

Given $t\ge 0$ we have

$$\begin{aligned} M_\infty P_t\varphi &= \lim_{T\to\infty}\frac{1}{T}\int_0^T P_{t+s}\varphi ds = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} P_s\varphi ds\\ &= \lim_{T\to\infty}\frac{1}{T}\Big(\int_0^T P_s\varphi ds + \int_T^{t+T} P_s\varphi ds - \int_0^t P_s\varphi ds\Big) = M_\infty\varphi. \end{aligned}$$

In a similar way, one can check that $P_t M_\infty\varphi = M_\infty\varphi$, so (5.1.11) is completely proved.

For all $\varphi\in L^2(\mathbb{R}^N,\mu)$, (5.1.11) implies that $M_\infty\varphi\in\Sigma$. Conversely, if $\varphi\in\Sigma$, then $M(T)\varphi=\varphi$ and consequently, taking the limit as $T\to\infty$, $M_\infty\varphi=\varphi\in M_\infty(L^2(\mathbb{R}^N,\mu))$. Since $P_tM_\infty = M_\infty P_t = M_\infty$, it follows that $M_\infty M(T) = M(T)M_\infty = M_\infty$, that yields $M_\infty = M_\infty^2$, letting $T\to\infty$. Finally, integrating (5.1.9) with respect to μ, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N}(M(T)\varphi)(x)\mu(dx) &= \frac{1}{T}\int_{\mathbb{R}^N}\int_0^T (P_s\varphi)(x)ds\,\mu(dx) = \frac{1}{T}\int_0^T\int_{\mathbb{R}^N}\varphi(x)\mu(dx)ds\\ &= \int_{\mathbb{R}^N}\varphi(x)\mu(dx). \end{aligned}$$

Letting $T\to\infty$ we conclude the proof. □

Remark 5.1.12 The Von Neumann Theorem gives information on the asymptotic behaviour of the semigroup (P_t), as $t\to\infty$. We note that, in general, the limit of $P_t\varphi(x)$ as $t\to\infty$ does not exist, if $\varphi\notin\Sigma$. For example in $\mathbb{R}^2$ consider the Cauchy problem

$$\begin{cases} \xi'(t) = -\eta(t)\\ \eta'(t) = \xi(t)\\ \xi(0)=x_1,\ \eta(0)=x_2 \end{cases}$$

Then $(\xi(t,x),\eta(t,x)) = (x_1\cos t - x_2\sin t, x_1\sin t + x_2\cos t)$, $t\ge 0$, $x\in\mathbb{R}^2$. The semigroup $P_t\varphi(x)=\varphi(\xi(t,x),\eta(t,x))$ is such that $\lim_{t\to\infty}P_t\varphi(x)$ exists only if $x=0$.

If (P_t) is regular, then it can be proved that $\lim_{t\to\infty}(P_t\varphi)(x) = \int_{\mathbb{R}^N}\varphi(y)\mu(dy)$, for all $\varphi\in L^2(\mathbb{R}^N,\mu)$ and $x\in\mathbb{R}^N$. This results, which is due to Doob, means that the underlying stochastic process is stable and $\int_{\mathbb{R}^N}\varphi(y)\mu(dy)$ is the equilibrium.

The next proposition contains the main properties of the subspace Σ. In particular it shows that Σ is a lattice. We remark that if $(A_2, D(A_2))$ is the generator of (P_t) in $L^2(\mathbb{R}^N,\mu)$, then $\Sigma=\ker A_2$.

Proposition 5.1.13 *Let $\varphi,\psi\in\Sigma$. Then the following assertions hold*

(i) $|\varphi|\in\Sigma$,

(ii) $\varphi^+,\varphi^-\in\Sigma$,

(iii) $\varphi\vee\psi, \varphi\wedge\psi\in\Sigma$,

(iv) for all $\lambda \in \mathbb{R}$, the characteristic function of the set $\{x \in \mathbb{R}^N \mid \varphi(x) > \lambda\}$ belongs to Σ.

Proof. Let us prove (i). By the positivity of (P_t) we infer $|\varphi(x)| = |P_t\varphi(x)| \le P_t|\varphi|(x)$. Assume, by contradiction, that there exists a Borel set Γ such that $\mu(\Gamma) > 0$ and $|\varphi(x)| < P_t|\varphi|(x)$, for $x \in \Gamma$. Then

$$\int_{\mathbb{R}^N} |\varphi(x)|\mu(dx) < \int_{\mathbb{R}^N} P_t|\varphi|(x)\mu(dx),$$

which contradicts the invariance of μ.

Assertions (ii) and (iii) follow easily from the identities

$$\varphi^+ = \frac{1}{2}(\varphi + |\varphi|), \quad \varphi^- = \frac{1}{2}(\varphi - |\varphi|),$$

$$\varphi \vee \psi = (\varphi - \psi)^+ + \psi, \quad \varphi \wedge \psi = -(\varphi - \psi)^+ + \varphi.$$

In order to prove (iv), it is sufficient to take $\lambda = 0$. Consider $\varphi_n(x) := (n\varphi^+ \wedge 1)(x)$. Then $\lim_{n\to\infty} \varphi_n(x) = \chi_{\{\varphi>0\}}(x)$ and $\varphi_n \in \Sigma$, by (ii) and (iii). By dominated convergence, $\chi_{\{\varphi>0\}}(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} P_t\varphi_n(x) = P_t\chi_{\{\varphi>0\}}(x)$. Hence the thesis follows. □

Now, we devote our attention to the case where the limit M_∞ provided by the Von Neumann Theorem is of a particular form.

Definition 5.1.14 *Let μ be an invariant measure for the semigroup (P_t). We say that μ is ergodic if*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P_t\varphi dt = \overline{\varphi},$$

in $L^2(\mathbb{R}^N, \mu)$, where $\overline{\varphi} = \int_{\mathbb{R}^N} \varphi(x)\mu(dx)$.

Proposition 5.1.15 *μ is ergodic if and only if the dimension of Σ, defined in (5.1.10), is equal to one.*

Proof. Assume that μ is ergodic. Then, from the Von Neumann Theorem it follows that $M_\infty\varphi = \overline{\varphi}$, for all $\varphi \in L^2(\mathbb{R}^N, \mu)$. Since M_∞ is a projection on Σ, it turns out that Σ is one dimensional.

Conversely, assume that the dimension of Σ is one. Then, there exists a linear continuous functional f on $L^2(\mathbb{R}^N, \mu)$ such that $M_\infty\varphi = f(\varphi)1 = f(\varphi)$, for all $\varphi \in L^2(\mathbb{R}^N, \mu)$. Moreover, the Riesz-Frechèt Theorem yields a function $\varphi_0 \in L^2(\mathbb{R}^N, \mu)$ satisfying $f(\varphi) = \int_{\mathbb{R}^N} \varphi(x)\varphi_0(x)\mu(dx)$. Integrating this identity with respect to μ and recalling the invariance of M_∞ (see Theorem 5.1.11) we find

$$\int_{\mathbb{R}^N} M_\infty\varphi(x)\mu(dx) = \int_{\mathbb{R}^N} \varphi(x)\mu(dx) = \int_{\mathbb{R}^N} \varphi(x)\varphi_0(x)\mu(dx),$$

for all $\varphi \in L^2(\mathbb{R}^N, \mu)$. This leads to $\varphi_0 = 1$ and consequently $M_\infty\varphi = f(\varphi) = \overline{\varphi}$. □

Let μ be an invariant measure for (P_t). A Borel set Γ is said to be *invariant* for the semigroup, if its characteristic function χ_Γ belongs to Σ. Γ is said to be *trivial* if $\mu(\Gamma)$ is equal to 0 or 1.

The next result is a characterization of the ergodicity of an invariant measure in terms of invariant sets.

Proposition 5.1.16 *Let μ be an invariant measure for (P_t). Then μ is ergodic if and only if each invariant set is trivial.*

Proof. Assume that μ is ergodic and let Γ be an invariant set. Then χ_Γ must be μ-a.e. constant in order to keep Σ one dimensional.

Conversely, suppose that all the invariant sets are trivial and, by contradiction, that μ is not ergodic. Then there exists a nonconstant function $\varphi \in \Sigma$. Therefore, for some $\lambda \in \mathbb{R}$ the set $\{\varphi > \lambda\}$, which is invariant by Proposition 5.1.13, is not trivial. □

An interesting relationship between uniqueness and ergodicity of an invariant measure is contained in the next proposition.

Proposition 5.1.17 *Suppose that there exists a unique invariant measure μ for (P_t). Then it is ergodic.*

Proof. Assume by contradiction that μ is not ergodic. Then there exists a non trivial invariant set Γ. Define

$$\mu_\Gamma(A) = \frac{\mu(A\cap\Gamma)}{\mu(\Gamma)},$$

for any A Borel set. Since Γ is not trivial, $\mu_\Gamma(\Gamma) \neq \mu(\Gamma)$, hence μ and μ_Γ are distinct. We claim that μ_T is an invariant measure for (P_t). To this aim, it is sufficient to show that

$$\mu_\Gamma(A) = \int_{\mathbb{R}^N} p_t(x,A)\mu_\Gamma(dx),$$

for any Borel set A (see (5.1.4)) or, equivalently, that

$$\mu(A\cap\Gamma) = \int_\Gamma p_t(x,A)\mu(dx).$$

Since Γ is invariant, for all $t \geq 0$ we have $P_t\chi_\Gamma = \chi_\Gamma$ μ-a.e. Then $p_t(x,\Gamma) = \chi_\Gamma(x)$ μ-a.e. and, as a consequence, $p_t(x,A\cap\Gamma) = 0$, μ-a.e. in Γ^c, since $p_t(x,A\cap\Gamma) \leq p_t(x,\Gamma)$. Analogously, $P_t\chi_{\Gamma^c} = \chi_{\Gamma^c}$ μ-a.e., because $P_t\mathbb{1} = \mathbb{1}$. Then $p_t(x,\Gamma^c) = \chi_{\Gamma^c}(x)$ and therefore $p_t(x,A\cap\Gamma^c) = 0$, μ-a.e. in Γ. So we have

$$\begin{aligned}\int_\Gamma p_t(x,A)\mu(dx) &= \int_\Gamma p_t(x,A\cap\Gamma)\mu(dx) + \int_\Gamma p_t(x,A\cap\Gamma^c)\mu(dx)\\ &= \int_\Gamma p_t(x,A\cap\Gamma)\mu(dx) = \int_{\mathbb{R}^N} p_t(x,A\cap\Gamma)\mu(dx)\\ &= \mu(A\cap\Gamma).\end{aligned}$$

Thus, we have established that μ_T is an invariant measure for (P_t) and this clearly contradicts the uniqueness of μ. □

Lemma 5.1.18 *Let μ, ν be two ergodic invariant measures of (P_t), with $\mu \neq \nu$. Then μ and ν are singular.*

Proof. Let Γ be a Borel set such that $\mu(\Gamma) \neq \nu(\Gamma)$. From the Von Neumann Theorem 5.1.11, it follows that $\lim_{T\to\infty}\frac{1}{T}\int_0^T P_s\varphi ds = M_\infty\varphi$ in $L^2(\mathbb{R}^N,\mu)$. In particular, choosing $\varphi = \chi_\Gamma$, we find that there exist a sequence $T_n \to \infty$ and a Borel set M such that $\mu(M) = 1$ and

$$\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} P_s\chi_\Gamma(x)ds = M_\infty\chi_\Gamma(x), \quad \forall x \in M.$$

Since μ is ergodic, $M_\infty\chi_\Gamma = \mu(\Gamma)$, hence

$$\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} P_s\chi_\Gamma(x)ds = \mu(\Gamma), \quad \forall x \in M.$$

Analogously, one can check that there exist a Borel set N with $\nu(N) = 1$ such that

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} P_s\chi_\Gamma(x)ds = \nu(\Gamma), \quad \forall x \in N$$

(without loss of generality we assume that the sequence T_n is the same for ν). Since $\mu(\Gamma) \neq \nu(\Gamma)$, we have that $M \cap N = \emptyset$, hence μ and ν are singular. □

Finally, we are ready to prove Theorem 5.1.8.

Proof of Theorem 5.1.8. Let μ be an invariant measure for the semigroup (P_t). First we show that μ is equivalent to $p_t(x, \cdot)$, for all $t > 0$ and $x \in \mathbb{R}^N$. Let $t_0 > 0$ and $x_0 \in \mathbb{R}^N$ be fixed. By identity (5.1.4) we have

$$\mu(\Gamma) = \int_{\mathbb{R}^N} p_t(x, \Gamma)\mu(dx), \tag{5.1.12}$$

for any Borel set Γ. Let Γ be such that $p_{t_0}(x_0, \Gamma) = 0$. Then, since (P_t) is regular, $p_t(x, \Gamma) = 0$, for all $t > 0$ and $x \in \mathbb{R}^N$. From the integral representation above it follows that $\mu(\Gamma) = 0$. Therefore $\mu << p_{t_0}(x_0, \cdot)$. Conversely, assume that $\mu(\Gamma) = 0$. Then, again from (5.1.12) $p_t(x, \Gamma) = 0$ for some x, hence for every x by the regularity of (P_t). As a consequence, $p_{t_0}(x_0, \cdot) << \mu$.

Let us prove that μ is ergodic. Using Proposition 5.1.16, we show that every invariant set is trivial. Let Γ be a Borel set such that $P_t\chi_\Gamma = \chi_\Gamma$, μ-a.e. Then $p_t(x, \Gamma) = \chi_\Gamma(x)$, μ-a.e. The regularity of (P_t) implies that either $p_t(x, \Gamma) = 0$ μ-a.e. or $p_t(x, \Gamma) = 1$ μ-a.e. From (5.1.12) it follows that $\mu(\Gamma)$ is either 0 or 1, as claimed.

If ν is another invariant measure, then the argument above proves that ν is equivalent to $p_t(x, \cdot)$, for all $t > 0$, $x \in \mathbb{R}^N$ and that ν is ergodic. It turns out that μ and ν are equivalent. If they were different, then Proposition 5.1.18 would imply that μ and ν are singular, which is a contradiction. We conclude that $\mu = \nu$, as stated. □

5.2 Feller semigroups and differential operators

Feller semigroups naturally arise when one deal with second order elliptic operators in spaces of continuous functions. Suppose we are given a second order partial differential operator

$$Au = \sum_{i,j=1}^{N} q_{ij}D_{ij}u + \sum_{i=1}^{N} F_iD_iu, \tag{5.2.1}$$

whose coefficients are locally α-Hölder continuous in $\mathbb{R}^N$, $0 < \alpha < 1$, and satisfy

$$q_{ij} = q_{ji}, \quad \sum_{i,j=1}^{N} q_{ij}(x)\xi_i\xi_j \geq \nu(x)|\xi|^2, \quad \text{for all } x, \xi \in \mathbb{R}^N,$$

with $\inf_K \nu(x) > 0$, for any compact set K of $\mathbb{R}^N$. Under these assumptions it is always possible to associate with A a semigroup $T(t)$ in $C_b(\mathbb{R}^N)$, which yields a bounded classical solution to the parabolic problem

$$\begin{cases} u_t - Au = 0 & \text{in } (0, \infty) \times \mathbb{R}^N \\ u(0, x) = f(x) & x \in \mathbb{R}^N \end{cases} \tag{5.2.2}$$

for every $f \in C_b(\mathbb{R}^N)$. The construction of such a semigroup is based on an approximation procedure which consists of finding a solution to problem (5.2.2) as limit of solutions of parabolic problems in cylinders $(0, \infty) \times B_\rho$, where A is uniformly elliptic. We have already used this construction in Chapters 2, 3 to solve parabolic problems with Neumann or Dirichlet boundary conditions. Here the situation is easier, since we do not have to take any boundary into consideration.

For the sake of completeness, we briefly recall the construction of $T(t)$. Then we give sufficient conditions for the existence of an invariant measure μ for $(T(t))$. We will see also that μ is unique and absolutely continuous with respect to the Lebesgue measure.

5.2.1 Preliminary results

We refer to [38] and the references therein for more details on this argument and the proofs of the results that we are going to show.

Let us fix a ball B_ρ and consider the domain

$$D_\rho(A) = \{u \in C(\overline{B}_\rho) \cap W^{2,p}(B_\rho) \text{ for all } p < \infty \mid u_{|\partial B_\rho} = 0 \text{ and } Au \in C(\overline{B}_\rho)\}. \tag{5.2.3}$$

Then the operator $(A, D_\rho(A))$ generates an analytic semigroup $(T_\rho(t))$ of positive contractions in the space $C(\overline{B}_\rho)$ (see [32, Corollary 3.1.21]) and, for every $f \in C(\overline{B}_\rho)$ the function $u_\rho(t,x) = T_\rho(t)f(x)$ satisfies

$$\begin{cases} D_t u_\rho(t,x) - Au_\rho(t,x) = 0 & \text{in } (0,\infty) \times B_\rho \\ u_\rho(0,x) = f(x) & x \in B_\rho \\ u_\rho(t,x) = 0 & \text{in } (0,\infty) \times \partial B_\rho. \end{cases} \tag{5.2.4}$$

Since the domain $D_\rho(A)$ is not dense in $C(\overline{B}_\rho)$, strong continuity at 0 fails: in fact, $T_\rho(t)f$ converges uniformly to f in $\overline{B}_\rho$, as $t \to 0$, if and only if f vanishes on ∂B_ρ. However, $T_\rho(t)f$ converges to f uniformly in $\overline{B}_{\rho'}$, as $t \to 0$, for every $\rho' < \rho$, hence pointwise in B_ρ. For all $\rho > 0$, there exists a kernel $p_\rho(t,x,y)$ that represents the semigroup $(T_\rho(t))$:

$$T_\rho(t)f(x) = \int_{B_\rho} p_\rho(t,x,y)f(y)dy,$$

for all $f \in C(\overline{B}_\rho)$. Moreover, $p_\rho(t,x,y) > 0$ for $t > 0, x, y \in B_\rho$, $p_\rho(t,x,y) = 0$ for $t > 0, x \in \partial B_\rho, y \in B_\rho$ and for every $y \in B_\rho$, $0 < \varepsilon < \tau$ it belongs to $C^{1+\alpha/2,2+\alpha}((\varepsilon,\tau) \times B_\rho)$ as function of (t,x), and satisfies $D_t p_\rho - Ap_\rho = 0$. If f is positive then $T_\rho(t)f$ is positive and $\|T_\rho(t)f\|_\infty \le \|f\|_\infty$. For all the properties of p_ρ we refer to [24, Chapter 3, Section 7].

An argument based on the classical maximum principle shows that for every $f \in C_b(\mathbb{R}^N)$ the limit $\lim_{\rho\to\infty} T_\rho(t)f$ exists uniformly on compact sets in $\mathbb{R}^N$ and defines a semigroup $(T(t))$ of positive contractions in $C_b(\mathbb{R}^N)$. The main properties of $(T(t))$ are listed in the proposition below.

Proposition 5.2.1 *For every $f \in C_b(\mathbb{R}^N)$, the function $u(t,x) = T(t)f(x)$ belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,\infty) \times \mathbb{R}^N)$ and satisfies the equation*

$$D_t u - Au = 0.$$

Moreover, $T(t)f$ can be represented in the form

$$T(t)f(x) = \int_{\mathbb{R}^N} f(y)p(t,x,y)dy, \tag{5.2.5}$$

where p is a positive function. For almost all $y \in \mathbb{R}^N$, $p(t,x,y)$, as function of (t,x), belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,\infty) \times \mathbb{R}^N)$ and solves $D_t p = Ap$. Finally, $T(t)f$ converges to f uniformly on compact sets of $\mathbb{R}^N$, as $t \to 0$, hence u belongs to $C([0,+\infty[\times\mathbb{R}^N)$ and solves (5.2.2).

We note that the previous proposition establishes, in particular, an integral representation for the semigroup $T(t)$ similar to (5.1.1). Here we get more, since all the measures are absolutely continuous with respect to the Lebesgue measure.

We note also that, since $(T(t))$ is contractive, we have $T(t)\mathbf{1} = \int_{\mathbb{R}^N} p(t,x,y)dy \le 1$ and there are cases where the strict inequality holds. We will see later a necessary and sufficient condition to have $T(t)\mathbf{1} = \mathbf{1}$ (see Proposition 5.2.7). Finally we observe that, as in the general setting, formula (5.2.5) makes sense also for bounded Borel functions.

As a consequence of the results above, we can prove that $(T(t))$ is irreducible and has the strong Feller property (see Definition 5.1.7).

Proposition 5.2.2 *The semigroup $T(t)$ is irreducible and has the strong Feller property.*

PROOF. The irreducibility of $T(t)$ is a consequence of the integral representation (5.2.5) and the positivity of the kernel p. Concerning the strong Feller property, let f be a Borel function and consider a bounded sequence (f_n) in $C_b(\mathbb{R}^N)$ such that $f_n(x)$ converges to $f(x)$, for almost all $x \in \mathbb{R}^N$. From (5.2.5) and the dominated convergence theorem it follows that $T(t)f_n(x)$ converges to $T(t)f(x)$ for all $x \in \mathbb{R}^N$, $t > 0$. Using the interior Schauder estimates (see [30, Theorem IV.10.1]), it turns out that for every fixed $t > 0$, $\rho > 0$ and for all $n \in \mathbb{N}$

$$\|T(t)f_n\|_{C^1(\overline{B}_\rho)} \le C\|T(t)f_n\|_{C(\overline{B}_{2\rho})} \le C\|f_n\|_\infty \le C',$$

with $C' > 0$ independent of n. This implies, by a compactness argument, that there exists a subsequence of $T(t)f_n$ which converges to $T(t)f$ uniformly on compact sets. Therefore $T(t)f \in C_b(\mathbb{R}^N)$. □

Even though $(T(t))$ is not strongly continuous one can define its generator following the approach of [48]. More precisely, let us introduce the operator

$$\begin{aligned}
\widehat{D} &= \Big\{ f \in C_b(\mathbb{R}^N) : \sup_{t\in(0,1)} \frac{\|T(t)f - f\|}{t} < \infty \text{ and } \exists g \in C_b(\mathbb{R}^N) \text{ such that} \\
&\qquad \lim_{t\to 0} \frac{(T(t)f)(x) - f(x)}{t} = g(x), \quad \forall\, x \in \mathbb{R}^N \Big\} \\
\widehat{A}f(x) &= \lim_{t\to 0} \frac{(T(t)f)(x) - f(x)}{t}, \qquad f \in \widehat{D},\ x \in \mathbb{R}^N
\end{aligned}$$

$(\widehat{A}, \widehat{D})$ is called the *weak generator* of $(T(t))$. It enjoys several properties which are well-known for generators of strongly continuous semigroups. In particular, if $f \in \widehat{D}$, then $T(t)f \in \widehat{D}$ and $\widehat{A}T(t)f = T(t)\widehat{A}f$, for all $t \ge 0$. Moreover, the map $t \to T(t)f(x)$ is continuously differentiable in $[0,\infty[$ for all $x \in \mathbb{R}^N$ and $D_tT(t)f(x) = T(t)\widehat{A}f(x)$. Besides, one can prove that $(0,+\infty) \subset \rho(\widehat{A})$, $\|R(\lambda,\widehat{A})\| \le 1/\lambda$ and

$$(R(\lambda,\widehat{A})f)(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)\,dt, \qquad f \in C_b(\mathbb{R}^N),\ x \in \mathbb{R}^N. \tag{5.2.6}$$

The notion of weak generator is quite general and it allows to study a large class of semigroups on $C_b(E)$ (the so called π-semigroups), for some separable metric space E. In our situation, since the semigroup $(T(t))$ has been constructed starting from a differential operator, it is interesting to point out the relationship existing between $\widehat{A}$ and our operator A. In fact, it can be proved that $\widehat{A}$ is a restriction of A, in the sense specified by the following proposition.

Proposition 5.2.3 *Let $D_{\max}(A)$ be the maximal domain of A in $C_b(\mathbb{R}^N)$:*

$$D_{\max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{\rm loc}(\mathbb{R}^N) \text{ for all } p < \infty \mid Au \in C_b(\mathbb{R}^N)\}. \tag{5.2.7}$$

Then $\widehat{D} \subset D_{\max}(A)$ and $\widehat{A}f = Af$, for $f \in \widehat{D}$. The equality $\widehat{D} = D_{\max}(A)$ holds if and only if $\lambda - A$ is injective on $D_{\max}(A)$ for some (hence for all) $\lambda > 0$.

Proof. Let $\lambda > 0$ be fixed. If $u \in \widehat{D}$, then there exists a unique $f \in C_b(\mathbb{R}^N)$ such that $u = R(\lambda, \widehat{A})f$. We claim that u belongs to $D_{\max}(A)$ and solves the equation $\lambda u - Au = f$. From identity (5.2.6) and the construction of the semigroup $T(t)$, it follows that for every $x \in \mathbb{R}^N$

$$u(x) = \int_0^{+\infty} e^{-\lambda t} \lim_{\rho \to +\infty} (T_\rho(t)f)(x)dt = \lim_{\rho \to +\infty} \int_0^{+\infty} e^{-\lambda t}(T_\rho(t)f)(x)dt,$$

where the last equality follows from the dominated convergence theorem. For each $\rho > 0$ we have

$$\int_0^{+\infty} e^{-\lambda t}(T_\rho(t)f)(x)dt = (R(\lambda, A_\rho)f)(x) =: u_\rho(x), \tag{5.2.8}$$

where A_ρ means the operator A endowed with the domain $D_\rho(A)$ defined in (5.2.3). Therefore the function $u_\rho \in D_\rho(A)$ satisfies

$$\begin{cases} \lambda u_\rho - Au_\rho = f & \text{in } B_\rho, \\ u_\rho = 0 & \text{on } \partial B_\rho. \end{cases}$$

Since $T_\rho(t)$ is contractive, we have

$$\|u_\rho\|_\infty \le \frac{\|f\|_\infty}{\lambda}. \tag{5.2.9}$$

Hence, by difference, we obtain

$$\|Au_\rho\|_\infty \le 2\|f\|_\infty. \tag{5.2.10}$$

For every $R > 0$, the classical interior L^p estimates (see [26, Theorem 9.11]) yield a constant $C > 0$ depending on p, R, N and the operator A such that

$$\|u_\rho\|_{W^{2,p}(B_R)} \le C(\|Au_\rho\|_{L^p(B_{2R})} + \|u_\rho\|_{L^p(B_{2R})}), \tag{5.2.11}$$

for all $\rho > 2R$. From (5.2.9) and (5.2.10) it follows that

$$\|u_\rho\|_{W^{2,p}(B_R)} \le C_1\|f\|_\infty, \tag{5.2.12}$$

with C_1 depending on R, p, N, λ, the operator A but independent of ρ. Choosing $p > N$, (5.2.12) gives a uniform estimate of (u_ρ) in $C^1(\overline{B}_R)$ which allows to apply Ascoli's Theorem and to deduce that a subsequence (u_{ρ_n}) of (u_ρ) converges uniformly to u on compact subsets of $\mathbb{R}^N$. From the equation $\lambda u_{\rho_n} - Au_{\rho_n} = f$ it follows that Au_{ρ_n} converges uniformly on compact sets as well. Therefore, applying (5.2.11) to the difference $u_{\rho_n} - u_{\rho_m}$, we find that u_{ρ_n} converges to u in $W^{2,p}_{\rm loc}(\mathbb{R}^N)$, hence $u \in W^{2,p}_{\rm loc}(\mathbb{R}^N)$. Taking the limit in the equation satisfied by u_{ρ_n} we deduce that $\lambda u - Au = f$ and, as a consequence, $u \in D_{\max}(A)$. Since $\lambda u - Au = f = \lambda u - \widehat{A}u$, we have $Au = \widehat{A}u$ and the first assertion is proved. As regards the second statement, clearly $\lambda - A$ is bijective from $\widehat{D}$ onto $C_b(\mathbb{R}^N)$. Assume that it is injective also in $D_{\max}(A)$. If $u \in D_{\max}(A)$, there exists $v \in \widehat{D}$ such that $\lambda v - Av = \lambda u - Au$. Therefore $u - v$ belongs to $D_{\max}(A)$ and $\lambda(u - v) - A(u - v) = 0$. From the injectivity of $\lambda - A$ on $D_{\max}(A)$ we deduce that $u = v$ and, consequently, $\widehat{D} = D_{\max}(A)$. □

As a consequence of Proposition 5.2.3, we can write $R(\lambda, A)$ instead of $R(\lambda, \widehat{A})$ (keeping the fact that $R(\lambda, A)$ maps $C_b(\mathbb{R}^N)$ onto $\widehat{D}$ and not onto $D_{\max}(A)$, in general). It is worth stating explicitly a result included in the proof of the above proposition.

Corollary 5.2.4 *For all $\lambda > 0$ and $f \in C_b(\mathbb{R}^N)$, there exists u belonging to $D_{\max}(A)$ such that $\lambda u - Au = f$ and $\|u\|_\infty \le \frac{1}{\lambda}\|f\|_\infty$. Moreover, $u \ge 0$ if $f \ge 0$.*

Remark 5.2.5 Let us consider $f \in C_b(\mathbb{R}^N)$, $f \ge 0$. Then $R(\lambda, A)f$ is a positive solution in $D_{\max}(A)$ of the equation $\lambda u - Au = f$, not unique, in general. In any case, it is the minimal among all the positive solutions of the same equation in $D_{\max}(A)$. Indeed, let $w \in D_{\max}(A)$ be positive and such that $\lambda w - Aw = f$. The function $u_\rho - w \in W^{2,p}(B_\rho) \cap C(\overline{B}_\rho)$, with u_ρ given by (5.2.8), is such that $A(u_\rho - w) \in C(\overline{B}_\rho)$ and satisfies

$$\begin{cases} \lambda(u_\rho - w) - A(u_\rho - w) = 0 & \text{in } B_\rho, \\ u_\rho - w \le 0 & \text{on } \partial B_\rho. \end{cases}$$

We claim that $u_\rho - w \le 0$ in B_ρ. Since $u_\rho - w \in C(\overline{B}_\rho)$, there exists a maximum point $x_0 \in \overline{B}_\rho$. Assume by contradiction that $u_\rho(x_0) - w(x_0) > 0$. Then $x_0 \in B_\rho$. From Corollary A.0.9 we deduce that $A(u_\rho - w)(x_0) \le 0$ and therefore

$$0 = \lambda(u_\rho - w)(x_0) - A(u_\rho - w)(x_0) \ge \lambda(u_\rho - w)(x_0) > 0,$$

which is impossible. Hence $u_\rho(x) - w(x) \le u_\rho(x_0) - w(x_0) \le 0$ for every $x \in B_\rho$. Letting $\rho \to +\infty$ and recalling that $\lim_{\rho \to +\infty} u_\rho = R(\lambda, A)f$, we have $R(\lambda, A)f \le w$, as claimed.

A sufficient condition for the injectivity of $\lambda - A$ on $D_{\max}(A)$ is the existence of a *Liapunov function*, i.e. a function $V \in C^2(\mathbb{R}^N)$, such that $\lim_{|x| \to +\infty} V(x) = +\infty$ and $\lambda V - AV \ge 0$. This assumption leads to growth conditions on the coefficients of A. Indeed, in order to find a Liapunov function, one often considers some simple function V which goes to $+\infty$ as $|x| \to +\infty$, plugs it into $\lambda - A$ and imposes that $\lambda V - AV \ge 0$. By taking for example $V(x) = 1 + |x|^2$, one requires that

$$\sum_{i=1}^N q_{ii}(x) + \sum_{i=1}^N F_i(x) x_i \le \lambda(1 + |x|^2), \quad x \in \mathbb{R}^N.$$

If $\lambda - A$ is injective on $D_{\max}(A)$ then the semigroup $T(t)$ yields the *unique* bounded classical solution to problem (5.2.2).

Proposition 5.2.6 *Suppose that $\lambda - A$ is injective on $D_{\max}(A)$ for some $\lambda > 0$ and let $w \in C^{1,2}((]0, \tau] \times \mathbb{R}^N) \cap C([0, \tau] \times \mathbb{R}^N)$ be a bounded solution of problem (5.2.2). Then $w(t, x) = T(t)f(x)$.*

Proof. By linearity, it is sufficient to prove the statement in the case where w solves problem (5.2.2) with $f = 0$. For $0 < \varepsilon < t \le \tau$ and $x \in \mathbb{R}^N$ we have

$$w(t, x) - w(\varepsilon, x) = \int_\varepsilon^t \frac{d}{ds} w(s, x) ds = \int_\varepsilon^t Aw(s, x) ds = A \int_\varepsilon^t w(s, x) ds. \tag{5.2.13}$$

Since $(A, D_{\max}(A))$ is the weak generator of $T(t)$ (see Proposition 5.2.3), from [48, Proposition 3.4] it follows that it is closed with respect to the π-convergence, defined as

$$f_n \xrightarrow{\pi} f \quad \Longleftrightarrow \quad f_n(x) \to f(x) \quad \text{and} \quad \|f_n\|_\infty \le C.$$

Since $w \in C([0, \tau] \times \mathbb{R}^N)$, we have that $\int_\varepsilon^t w(s, x) ds$ converges to $\int_0^t w(s, x) ds$ as $\varepsilon \to 0$, for every $x \in \mathbb{R}^N$. Moreover $\left\| \int_\varepsilon^t w(s, \cdot) ds \right\|_\infty \le \|w\|_\infty t$, which implies that

$$\int_\varepsilon^t w(s, \cdot) ds \xrightarrow{\pi} \int_0^t w(s, \cdot) ds, \quad \text{as } \varepsilon \to 0.$$

From (5.2.13) we infer that $A\int_\varepsilon^t w(s,x)ds$ converges to $w(t,x)$ when ε goes to zero, for every $x\in\mathbb{R}^N$, and

$$\left\|A\int_\varepsilon^t w(s,\cdot)ds\right\|_\infty = \|w(t,\cdot)-w(\varepsilon,\cdot)\|_\infty \le 2\|w\|_\infty,$$

i.e. $A\int_\varepsilon^t w(s,\cdot)ds \xrightarrow{\pi} w(t,\cdot)$. The closedness of $(A, D_{\max}(A))$ yields

$$\int_0^t w(s,\cdot)ds \in D_{\max}(A) \qquad \text{and} \qquad w(t,x) = A\int_0^t w(s,x)ds, \tag{5.2.14}$$

for $t\le\tau$. Setting $w(\tau+s,x) = T(s)w(\tau,\cdot)(x)$ we obtain a bounded function w which belongs to $C([0,+\infty[\times\mathbb{R}^N)$ and such that (5.2.14) holds for every $t>0$. Indeed, it is clear that the extended function is bounded in $[0,\infty[\times\mathbb{R}^N$. As regards the continuity, by the semigroup law, it is sufficient to show that if $s_n\to 0$ and $x_n\to x$ then $w(\tau+s_n,x_n)\to w(\tau,x)$. To this aim we observe that

$$\begin{aligned}|w(\tau+s_n,x_n)-w(\tau,x)| &= |T(s_n)w(\tau,\cdot)(x_n)-w(\tau,x)|\\ &\le |T(s_n)w(\tau,\cdot)(x_n)-w(\tau,x_n)|+|w(\tau,x_n)-w(\tau,x)|\\ &\le \sup_{y\in K}|T(s_n)w(\tau,\cdot)(y)-w(\tau,y)|+|w(\tau,x_n)-w(\tau,x)|,\end{aligned}$$

where K is a compact subset of $\mathbb{R}^N$ such that $x_n\in K$ for all $n\in\mathbb{N}$. Since the semigroup $T(t)$ is strongly continuous with respect to the uniform convergence on compact sets (see Proposition 5.2.1), the first term tends to zero as $n\to\infty$. The second one is infinitesimal, too, by the continuity of w. Now, we claim that (5.2.14) is true for every $t>\tau$. Since

$$\int_0^t w(s,x)ds = \int_0^\tau w(s,x)ds + \int_0^{t-\tau}(T(\sigma)w(\tau,\cdot))(x)d\sigma$$

the claim is proved, because $\int_0^\tau w(s,\cdot)ds \in D_{\max}(A)$ by (5.2.14) and $\int_0^{t-\tau}(T(\sigma)w(\tau,\cdot))(x)d\sigma \in D_{\max}(A)$ by [48, Proposition 3.4].

Using again the closedness of $(A, D_{\max}(A)$ with respect to the π-convergence and Fubini's Theorem we obtain

$$\begin{aligned}\int_0^{+\infty}e^{-\lambda t}w(t,x)dt &= A\Big(\int_0^{+\infty}e^{-\lambda t}\int_0^t w(s,x)ds\,dt\Big)\\ &= A\Big(\int_0^{+\infty}w(s,x)\int_s^{+\infty}e^{-\lambda t}dt\,ds\Big)\\ &= \frac{1}{\lambda}A\Big(\int_0^{+\infty}e^{-\lambda s}w(s,x)ds\Big).\end{aligned}$$

It follows that the function $v(x)=\int_0^{+\infty}e^{-\lambda s}w(s,x)ds$ belong to $D_{\max}(A)$ and satisfies $\lambda v - Av = 0$. Since $\lambda - A$ is injective on $D_{\max}(A)$ we infer that $v=0$. This means that the Laplace transform of $w(\cdot,x)$ is identically zero, hence $w=0$. □

Moreover, the following result can be proved.

Proposition 5.2.7 *$\lambda - A$ is injective on $D_{\max}(A)$ if and only if $T(t)\mathbb{1}=\mathbb{1}$, for all $t\ge 0$.*

PROOF. If $\lambda - A$ is injective on $D_{\max}(A)$, then from Proposition 5.2.6 it follows that the semigroup $T(t)$ yields the unique bounded classical solution to problem (5.2.2). Since $\mathbb{1}$ is in fact a bounded classical solution of problem (5.2.2) with initial datum $f=\mathbb{1}$, by uniqueness it turns out that $T(t)\mathbb{1}=\mathbb{1}$.

Conversely, if $T(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$, then $R(1,A)\mathbb{1} = \mathbb{1}$ (see (5.2.6)). Let $u \in D_{\max}(A)$ be such that $u - Au = 0$ and $\|u\|_\infty \leq 1$. The function $v = \mathbb{1} - u \in D_{\max}(A)$ is nonnegative and satisfies $v - Av = \mathbb{1}$. On the other hand, by Remark 5.2.5, $R(1,A)\mathbb{1} = \mathbb{1}$ is the minimal positive solution of $w - Aw = \mathbb{1}$, hence $\mathbb{1} \leq \mathbb{1} - u$, i.e. $u \leq 0$. The same argument applied to $-u$ proves that $u \geq 0$ and therefore $u = 0$. □

If $T(t)\mathbb{1} = \mathbb{1}$ then, collecting all the results so far, we have that $(T(t))$ is a Feller semigroup, according to the terminology introduced in the previous section.

5.2.2 Invariant measures

Our aim is to establish now some criteria for the existence of an invariant measure for $T(t)$ in terms of the coefficients of the operator A. Since $T(t)$ is irreducible and has the strong Feller property (see Proposition 5.2.2) we already know that if an invariant measure exists, then it is unique and ergodic (see Theorem 5.1.8). Therefore, we limit our study to the existence part.

We start by a preliminary lemma which is similar to Proposition 5.1.2. We note, however, that here the semigroup is not strongly continuous and A is only its weak generator. For the proof see [38].

Lemma 5.2.8 *Assume that $\lambda - A$ is injective on $D_{\max}(A)$. Then a probability measure μ is invariant for $(T(t))$ if and only if $\int_{\mathbb{R}^N} Af\mu(dx) = 0$, for all $f \in D_{\max}(A)$.*

Proof. Since $(A, D_{\max}(A))$ is the weak generator of $T(t)$, if $u \in D_{\max}(A)$, we have that $T(t)u \in D_{\max}(A)$ and $\frac{d}{dt}T(t)u(x) = (AT(t)u)(x) = (T(t)Au)(x)$. Therefore $\left\|\frac{d}{dt}T(t)u\right\|_\infty \leq \|Au\|_\infty$ and by dominated convergence

$$\frac{d}{dt}\int_{\mathbb{R}^N} T(t)u(x)\mu(dx) = \int_{\mathbb{R}^N} AT(t)u(x)\mu(dx).$$

This shows that μ is an invariant measure for the restriction of $T(t)$ to $D_{\max}(A)$ if and only if $\int_{\mathbb{R}^N} Au\,\mu(dx) = 0$, for every $u \in D_{\max}(A)$. If this is the case and $f \in C_b(\mathbb{R}^N)$, then $f_n = n\int_0^{1/n} T(s)f\,ds$ belongs to $D_{\max}(A)$ and satisfies $\|f_n\|_\infty \leq \|f\|_\infty$, $f_n(x) \to f(x)$, for every $x \in \mathbb{R}^N$ (see [48, Proposition 3.4]). It follows that $T(t)f_n(x)$ converges to $T(t)f(x)$ (see [38, Proposition 4.6]) and $\|T(t)f_n\|_\infty \leq \|f_n\|_\infty \leq \|f\|_\infty$. Since

$$\int_{\mathbb{R}^N} T(t)f_n(x)\mu(dx) = \int_{\mathbb{R}^N} f_n(x)\mu(dx),$$

by dominated convergence we have

$$\int_{\mathbb{R}^N} T(t)f(x)\mu(dx) = \int_{\mathbb{R}^N} f(x)\mu(dx)$$

and the proof is complete. □

The following result is due to Khas'minskii.

Theorem 5.2.9 (Khas'minskii) *Assume that there exists a function $V \in C^2(\mathbb{R}^N)$ such that $\lim_{|x|\to+\infty} V(x) = +\infty$ and $\lim_{|x|\to+\infty} AV(x) = -\infty$. Then there is an invariant measure μ for $(T(t))$.*

Proof. We observe preliminarily that the existence of a function V satisfying the stated properties implies that $\lambda - A$ is injective on $D_{\max}(A)$, hence $T(t)\mathbb{1} = \mathbb{1}$ (see Proposition 5.2.7) and $(A, D_{\max}(A))$ is the generator of $T(t)$ (see Proposition 5.2.3). Without loss of generality, we can assume that $V \geq 0$ (otherwise we consider $V+c$ instead of V, for a suitable constant c). Recalling Theorem 5.1.6, it is sufficient to prove that the family of measures

$$\frac{1}{T}\int_0^T p(t,x_0,\cdot)dt, \quad T > T_0 \tag{5.2.15}$$

is tight for some $x_0 \in \mathbb{R}^N$ and $T_0 > 0$. Let $M > 0$ be such that $AV(x) \leq M$ for all $x \in \mathbb{R}^N$. Consider $\psi_n \in C^\infty(\mathbb{R})$ such that $\psi_n(t) = t$ for $t \leq n$, ψ_n is constant in $[n+1, +\infty[$ and $\psi_n' \geq 0, \psi_n'' \leq 0$. It is easily seen that $\psi_n \circ V$ belongs to $D_{\max}(A)$. Indeed, $\psi_n \circ V$ is obviously continuous in $\mathbb{R}^N$ and $\sup_{x\in\mathbb{R}^N} |\psi_n(V(x))| \leq \sup_{t\geq 0}\psi_n(t) < +\infty$. It is also clear that $\psi_n \circ V$ and its first and second order derivatives

$$\begin{aligned}
D_i(\psi_n \circ V)(x) &= \psi_n'(V(x))D_iV(x),\\
D_{ij}(\psi_n \circ V)(x) &= \psi_n''(V(x))D_iV(x)D_jV(x) + \psi_n'(V(x))D_{ij}V(x)
\end{aligned}$$

are locally p-summable, for every $p < \infty$. It remains to show that $A(\psi_n \circ V)$ is bounded in $\mathbb{R}^N$. To this aim, we observe that, by the assumption, there exists $R > 0$ such that $V(x) > n+1$ if $|x| > R$. It follows that $\psi_n'(V(x)) = \psi_n''(V(x)) = 0$, if $|x| > R$ and therefore

$$\begin{aligned}
\sup_{x\in\mathbb{R}^N} |A(\psi_n \circ V)(x)| &= \sup_{x\in\mathbb{R}^N}\left|\psi_n'(V(x))AV(x) + \psi_n''(V(x))\sum_{i,j=1}^N q_{ij}(x)D_iV(x)D_jV(x)\right|\\
&= \sup_{|x|\leq R}\left|\psi_n'(V(x))AV(x) + \psi_n''(V(x))\sum_{i,j=1}^N q_{ij}(x)D_iV(x)D_jV(x)\right|\\
&< +\infty.
\end{aligned}$$

Hence we deduce that $u_n(t,\cdot) = T(t)(\psi_n \circ V)(\cdot) \in D_{\max}(A)$ and

$$\begin{aligned}
D_tu_n(t,x) &= T(t)A(\psi_n \circ V)(x) = \int_{\mathbb{R}^N} p(t,x,y)A(\psi_n \circ V)(y)dy\\
&= \int_{\mathbb{R}^N} p(t,x,y)\Big(\psi_n'(V(y))AV(y) + \psi_n''(V(y))\sum_{i,j=1}^N q_{ij}(y)D_iV(y)D_jV(y)\Big)dy
\end{aligned}$$

Integrating this identity and recalling that $\psi_n'' \leq 0$ we have

$$\begin{aligned}
u_n(T,x) - \psi_n(V(x)) &\leq \int_0^T\int_{\mathbb{R}^N} p(t,x,y)\psi_n'(V(y))AV(y)dy\,dt\\
&= \int_0^T\int_E p(t,x,y)\psi_n'(V(y))AV(y)dy\,dt\\
&\quad + \int_0^T\int_{\mathbb{R}^N\setminus E} p(t,x,y)\psi_n'(V(y))AV(y)dy\,dt,
\end{aligned}$$

where $E = \{y \in \mathbb{R}^N \mid 0 \leq AV(y) \leq M\}$. In the first integral we can use dominated convergence since $p(t,x,y)\psi_n'(V(y))AV(y) \leq p(t,x,y)M$. In the second one, where AV is unbounded but negative, we use monotone convergence because $\psi_n' \leq \psi_{n+1}'$. Letting $n \to \infty$ we deduce that

$$\int_{\mathbb{R}^N} p(T,x,y)V(y)dy - V(x) \leq \int_0^T\int_{\mathbb{R}^N} p(t,x,y)AV(y)dy\,dt.$$

Let $\varepsilon, \rho > 0$ be such that $AV(y) \leq -1/\varepsilon$ if $|y| \geq \rho$. It follows that

$$\begin{aligned}
-\frac{V(x)}{T} &\leq \frac{1}{T}\int_0^T\int_{\mathbb{R}^N} p(t,x,y)AV(y)dy\,dt \leq -\frac{1}{\varepsilon T}\int_0^T\int_{\mathbb{R}^N\setminus B_\rho} p(t,x,y)dy\,dt\\
&\quad + \frac{1}{T}\int_0^T\int_{B_\rho} p(t,x,y)AV(y)dy\,dt \leq -\frac{1}{\varepsilon T}\int_0^T\int_{\mathbb{R}^N\setminus B_\rho} p(t,x,y)dy\,dt + M
\end{aligned}$$

hence

$$\frac{1}{T}\int_0^T p(t,x,\mathbb{R}^N\setminus B_\rho)dt \le \varepsilon\left(M+\frac{V(x)}{T}\right),$$

where we have set $p(t,x,\mathbb{R}^N\setminus B_\rho)=\int_{\mathbb{R}^N\setminus B_\rho} p(t,x,y)dy$. Therefore, we have established that the set of the measures (5.2.15) is tight for every fixed $x_0\in\mathbb{R}^N$ and $T_0>0$ and this completes the proof. □

Khas'minkii's Theorem relies upon the existence of suitable supersolutions of the equation $\lambda u - Au = 0$. Next, we give a different criterion, due to Varadhan, to establish the existence of an invariant measure for a special class of operators (see [40, Proposition 2.1]).

Theorem 5.2.10 *Consider the operator*

$$A=\Delta-\langle D\Phi+G,D\rangle,$$

where $\Phi\in C^1(\mathbb{R}^N)$ *and* $G\in C^1(\mathbb{R}^N;\mathbb{R}^N)$. *Assume that* $e^{-\Phi}\in L^1(\mathbb{R}^N)$ *and* $|G|\in L^1(\mathbb{R}^N,\mu)$, *with* $\mu(dx)=ae^{-\Phi(x)}dx$, $a=\|e^{-\Phi}\|_{L^1}^{-1}$. *Suppose also that*

$$\operatorname{div} G=\langle G,D\Phi\rangle, \tag{5.2.16}$$

i.e. $\operatorname{div}(Ge^{-\Phi})=0$. *If* $(T(t))$ *denotes the semigroup associated with* A, *then* $(T(t))$ *is generated by* $(A,D_{\max}(A))$ *and* μ *is its unique invariant measure.*

Proof. Uniqueness follows immediately from the irreducibility and the strong Feller property (see Proposition 5.2.2 and Theorem 5.1.8). For the existence part, we split the proof in two steps.

Step1. The closure $(B,D(B))$ of $(A,C_c^\infty(\mathbb{R}^N))$ generates a strongly continuous semigroup $(S(t))$ in $L^1(\mathbb{R}^N,\mu)$.

Let us prove that $(A,C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^1(\mathbb{R}^N,\mu)$. Let $\lambda>0$ and $u\in C_c^\infty(\mathbb{R}^N)$ be fixed. Multiplying the equation $\lambda u-Au=f$ by sign$u$ and integrating on $\mathbb{R}^N$ with respect to μ we obtain

$$\begin{aligned}\lambda\int_{\mathbb{R}^N}|u|e^{-\Phi}dx-\int_{\mathbb{R}^N}(\Delta u-\langle D\Phi,Du\rangle)\,\mathrm{sign}u\,e^{-\Phi}dx+\int_{\mathbb{R}^N}\langle G,Du\rangle\,\mathrm{sign}u\,e^{-\Phi}dx\\ =\int_{\mathbb{R}^N} f\,\mathrm{sign}u\,e^{-\Phi}dx.\end{aligned}$$

Since $\Delta u-\langle D\Phi,Du\rangle=e^{\Phi}\operatorname{div}(e^{-\Phi}Du)$ and $(Du)\,\mathrm{sign}u=D|u|$ we get

$$\lambda\int_{\mathbb{R}^N}|u|e^{-\Phi}dx-\int_{\mathbb{R}^N}\operatorname{div}(e^{-\Phi}Du)\,\mathrm{sign}u\,dx+\int_{\mathbb{R}^N}\langle G,D|u|\rangle e^{-\Phi}dx=\int_{\mathbb{R}^N} f\,\mathrm{sign}u\,e^{-\Phi}dx.$$

We claim that $\int_{\mathbb{R}^N}\operatorname{div}(e^{-\Phi}Du)\,\mathrm{sign}u\,dx\le 0$. Let $\varphi_n\in C^1(\mathbb{R})$ be such that $|\varphi_n|\le 1$, $\varphi_n'\ge 0$ and $\varphi_n(t)\to \mathrm{sign}t$ for all $t\ne 0$. Then, by dominated convergence, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\operatorname{div}(e^{-\Phi}Du)\,\mathrm{sign}u\,dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\operatorname{div}(e^{-\Phi}Du)\varphi_n(u)dx\\ &= -\lim_{n\to\infty}\int_{\mathbb{R}^N}e^{-\Phi}|Du|^2\varphi_n'(u)dx\le 0,\end{aligned}$$

as claimed. Integrating by parts and taking (5.2.16) into account we deduce that

$$\int_{\mathbb{R}^N}\langle G,D|u|\rangle e^{-\Phi}dx=0.$$

It follows that

$$\lambda\int_{\mathbb{R}^N}|u|e^{-\Phi}dx\le\int_{\mathbb{R}^N}|f|e^{-\Phi}dx,$$

which means $\lambda\|u\|_{L^1(\mathbb{R}^N,\mu)} \le \|\lambda u - Au\|_{L^1(\mathbb{R}^N,\mu)}$.

Next we show that $(I-A)C_c^\infty(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N,\mu)$. Let $g \in L^\infty(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N} (u - Au) g e^{-\Phi} dx = 0, \qquad \forall\, u \in C_c^\infty(\mathbb{R}^N). \tag{5.2.17}$$

Since, in particular, $ge^{-\Phi} \in L^2_{\text{loc}}(\mathbb{R}^N)$, from a classical result of local regularity for distributional solutions of elliptic equations (see [1] for $p=2$ and [2] for general p and also [5]) it follows that $ge^{-\Phi} \in H^1_{\text{loc}}(\mathbb{R}^N)$, if λ is sufficiently large and, as a consequence, $g \in H^1_{\text{loc}}(\mathbb{R}^N)$. This leads to

$$\int_{\mathbb{R}^N} u\, g\, e^{-\Phi} dx + \int_{\mathbb{R}^N} \langle Du, Dg\rangle e^{-\Phi} dx + \int_{\mathbb{R}^N} \langle G, Du\rangle g\, e^{-\Phi} dx = 0 \tag{5.2.18}$$

for every $u \in H^1(\mathbb{R}^N)$ with compact support. Indeed, if u is such a function, set $u_n = \varrho_n * u$, where ϱ_n is a standard sequence of mollifiers. Then $u_n \in C_c^\infty(\mathbb{R}^N)$ and u_n converges to u in $H^1(\mathbb{R}^N)$, as $n \to \infty$. Moreover, we can find $R > 0$ sufficiently large in such a way that $\operatorname{supp} u_n$ and $\operatorname{supp} u$ are contained in B_R, for every $n \in \mathbb{N}$. Now, each u_n satisfies (5.2.17), hence, integrating by parts, we have

$$\int_{\mathbb{R}^N} u_n\, g\, e^{-\Phi} dx + \int_{\mathbb{R}^N} \langle Du_n, Dg\rangle e^{-\Phi} dx + \int_{\mathbb{R}^N} \langle G, Du_n\rangle g\, e^{-\Phi} dx = 0$$

Letting $n \to \infty$, we obtain (5.2.18). Let η be in $C_c^\infty(\mathbb{R}^N)$ such that $\eta \equiv 1$ in B_1, $0 \le \eta \le 1$, $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_2$ and set $\eta_n(x) = \eta(\frac{x}{n})$. Plugging $g\,\eta_n^2$ into (5.2.18) we find

$$\begin{aligned}\int_{\mathbb{R}^N} g^2\eta_n^2 e^{-\Phi} dx \quad &+ \int_{\mathbb{R}^N} \eta_n^2 |Dg|^2 e^{-\Phi} dx + 2\int_{\mathbb{R}^N} \langle D\eta_n, Dg\rangle \eta_n g\, e^{-\Phi} dx \\ &+ \int_{\mathbb{R}^N} \langle G, Dg\rangle g\, \eta_n^2 e^{-\Phi} dx + 2\int_{\mathbb{R}^N} \langle G, D\eta_n\rangle g^2 \eta_n e^{-\Phi} dx = 0.\end{aligned} \tag{5.2.19}$$

Integrating by parts and recalling (5.2.16) it follows that

$$\int_{\mathbb{R}^N} \langle G, Dg\rangle g\, \eta_n^2 e^{-\Phi} dx = -\int_{\mathbb{R}^N} \langle G, D\eta_n\rangle g^2 \eta_n e^{-\Phi} dx,$$

therefore from (5.2.19) we deduce

$$\begin{aligned}\int_{\mathbb{R}^N} g^2\eta_n^2 e^{-\Phi} dx + \int_{\mathbb{R}^N} \eta_n^2 |Dg|^2 e^{-\Phi} dx \;&=\; -2\int_{\mathbb{R}^N} \langle D\eta_n, Dg\rangle \eta_n g\, e^{-\Phi} dx \\ &\quad - \int_{\mathbb{R}^N} \langle G, D\eta_n\rangle g^2 \eta_n e^{-\Phi} dx \\ &\le\; \frac{2c}{n}\int_{\mathbb{R}^N} \eta_n |g|\,|Dg| e^{-\Phi} dx + \frac{c}{n}\int_{\mathbb{R}^N} |g|^2\,|G| e^{-\Phi} dx \\ &\le\; \frac{c}{n}\int_{\mathbb{R}^N} \eta_n^2 |Dg|^2 e^{-\Phi} dx + \frac{c}{n}\,\|g\|_\infty^2 \int_{\mathbb{R}^N} e^{-\Phi} dx \\ &\quad + \frac{c}{n}\,\|g\|_\infty^2 \int_{\mathbb{R}^N} |G| e^{-\Phi} dx.\end{aligned}$$

For n large $1 - \frac{c}{n} > 0$, hence

$$\int_{\mathbb{R}^N} g^2\eta_n^2 e^{-\Phi} dx \le \frac{c}{n}\|g\|_\infty^2 \int_{\mathbb{R}^N} e^{-\Phi} dx + \frac{c}{n}\|g\|_\infty^2 \int_{\mathbb{R}^N} |G| e^{-\Phi} dx.$$

Letting $n \to \infty$ and using monotone convergence, we find that $g = 0$, which implies that $I - A$ has dense range.

Since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N,\mu)$, from the Lumer-Phillips Theorem (see e.g. [21, Theorem II.3.15]) Step1 follows. We observe that, by construction, $C_c^\infty(\mathbb{R}^N)$ is a core for B. Then μ is an invariant measure for the generated semigroup $(S(t))$, since, integrating by parts we have $\int_{\mathbb{R}^N} Au\,\mu(dx) = 0$, for all $u \in C_c^\infty(\mathbb{R}^N)$ (see Proposition 5.1.2).

Step2. The semigroups $(T(t))$ and $(S(t))$ coincide on $C_b(\mathbb{R}^N)$.

Let first $f \in C_c^\infty(\mathbb{R}^N)$, $f \geq 0$. By construction, $u(t,x) = T(t)f(x)$ is the limit of $u_\rho(t,x)$, as $\rho \to +\infty$, where u_ρ solves (5.2.4). Since f is positive, the classical maximum principle implies that the sequence (u_ρ) increases with ρ. Moreover, if supp$f \subset B_R$, then $u_R \in C^{1,2}([0,T] \times \overline{B}_R)$. Integrating the equation $D_t u_R = Au_R$ on B_R with respect to μ and using (5.2.16), we find

$$\begin{aligned} D_t \int_{B_R} u_R(t,x)\mu(dx) &= \int_{B_R} Au_R(t,x)\mu(dx) = a\int_{B_R} \mathrm{div}(e^{-\Phi} Du_R)dx \\ &\quad -a\int_{B_R} \langle G, Du_R\rangle e^{-\Phi} dx \\ &= a\int_{\partial B_R} \frac{\partial u_R}{\partial \nu}(t,x)e^{-\Phi}\sigma(dx) + a\int_{B_R} \mathrm{div}G\, u_R\, e^{-\Phi}dx \\ &\quad -a\int_{B_R} \langle G, D\Phi\rangle u_R\, e^{-\Phi}dx - a\int_{\partial B_R} \langle G, \nu\rangle u_R\, e^{-\Phi}dx \\ &= a\int_{\partial B_R} \frac{\partial u_R}{\partial \nu}(t,x)e^{-\Phi}\sigma(dx) \end{aligned}$$

where σ is the surface measure on ∂B_R and ν the outward unit normal vector to B_R. Since $u_R \geq 0$ in B_R and $u_R = 0$ on ∂B_R, it follows that $\frac{\partial u_R}{\partial \nu}(t,x) \leq 0$, hence the map $t \to \int_{B_R} u_R(t,x)\mu(dx)$ is decreasing. This yields

$$\int_{B_R} u_R(t,x)\mu(dx) \leq \int_{B_R} f(x)\mu(dx), \qquad t > 0$$

and, by monotone convergence, $\|T(t)f\|_{L^1(\mathbb{R}^N,\mu)} \leq \|f\|_{L^1(\mathbb{R}^N,\mu)}$. If $f \in C_b(\mathbb{R}^N)$ and $f \geq 0$, let $f_n \in C_c^\infty(\mathbb{R}^N)$ be such that $f_n \geq 0$, $\|f_n\|_\infty \leq \|f\|_\infty$ and $f_n(x) \to f(x)$, for every $x \in \mathbb{R}^N$. Then $T(t)f_n(x) \to T(t)f(x)$ and the same estimate holds by dominated convergence. Finally, since $T(t)$ is positive, we have $|T(t)f| \leq T(t)|f|$, for every $f \in C_b(\mathbb{R}^N)$, hence $\|T(t)f\|_{L^1(\mathbb{R}^N,\mu)} \leq \|f\|_{L^1(\mathbb{R}^N,\mu)}$. It follows that $(T(t))$ can be extended to a strongly continuous semigroup of positive contractions on $L^1(\mathbb{R}^N,\mu)$, denoted by $(\widetilde{T}(t))$, with generator $(\widetilde{A}, D(\widetilde{A}))$.

Let $f \in C_c^\infty(\mathbb{R}^N)$. Then f belongs to $\widehat{D}$, where $\widehat{D}$ is the domain of A as weak generator of $(T(t))$. This means that $\sup_{t>0} \frac{\|T(t)f - f\|_\infty}{t}$ is finite and $\lim_{t\to 0} \frac{T(t)f(x) - f(x)}{t} = Af(x)$, for every $x \in \mathbb{R}^N$. By dominated convergence, the above equality is also true in $L^1(\mathbb{R}^N,\mu)$. Therefore $f \in D(\widetilde{A})$ and $\widetilde{A}f = Af = Bf$. Hence, $C_c^\infty(\mathbb{R}^N)$ is contained in $D(\widetilde{A})$ and $\widetilde{A}$ coincide with B on $C_c^\infty(\mathbb{R}^N)$. If $f \in D(B)$, since $C_c^\infty(\mathbb{R}^N)$ is a core of B, we can find a sequence (f_n) in $C_c^\infty(\mathbb{R}^N)$ such that $f_n \to f$ and $\widetilde{A}f_n = Bf_n \to Bf$ in $L^1(\mathbb{R}^N,\mu)$. Since $(\widetilde{A}, D(\widetilde{A})$ is closed in $L^1(\mathbb{R}^N,\mu)$ it turns out that $f \in D(\widetilde{A})$ and $\widetilde{A}f = Bf$. Thus we have established that $\widetilde{A}$ is an extension of B. Since they are both generators, they must coincide, hence $\widetilde{T}(t) = S(t)$ on $L^1(\mathbb{R}^N,\mu)$. In particular $T(t) = S(t)$ on $C_b(\mathbb{R}^N)$, as claimed. Concerning the last assertion, we observe that $T(t)\mathbf{1} = \mathbf{1}$, since $T(t)\mathbf{1} \leq \mathbf{1}$ and $\int_{\mathbb{R}^N}(T(t)\mathbf{1} - \mathbf{1})e^{-\Phi}dx = 0$. Proposition 5.2.7 concludes the proof. □

Let us consider again A as in (5.2.1). Our next result shows that the invariant measure of $T(t)$, when exists, is absolutely continuous with respect to the Lebesgue measure $|\cdot|$. In this way, we extend the situation of Theorem 5.2.10 to the general case, even though it is not possible any more to know the density explicitly.

Proposition 5.2.11 *Assume that μ is the invariant measure of $T(t)$. Then μ is absolutely continuous with respect to $|\cdot|$ and its density $\varrho(x)$ is strictly positive $|\cdot|$ almost everywhere.*

Proof. Since $(T(t))$ is regular (see Propositions 5.2.2 and 5.1.9) all the probability measures $p(t,x,\cdot)$ are equivalent. Moreover, μ is equivalent to $p(t,x,\cdot)$ for all $t>0$ and $x\in\mathbb{R}^N$ (see Theorem 5.1.8). Since $p(t,x,\cdot)$ is absolutely continuous with respect to the Lebesgue measure $|\cdot|$ (see (5.2.5)), it follows that μ is absolutely continuous with respect to $|\cdot|$, too. Let $\varrho\in L^1(\mathbb{R}^N)$ be its density. It is clear that $\varrho\geq 0$. We prove that ϱ is strictly positive $|\cdot|$-a.e. If Γ is a Borel set such that $|\Gamma|>0$, then $\int_\Gamma \varrho(x)dx=\mu(\Gamma)=P_t\chi_\Gamma=\int_\Gamma p(t,x,y)dy>0$ since p is positive. Since Γ was arbitrary the thesis follows. □

Remark 5.2.12 As a consequence of the above proposition, we have that if an invariant measure of $(T(t))$ exists, then $T(t)\mathbb{1}=\mathbb{1}$ and therefore $T(t)$ is generated by $(A,D_{\max}(A))$ (see Propositions 5.2.3 and 5.2.7). Indeed, one has $T(t)\mathbb{1}\leq\mathbb{1}$ and $\int_{\mathbb{R}^N}(T(t)\mathbb{1}-\mathbb{1})\varrho(x)dx=0$, with $\varrho(x)>0$ $|\cdot|$-a.e. from Proposition 5.2.11.

Moreover, recalling Proposition 5.1.2, we have that $(T(t))$ extends to a strongly continuous semigroup in $L^p(\mathbb{R}^N,\mu)$, for every $1\leq p<\infty$. Here we have more information, since we can identify the generator $(A_p,D(A_p))$, relating it to the original operator A.

Proposition 5.2.13 *Assume that μ is an invariant measure of $(T(t))$. Then $D_{\max}(A)$ is a core of $(A_p,D(A_p))$ in $L^p(\mathbb{R}^N,\mu)$, hence $(A_p,D(A_p))$ is the closure of $(A,D_{\max}(A))$ in $L^p(\mathbb{R}^N,\mu)$.*

Proof. We continue to denote by $(T(t))$ the extended semigroup in $L^p(\mathbb{R}^N,\mu)$. In order to prove that $D_{\max}(A)$ is a core of $(A_p,D(A_p))$, it is sufficient to show that

(i) $D_{\max}(A)\subset D(A_p)$;

(ii) $D_{\max}(A)$ is dense in $L^p(\mathbb{R}^N,\mu)$;

(iii) $D_{\max}(A)$ is invariant under the semigroup.

Let $f\in D_{\max}(A)$. Then $\sup_{t>0}\frac{\|T(t)f-f\|_\infty}{t}$ is finite and $\lim_{t\to 0}\frac{T(t)f(x)-f(x)}{t}=Af(x)$, for every $x\in\mathbb{R}^N$. By dominated convergence, we have easily that

$$\left\|\frac{T(t)f-f}{t}-Af\right\|_{L^p(\mathbb{R}^N,\mu)}\to 0,\quad \text{as } t\to 0.$$

Therefore $f\in D(A_p)$ and $A_pf=Af$. Concerning *(ii)*, we show that $C_c^\infty(\mathbb{R}^N)$, which is contained in $D_{\max}(A)$, is dense in $L^p(\mathbb{R}^N,\mu)$. Let first $u\in C_c(\mathbb{R}^N)$. If (ϱ_n) is a standard sequence of mollifiers, then $\varrho_n * u\in C_c^\infty(\mathbb{R}^N)$ converges uniformly to u as $n\to\infty$. Since

$$\int_{\mathbb{R}^N}|\varrho_n*u(x)-u(x)|^p\varrho(x)dx\leq\|\varrho_n*u-u\|_\infty^p,$$

it follows that ϱ_n*u converges to u in $L^p(\mathbb{R}^N,\mu)$, too. This proves that $C_c^\infty(\mathbb{R}^N)$ is dense in $C_c(\mathbb{R}^N)$ with respect to the norm of $L^p(\mathbb{R}^N,\mu)$. Since $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N,\mu)$ (see [51, Theorem III.3.14]), assertion *(ii)* follows.

Finally, taking into account the fact that $(A,D_{\max}(A))$ generates $(T(t))$ in $C_b(\mathbb{R}^N)$, *(iii)* is clear. At this point, [21, Proposition II.1.7] leads to the conclusion. □

5.3 Characterization of the domain of a class of elliptic operators in $L^p(\mathbb{R}^N,\mu)$

The aim of the present section is to study the following class of operators

$$B=\operatorname{div}(qD)-\langle qD\Phi,D\rangle+\langle G,D\rangle$$

in the space $L^p(\mathbb{R}^N,\mu)$, $1<p<\infty$, where $d\mu = e^{-\Phi}dx$. In particular, our purpose is to provide an explicit description of the domain under which B generates a strongly continuous semigroup in $L^p(\mathbb{R}^N,\mu)$. Our main tools are the results of Chapter 1, where the same problem has been studied for differential operators in $L^p(\mathbb{R}^N)$. In fact, via the transformation $v = e^{-\frac{\Phi}{p}}u$, the operator B on $L^p(\mathbb{R}^N,\mu)$ is similar to an operator A of the form (1.0.1) in the unweighted space $L^p(\mathbb{R}^N)$. Suitable assumptions on the coefficients q, Φ, G allow to apply the generation results of Chapter 1 to the transformed operator so that, via the inverse transformation, we can deduce that B, endowed with the domain

$$\mathcal{D}_\mu = \{u \in W^{2,p}(\mathbb{R}^N,\mu) \mid \langle G, Du\rangle \in L^p(\mathbb{R}^N,\mu)\} \tag{5.3.1}$$

generates a strongly continuous semigroup $(T(t))$ on $L^p(\mathbb{R}^N,\mu)$. We note that, in particular, the measure μ can be the invariant measure of $(T(t))$. This is the case if an additional condition is satisfied (see (A4') below). By $W^{k,p}(\mathbb{R}^N,\mu)$ we mean the weighted Sobolev space

$$W^{k,p}(\mathbb{R}^N,\mu) = \left\{u \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^N) \mid D^\alpha u \in L^p(\mathbb{R}^N,\mu),\ |\alpha| \le k\right\}.$$

In order to prove that $(B,\mathcal{D}_\mu)$ is a generator, we make the following assumptions on the coefficients:

(A1) $q = (q_{ij})$ is a symmetric matrix, with $q_{ij} \in C^1_b(\mathbb{R}^N)$ and there exists $\nu > 0$ such that $\langle q\xi,\xi\rangle = \sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2$, for all $x, \xi \in \mathbb{R}^N$,

(A2) $\Phi \in C^2(\mathbb{R}^N)$, $G \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} e^{-\Phi(x)}dx < \infty$,

(A3) for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $|G| + |DG| + |D^2\Phi|^2 \le \varepsilon|D\Phi|^2 + c_\varepsilon$,

(A4) $|\mathrm{div} G - \langle G, D\Phi\rangle| \le \varepsilon|D\Phi|^2 + c_\varepsilon$,

(A4') $\mathrm{div}\, G = \langle G, D\Phi\rangle$.

Since $|\mathrm{div} G| \le \sqrt{N}|DG|$ and (A3) holds, (A4) actually says that $|\langle G, D\Phi\rangle| \le \varepsilon|D\Phi|^2 + c_\varepsilon$. Here and in the sequel, c_ε denotes a nonnegative constant which may go to infinity when ε goes to zero. It may change from line to line, but this is irrelevant to our interests.

We observe that the condition on Φ included in (A3) is satisfied by any polynomial whose homogeneous part of maximal degree is positive definite. However, it fails in $\mathbb{R}^2$ for the function x^2y^2. Moreover, we note that it implies the weaker condition $|D^2\Phi| \le \varepsilon|D\Phi|^2 + c_\varepsilon$, which is assumed in [41] together with a more restrictive assumption on G. If $q_{ij} = \delta_{ij}$ and $\Phi = |x|^2/2$ then we obtain the Ornstein Uhlenbeck operator perturbed with a non symmetric drift G:

$$\Delta - \langle x, D\rangle + \langle G, D\rangle.$$

If G is such that $\langle G(x), x\rangle = 0$ for every $x \in \mathbb{R}^N$, then assumption (A3) is verified if

$$|G(x)| + |DG(x)| \le \varepsilon|x|^2 + c_\varepsilon,$$

i.e. if G and its derivatives grow a little bit less than quadratically. Since $\langle G(x), x\rangle = 0$, this implies automatically (A4). For example, in $\mathbb{R}^2$ one can consider $G(x_1,x_2) = (-x_2, x_1)\times h(x_1,x_2)$, where $h \in C^1(\mathbb{R}^2)$. Since $|G| = |x|\,|h|$, and $|DG|^2 = |x|^2|Dh|^2 + 2h^2 + 2h\langle x, Dh\rangle$, the function h has to satisfy the condition $|h(x)| \le \varepsilon|x| + c_\varepsilon$, for every $\varepsilon > 0$. Then a possible choice is $h(x) = (|x|^2+1)^{\alpha/2}$, with $0 < \alpha < 1$. This situation is excluded in [41].

Replacing (A4) with (A4') we obtain that μ is the invariant measure for the generated semigroup, as we will see in Proposition 5.3.4.

We first need some technical lemmas. These results are completely similar to those of [41] and we give the proof for the sake of completeness. It is useful to observe that one can easily check, as in Lemma 1.3.1, that $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{k,p}(\mathbb{R}^N,\mu)$.

Lemma 5.3.1 *Let $1 < p < \infty$ and assume that $\Phi \in C^2(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} e^{-\Phi(x)}dx < \infty$. If for some $\varepsilon < 1$ there exists $c_\varepsilon > 0$ such that*

$$\Delta\Phi + (p-2)(1+|D\Phi|^2)^{-1}\langle D^2\Phi D\Phi, D\Phi\rangle \le \varepsilon|D\Phi|^2 + c_\varepsilon \tag{5.3.2}$$

then the map $u \to u|D\Phi|$ is bounded from $W^{1,p}(\mathbb{R}^N,\mu)$ to $L^p(\mathbb{R}^N,\mu)$ and the map $u \to |Du|\,|D\Phi|$ is bounded from $W^{2,p}(\mathbb{R}^N,\mu)$ to $L^p(\mathbb{R}^N,\mu)$. Therefore, the operator B is bounded from $\mathcal{D}_\mu$ in $L^p(\mathbb{R}^N,\mu)$.

Proof. Let $1 < p < \infty$ be fixed. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N,\mu)$, it is sufficient to prove that

$$\|u|D\Phi|\,\|_{L^p(\mathbb{R}^N,\mu)} \le c(\|u\|_{L^p(\mathbb{R}^N,\mu)} + \|Du\|_{L^p(\mathbb{R}^N,\mu)}),$$

for $u \in C_c^\infty(\mathbb{R}^N)$ and for some constant $c > 0$. Since $t^p \le a(1+t^2)^{\frac{p}{2}-1}t^2 + b$ for all $t \ge 0$ and for some suitable constants $a, b > 0$, we have only to estimate $\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p e^{-\Phi}dx$. Integrating by parts and using (5.3.2) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p e^{-\Phi}dx = -\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}\langle D\Phi, De^{-\Phi}\rangle|u|^p dx = \\
&(p-2)\int_{\mathbb{R}^N}|u|^p(1+|D\Phi|^2)^{\frac{p}{2}-2}\langle D^2\Phi D\Phi, D\Phi\rangle e^{-\Phi}dx + \int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}\Delta\Phi|u|^p e^{-\Phi}dx \\
&\quad +p\int_{\mathbb{R}^N}|u|^{p-2}u(1+|D\Phi|^2)^{\frac{p}{2}-1}\langle D\Phi, Du\rangle e^{-\Phi}dx \le \varepsilon\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p\,d\mu + \\
&\quad c_\varepsilon\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|u|^p\,d\mu + p\int_{\mathbb{R}^N}|u|^{p-1}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|\,|Du|d\mu.
\end{aligned}$$

Applying the inequality $(1+t^2)^{\frac{p}{2}-1} \le \eta(1+t^2)^{\frac{p}{2}-1}t^2 + c_\eta$, which holds for all $\eta > 0$, we deduce

$$\begin{aligned}
(1-\varepsilon)\int_{\mathbb{R}^N}&(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p e^{-\Phi}dx \le c_\varepsilon\,\eta\int_{\mathbb{R}^N}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p d\mu + \\
&c_\varepsilon\,c_\eta\int_{\mathbb{R}^N}|u|^p d\mu + p\int_{\mathbb{R}^N}\left(|u|^{p-1}(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|\right)|Du|d\mu.
\end{aligned} \tag{5.3.3}$$

Choosing $\eta = \frac{1-\varepsilon}{2c_\varepsilon}$ and using Young's inequality to estimate the last term in (5.3.3), we find that for all $\delta > 0$

$$\begin{aligned}
\frac{1-\varepsilon}{2}\int_{\mathbb{R}^N}&(1+|D\Phi|^2)^{\frac{p}{2}-1}|D\Phi|^2|u|^p e^{-\Phi}dx \le c_\varepsilon\,c_\eta\int_{\mathbb{R}^N}|u|^p d\mu \\
&+\delta\int_{\mathbb{R}^N}|u|^p(1+|D\Phi|^2)^{\left(\frac{p}{2}-1\right)p'}|D\Phi|^{p'}d\mu + c_\delta\int_{\mathbb{R}^N}|Du|^p d\mu,
\end{aligned}$$

where p' is the conjugate exponent of p. Now, the inequality $(1+t^2)^{\left(\frac{p}{2}-1\right)p'}t^{p'} \le k_1(1+t^2)^{\frac{p}{2}-1}t^2 + k_2$, which holds for certain constants $k_1, k_2 > 0$, and a suitable choice of δ conclude the proof. □

Lemma 5.3.2 *Let $1 < p < \infty$ and assume that $\Phi \in C^2(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} e^{-\Phi(x)}dx < \infty$ and such that for all $\varepsilon > 0$ there exists c_ε with the following property*

$$|D^2\Phi| \le \varepsilon|D\Phi|^2 + c_\varepsilon. \tag{5.3.4}$$

Then the map $u \to u\,|D\Phi|^2$ is bounded from $W^{2,p}(\mathbb{R}^N,\mu)$ to $L^p(\mathbb{R}^N,\mu)$.

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$. Then the vector function $uD\Phi \in W^{1,p}(\mathbb{R}^N)$ and from Lemma 5.3.1 it follows that

$$\begin{aligned}
\|u|D\Phi|^2\|_{L^p(\mathbb{R}^N,\mu)} &\le C\|uD\Phi\|_{W^{1,p}(\mathbb{R}^N,\mu)} \\
&\le C(\|uD\Phi\|_{L^p(\mathbb{R}^N,\mu)} + \|\langle Du, D\Phi\rangle\|_{L^p(\mathbb{R}^N,\mu)} + \|uD^2\Phi\|_{L^p(\mathbb{R}^N,\mu)}).
\end{aligned}$$

Using again Lemma 5.3.1 and applying (5.3.4) we have

$$\|u|D\Phi|^2\|_{L^p(\mathbb{R}^N,\mu)} \le C'(\|u\|_{W^{1,p}(\mathbb{R}^N,\mu)} + \|u\|_{W^{2,p}(\mathbb{R}^N,\mu)} + \varepsilon\|u|D\Phi|^2\|_{L^p(\mathbb{R}^N,\mu)} + c_\varepsilon\|u\|_{L^p(\mathbb{R}^N,\mu)}).$$

Choosing ε sufficiently small we get the statement for $u \in C_c^\infty(\mathbb{R}^N)$. The general case follows by density. □

We observe that under assumptions (A2) and (A3) Lemmas 5.3.1 and 5.3.2 hold.

Now, we are ready to prove the main result of this section. It is useful to introduce the quantities

$$L = \sup_{x\in\mathbb{R}^N} \Big(\sum_{i,j=1}^N |Dq_{ij}(x)|^2 \Big)^{\frac{1}{2}} \tag{5.3.5}$$

$$M = \sup_{x\in\mathbb{R}^N} \max_{|\xi|=1} \langle q\xi,\xi\rangle = \sup_{\mathbb{R}^N} \Big(\sum_{i,j=1}^N (q_{ij}(x))^2 \Big)^{\frac{1}{2}}$$

Theorem 5.3.3 *Let $1 < p < \infty$ and assume that hypotheses (A1), (A2), (A3) and (A4) are satisfied. Then the operator $(B, \mathcal{D}_\mu)$ generates a positive strongly continuous semigroup $(T(t))$ in $L^p(\mathbb{R}^N, \mu)$.*

Proof. Fix $p \in (1,\infty)$. As pointed out at the beginning of the section, we introduce a transformation in order to deal with an operator in the unweighted space $L^p(\mathbb{R}^N)$. Let us define the isometry

$$J: \quad L^p(\mathbb{R}^N,\mu) \longrightarrow L^p(\mathbb{R}^N)$$

$$u \longmapsto Ju = e^{-\frac{\Phi}{p}}u.$$

A straightforward computation shows that $Bu = J^{-1}\widetilde{B}\,Ju$, for $u \in C_c^\infty(\mathbb{R}^N)$, where

$$\widetilde{B} = \mathrm{div}\,(qD) + \langle F, D\rangle - V$$

with

$$F = \left(\frac{2}{p} - 1\right) qD\Phi + G,$$

$$V = \frac{1}{p}\left[\left(1 - \frac{1}{p}\right)\langle qD\Phi, D\Phi\rangle - \mathrm{Tr}\,(qD^2\Phi) - \langle G, D\Phi\rangle - \sum_{i,j=1}^N D_i q_{ij} D_j \Phi\right].$$

The proof is structured as follows. Setting $U = \frac{1}{p}\left(1 - \frac{1}{p}\right)\langle qD\Phi, D\Phi\rangle$, we first prove that

Step1 $A = \mathrm{div}\,(qD) + \langle F, D\rangle - U$, endowed with the domain

$$\mathcal{D}_p = \{u \in W^{2,p}(\mathbb{R}^N) \mid \langle F, Du\rangle,\ Uu \in L^p(\mathbb{R}^N)\}, \tag{5.3.6}$$

generates a positive strongly continuous semigroup in $L^p(\mathbb{R}^N)$.

Then we deduce that

Step2 $(\widetilde{B}, \mathcal{D}_p)$ generates a positive C_0 semigroup in $L^p(\mathbb{R}^N)$.

Finally, we show that

Step3 $(B, \mathcal{D}_\mu)$ generates a positive C_0 semigroup in $L^p(\mathbb{R}^N, \mu)$.

Proof of Step1. We want to show that under assumptions (A1)-(A4) the coefficients of A satisfy the hypotheses of Theorem 1.1.2 with $\sigma = 1$ and $\mu = 0$. More precisely, we claim that there exist a constant $\alpha > 0$, sufficiently small constants $\beta, \theta > 0$, and constants $c_\alpha, c_\beta, c_\theta \geq 0$ such that

(i) $|DU| \leq \alpha U + c_\alpha$,

(ii) $|DF| \leq \beta U + c_\beta$,

(iii) $|F| \leq \theta U + c_\theta$.

As far as (i) is concerned, we have

$$\begin{aligned}
|D_k U| &= \left|\frac{1}{p}\left(1-\frac{1}{p}\right)\sum_{i,j=1}^{N} D_k q_{ij} D_i\Phi D_j\Phi + \frac{2}{p}\left(1-\frac{1}{p}\right)\sum_{i,j=1}^{N} q_{ij} D_{ik}\Phi D_j\Phi\right| \\
&\leq \frac{1}{p}\left(1-\frac{1}{p}\right)|D\Phi|^2 \sup_{\mathbb{R}^N}\left(\sum_{i,j=1}^{N}|D_k q_{ij}|^2\right)^{\frac{1}{2}} \\
&\quad + \frac{2}{p}\left(1-\frac{1}{p}\right)|D\Phi|\left(\sum_{i=1}^{N}|D_{ik}\Phi|^2\right)^{\frac{1}{2}} \sup_{\mathbb{R}^N}\left(\sum_{i,j=1}^{N}|q_{ij}|^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{p}\left(1-\frac{1}{p}\right)L|D\Phi|^2 + \frac{2}{p}\left(1-\frac{1}{p}\right)M|D^2\Phi||D\Phi|,
\end{aligned}$$

where L and M are given in (5.3.5). From (A1) and (A3) and applying the inequality $t \leq \eta\, t^2 + c_\eta$, which holds for every $\eta > 0$, it follows that

$$\begin{aligned}
|D_k U| &\leq \frac{1}{p\,\nu}\left(1-\frac{1}{p}\right)L\langle qD\Phi, D\Phi\rangle + \frac{2}{p}\left(1-\frac{1}{p}\right)M(\varepsilon|D\Phi|^2 + c_\varepsilon|D\Phi|) \\
&\leq \frac{1}{p\,\nu}\left(1-\frac{1}{p}\right)L\langle qD\Phi, D\Phi\rangle + \frac{2}{p\,\nu}\left(1-\frac{1}{p}\right)M\varepsilon\langle qD\Phi, D\Phi\rangle \\
&\quad + \frac{2}{p\,\nu}\left(1-\frac{1}{p}\right)Mc_\varepsilon\eta\langle qD\Phi, D\Phi\rangle + \frac{2}{p}\left(1-\frac{1}{p}\right)Mc_\varepsilon c_\eta \\
&= \alpha U + c_\alpha
\end{aligned}$$

where $\alpha = \frac{L+2M(\varepsilon+c_\varepsilon\eta)}{\nu}$ and $c_\alpha = \frac{2}{p}\left(1-\frac{1}{p}\right)Mc_\varepsilon c_\eta$, for arbitrary $\varepsilon, \eta > 0$. This leads to (i). Now, similar computations yield

$$\begin{aligned}
|DF| &\leq \sqrt{3}\left(\left|\frac{2}{p}-1\right|L|D\Phi| + \left|\frac{2}{p}-1\right|M|D^2\Phi| + |DG|\right) \\
&\leq \sqrt{3}\left(\left|\frac{2}{p}-1\right|\varepsilon(L+M+1)|D\Phi|^2 + c_\varepsilon\right),
\end{aligned}$$

where c_ε depends on ε, p, L, M. Therefore $|DF| \leq \beta U + c_\beta$, with $\beta = O(\varepsilon)$ and $c_\beta > 0$ depending on ε, p, M, L. Finally, condition (iii) follows easily from (A3). Indeed, one has

$$\begin{aligned}
|F| &\leq \sqrt{2}\left(\left|\frac{2}{p}-1\right|M|D\Phi| + |G|\right) \\
&\leq \sqrt{2}\left(\left|\frac{2}{p}-1\right|M\varepsilon|D\Phi|^2 + \left|\frac{2}{p}-1\right|Mc_\varepsilon + \varepsilon|D\Phi|^2 + c_\varepsilon\right) \\
&= \theta U + c_\theta
\end{aligned}$$

with $\theta = O(\varepsilon)$ and c_θ depending on ε, M, p. At this point, assumptions (H1'), (H2'), (H4') and (H5) of Theorem 1.1.2 are satisfied with $\sigma = 1$ and $\mu = 0$. The smallness condition (1.1.7) is guaranteed by a suitable choice of ε and η. Note that the product $\alpha\,\theta$, and not α itself, has to be small. Then Theorem 1.1.2 applies and we find that $(A, \mathcal{D}_p)$ generates a positive, strongly continuous semigroup in $L^p(\mathbb{R}^N)$, with $\mathcal{D}_p$ given by (5.3.6). This concludes the proof of Step1.

Proof of Step2. Let us prove that $\mathrm{div}F + p(V + \lambda_0) \geq 0$, for a suitable $\lambda_0 > 0$. From assumption (A3) we infer

$$\begin{aligned}
\mathrm{div}F + pV \;&= 2\left(\frac{1}{p}-1\right)\sum_{j,k=1}^{N} D_k q_{jk} D_j \Phi + 2\left(\frac{1}{p}-1\right)\mathrm{Tr}(qD^2\Phi) + \left(1-\frac{1}{p}\right)\langle qD\Phi, D\Phi\rangle \\
&\quad +\mathrm{div}G - \langle G, D\Phi\rangle \\
&\geq 2\left(\frac{1}{p}-1\right)\sqrt{N}L|D\Phi| + 2\left(\frac{1}{p}-1\right)M|D^2\Phi| + \left(1-\frac{1}{p}\right)\nu|D\Phi|^2 \\
&\quad -\varepsilon|D\Phi|^2 - c_\varepsilon \\
&\geq \left[\left(2\left(\frac{1}{p}-1\right)(\sqrt{N}L+M)-1\right)\varepsilon + \left(1-\frac{1}{p}\right)\nu\right]|D\Phi|^2 \\
&\quad +2\left(\frac{1}{p}-1\right)(\sqrt{N}L+M)c_\varepsilon - c_\varepsilon.
\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, we obtain $\mathrm{div}F + pV \geq -p\,\lambda_0$, where $\lambda_0 > 0$ depends on p, ν, L, M. Under this condition, the operator $(\widetilde{B}, \mathcal{D}_p)$ is quasi-dissipative in $L^p(\mathbb{R}^N)$ (see Lemma 1.3.2 and Remark 1.3.4). Moreover, we observe that, setting $W = V - U$, by (A4), (A3) and (A1) respectively, we have

$$\begin{aligned}
|W| \;&\leq\; \frac{1}{p}M|D^2\Phi| + \frac{1}{p}|\langle G, D\Phi\rangle| + \frac{\sqrt{N}L}{p}|D\Phi| \\
&\leq\; \frac{1}{p}(M+1)\varepsilon|D\Phi|^2 + \frac{1}{p}(M+1)c_\varepsilon + \frac{\sqrt{N}}{p}|DG| + \frac{\sqrt{N}L}{p}\varepsilon|D\Phi|^2 + \frac{\sqrt{N}L}{p}c_\varepsilon \\
&\leq\; \frac{\varepsilon}{p}\left(M+\sqrt{N}+\sqrt{N}L\right)|D\Phi|^2 + \frac{1}{p}\left(M+\sqrt{N}+\sqrt{N}L\right)c_\varepsilon,
\end{aligned}$$

which means

$$|W| \leq \eta U + c_\eta, \tag{5.3.7}$$

for all $\eta > 0$. Then, if $u \in \mathcal{D}_p$ one deduces

$$\|Wu\|_p \leq 2^{1-\frac{1}{p}}(\eta\|Uu\|_p + c_\eta\|u\|_p)$$

and applying estimate (1.3.10) to the operator A we obtain

$$\|Wu\|_p \leq 2^{1-\frac{1}{p}}(\eta c\|Au\|_p + \eta c\|u\|_p + c_\eta\|u\|_p) = \delta\|Au\|_p + c_\delta\|u\|_p \tag{5.3.8}$$

with $\delta > 0$ arbitrarily small. Now, if $\lambda > 0$ is large enough, then $\lambda \in \rho(A)$, since A is the generator of a strongly continuous semigroup. This means that $\lambda - A : \mathcal{D}_p \to L^p(\mathbb{R}^N)$ is invertible, therefore we may write

$$\lambda - \widetilde{B} = \lambda - A + W = [I + WR(\lambda, A)](\lambda - A).$$

It follows that $\lambda - \widetilde{B}$ is invertible on $\mathcal{D}_p$ if and only if $I + WR(\lambda, A)$ is invertible on $L^p(\mathbb{R}^N)$. This is the case if $\|WR(\lambda, A)\| < 1$. Let $f \in L^p(\mathbb{R}^N)$. Applying (5.3.8) with $u = R(\lambda, A)f$ and considering the fact that $(A, \mathcal{D}_p)$ is quasi-dissipative, (see Lemma 1.6.1), we deduce

$$\begin{aligned}
\|WR(\lambda, A)f\|_p \;&\leq\; \delta\|AR(\lambda, A)f\|_p + c_\delta\|R(\lambda, A)f\|_p \\
&\leq\; \delta\lambda\|R(\lambda, A)f\|_p + \delta\|f\|_p + c_\delta\|R(\lambda, A)f\|_p \\
&\leq\; \left(\frac{\delta\,\lambda}{\lambda - \lambda_p} + \delta + \frac{c_\delta}{\lambda - \lambda_p}\right)\|f\|_p,
\end{aligned}$$

if $\lambda > \lambda_p$, for a suitable λ_p. Choose $\delta < \frac{1}{6}$. Then $\delta\left(1+\frac{\lambda}{\lambda-\lambda_p}\right) < \frac{1}{2}$ for all $\lambda \geq 2\lambda_p$. Let $\lambda \geq 2\lambda_p$ be such that $\lambda > \lambda_p + 2c_\delta$. This implies that $\|WR(\lambda, A)f\|_p \leq a\|f\|_p$, with $a < 1$. Thus, we have established that if λ is large enough, then $\lambda - \widetilde{B}$ is invertible on $\mathcal{D}_p$. This implies also that $(B, \mathcal{D}_p)$ is closed and Step2 follows from the Hille Yosida Theorem [21].

Proof of Step3. As a consequence of Step2, $B = J^{-1}\widetilde{B}J$ with domain $D(B) = \{u \in L^p(\mathbb{R}^N, \mu) \mid Ju \in \mathcal{D}_p\}$ generates a positive C_0-semigroup $(T(t))$ in $L^p(\mathbb{R}^N, \mu)$. We have to show that $D(B) = \mathcal{D}_\mu$.

Let $u \in D(B)$. Then $v = Ju \in \mathcal{D}_p$, so in particular $v \in W^{2,p}(\mathbb{R}^N)$ and $Uv \in L^p(\mathbb{R}^N)$. Since $|D\Phi|^2 \leq \frac{p^2}{(p-1)\nu}U$, we have that $|D\Phi|^2 v \in L^p(\mathbb{R}^N)$. Therefore

$$e^{-\frac{\Phi}{p}} D_j u = \frac{1}{p} v D_j \Phi + D_j v \in L^p(\mathbb{R}^N),$$

since $|D\Phi| \leq |D\Phi|^2 + 1$. Moreover, (A3) and the estimate

$$\|U^{\frac{1}{2}} Dv\|_p \leq K(\|\Delta v\|_p + \|Uv\|_p)$$

(see [41, Proposition 2.3]) yield

$$e^{-\frac{\Phi}{p}} D_{ij} u = \frac{1}{p} v D_{ij}\Phi + D_{ij} v + \frac{1}{p} D_j v D_i \Phi + \frac{1}{p} D_i v D_j \Phi + \frac{1}{p^2} v D_i \Phi D_j \Phi \in L^p(\mathbb{R}^N),$$

i.e. $u \in W^{2,p}(\mathbb{R}^N, \mu)$. Recalling (5.3.7), we have

$$|V| \leq (\eta + 1)U + c_\eta, \tag{5.3.9}$$

hence $Vv \in L^p(\mathbb{R}^N)$. Since $v \in \mathcal{D}_p$, we have that $v \in W^{2,p}(\mathbb{R}^N)$ and $\langle F, Dv\rangle \in L^p(\mathbb{R}^N)$, then $\widetilde{B}v \in L^p(\mathbb{R}^N)$, which implies that $Bu = J^{-1}\widetilde{B}v \in L^p(\mathbb{R}^N, \mu)$. From Lemma 5.3.1 and the fact that $q_{ij} \in C^1_b(\mathbb{R}^N)$ it follows that $\mathrm{div}(qDu)$, $\langle qD\Phi, Du\rangle \in L^p(\mathbb{R}^N, \mu)$. By difference, $\langle G, Du\rangle \in L^p(\mathbb{R}^N, \mu)$ and then $u \in \mathcal{D}_\mu$.

Conversely, let $u \in \mathcal{D}_\mu$ and set $v = Ju$. Then, by Lemma 5.3.1

$$D_j v = e^{-\frac{\Phi}{p}}\left(-\frac{1}{p} u D_j \Phi + D_j u\right) \in L^p(\mathbb{R}^N).$$

Now, Lemma 5.3.2 implies that $|D\Phi|^2 v \in L^p(\mathbb{R}^N)$. Then, since $U \leq \frac{1}{p}\left(1 - \frac{1}{p}\right) M|D\Phi|^2$ and (5.3.9) holds, we obtain that $Uv, Vv \in L^p(\mathbb{R}^N)$. Using again Lemma 5.3.1 and (A3) we get

$$D_{ij} v = e^{-\frac{\Phi}{p}}\left(D_{ij} u - \frac{1}{p} u D_{ij}\Phi + \frac{1}{p^2} u D_i \Phi D_j \Phi - \frac{1}{p} D_j u D_i \Phi - \frac{1}{p} D_i u D_j \Phi\right) \in L^p(\mathbb{R}^N).$$

Therefore $v \in W^{2,p}(\mathbb{R}^N)$. Since $Bu \in L^p(\mathbb{R}^N, \mu)$, we have that $\widetilde{B}v = JBu \in L^p(\mathbb{R}^N)$. By difference, it follows that $\langle F, Dv\rangle \in L^p(\mathbb{R}^N)$. Therefore $u \in D(B)$ and we have proved that $D(B) = \mathcal{D}_\mu$. This concludes the proof. □

In the proposition below, we show that assuming (A4') instead of (A4), the measure μ turns out to be the invariant measure of the semigroup yielded by Theorem 5.3.3.

Proposition 5.3.4 *Assume that (A1), (A2), (A3), (A4') hold. Then μ is, up to a multiplicative constant, the unique invariant measure of the semigroup $(T(t))$ generated by $(B, \mathcal{D}_\mu)$.*

Proof. We claim that $C^2_c(\mathbb{R}^N)$ is a core of B. Recalling the notation introduced in the proof of Theorem 5.3.3, from Lemma 1.3.1 it follows that $C^\infty_c(\mathbb{R}^N)$ is a core of $\widetilde{B}$. This easily implies that $J^{-1}(C_c^\infty(\mathbb{R}^N))$ is a core of B. Indeed, take $u \in \mathcal{D}_\mu$ and consider $v = Ju \in \mathcal{D}_p$. Let (v_n) be a sequence in $C_c^\infty(\mathbb{R}^N)$ such that $v_n \to v$ and $\widetilde{B}v_n \to \widetilde{B}v$ in $L^p(\mathbb{R}^N)$. Set $u_n = J^{-1}v_n$. Then $u_n \in J^{-1}(C_c^\infty(\mathbb{R}^N))$ and $u_n \to u$, $Bu_n = J^{-1}\widetilde{B}v_n \to J^{-1}\widetilde{B}v = Bu$ in $L^p(\mathbb{R}^N,\mu)$. Now, since $J^{-1}(C_c^\infty(\mathbb{R}^N)) \subset C_c^2(\mathbb{R}^N) \subset \mathcal{D}_\mu$ the statement follows. Therefore, in order to show that μ is an invariant measure of $(T(t))$, it is sufficient to prove that $\int_{\mathbb{R}^N} Bu\, d\mu = 0$, for all $u \in C_c^2(\mathbb{R}^N)$ (see Proposition 5.1.2). This follows easily integrating by parts and taking condition (A4') into account. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^N} Bu\, d\mu &= \int_{\mathbb{R}^N} \operatorname{div}(e^{-\Phi} qDu)dx + \int_{\mathbb{R}^N} \langle G, Du\rangle e^{-\Phi} dx \\ &= -\int_{\mathbb{R}^N} \operatorname{div} G\, u\, e^{-\Phi} dx + \int_{\mathbb{R}^N} \langle G, D\Phi\rangle u\, e^{-\Phi} dx = 0.\end{aligned}$$

To see that μ is the *unique* invariant measure of $T(t)$, we first note that $T(t)$ is the extension to $L^p(\mathbb{R}^N,\mu)$ of the semigroup generated by $(B, D_{\max}(B))$ in $C_b(\mathbb{R}^N)$, where $D_{\max}(B) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $q < \infty \mid Au \in C_b(\mathbb{R}^N)\}$ (see Section 5.2). Indeed, since $C_c^2(\mathbb{R}^N)$ is a core for $(B, \mathcal{D}_\mu)$ and since $C_c^2(\mathbb{R}^N)$ is contained in $D_{\max}(B)$, we deduce that $D_{\max}(B)$ is also a core for $(B, \mathcal{D}_\mu)$, hence $(B, \mathcal{D}_\mu)$ is the closure of $(B, D_{\max}(B))$ in $L^p(\mathbb{R}^N,\mu)$. Recalling Proposition 5.2.13, we get that the semigroup generated by $(B, \mathcal{D}_\mu)$ is the extension of that generated in $C_b(\mathbb{R}^N)$, as claimed. At this point, the uniqueness of μ as invariant measure follows, as usual, from the irreducibility and the strong Feller property (see Proposition 5.2.2). □

Appendix A

Maximum principles

In this appendix we state and prove the maximum principles used in the previous chapters. They are not classical, since the coefficients of the involved operator are unbounded. More precisely, let us consider

$$A = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i=1}^{N} F_i \, D_i - V, \tag{A.0.1}$$

with $q_{ij} = q_{ji}$, F_i, V continuous real-valued functions in $\mathbb{R}^N$, satisfying

$$V \geq 0, \qquad \sum_{i,j=1}^{N} q_{ij}(x)\xi_i\xi_j \geq \nu_0|\xi|^2, \ \nu_0 > 0.$$

To overcome the unboundedness of the coefficients, we make the following assumption

(H) *there exists a positive function* $\varphi \in C^2(\mathbb{R}^N)$*, such that* $\lim_{|x|\to+\infty} \varphi(x) = +\infty$ *and* $A\varphi - \lambda_0 \varphi \leq 0$*, for some* $\lambda_0 > 0$.

φ is called a *Liapunov function.* Clearly, assumption (H) gives growth bounds on the coefficients of A. If for instance $\varphi(x) = 1 + |x|^2$, then (H) is satisfied if there exists $\lambda_0 > 0$ such that

$$\operatorname{Tr} Q(x) + \langle F(x), x\rangle \leq \lambda_0(1 + |x|^2).$$

It can be assumed that $\sup_{\mathbb{R}^N}(A\varphi - \lambda_0\varphi) < +\infty$. This does not make any difference since replacing φ with $\varphi + C$ for a suitable constant C, we return exactly to (H). Moreover, when one deals with parabolic problems, it is possible to consider φ dependent also on time and to require that $\varphi \in C^2([0,T] \times \mathbb{R}^N)$, $\varphi \geq 0$, $\lim_{|x|\to+\infty} \varphi(t,x) = +\infty$ uniformly in $[0,T]$ and $(D_t - A + \lambda_0)\varphi \geq 0$. Since we are concerned both with parabolic and elliptic problems and since the coefficients of A do not depend on t, we keep assumption (H) throughout the manuscript.

We start by proving maximum principles for parabolic and elliptic problems in a regular, (possibly) unbounded open set Ω of $\mathbb{R}^N$ with Neumann boundary conditions. Such results have been used in Chapter 2. In this case it is sufficient for φ to be defined in $\overline{\Omega}$, but we have to require an additional condition concerning its normal derivatives on $\partial\Omega$. The proof is similar to [34, Proposition 2.1].

Proposition A.0.5 *Let* Ω *be an open set in* $\mathbb{R}^N$ *with* C^1 *boundary. Assume (H) and in addition suppose that* $\dfrac{\partial\varphi}{\partial\eta} \geq 0$ *on* $\partial\Omega$*, where* η *is the outward unit normal vector to* $\partial\Omega$*. Let* $z \in C([0,T] \times$

$\overline{\Omega}) \cap C^1(]0,T] \times \overline{\Omega}) \cap C^{1,2}(]0,T] \times \Omega)$ *be a bounded function satisfying*

$$\begin{cases} z_t(t,x) - Az(t,x) \le 0, & 0 < t \le T,\ x \in \Omega, \\ \dfrac{\partial z}{\partial \eta}(t,x) \le 0, & 0 < t \le T,\ x \in \partial\Omega, \\ z(0,x) \le 0 & x \in \Omega. \end{cases}$$

Then $z \le 0$.

PROOF. Set $v(t,x) = e^{-\lambda_0 t} z(t,x)$; we prove that $v \le 0$, then the statement follows. We consider the sequence

$$v_n(t,x) = v(t,x) - \frac{1}{n}\varphi(x), \quad 0 \le t \le T,\ x \in \Omega,$$

and we observe that

$$\begin{cases} D_t v_n(t,x) - (A - \lambda_0) v_n(t,x) \le 0, & 0 < t \le T,\ x \in \Omega, \\ \dfrac{\partial v_n}{\partial \eta}(t,x) \le 0, & 0 < t \le T,\ x \in \partial\Omega, \\ v_n(0,x) \le 0, & x \in \overline{\Omega}. \end{cases}$$

For every $n \in \mathbb{N}$ the function v_n attains its maximum in $[0,T] \times \overline{\Omega}$ at some point (t_n, x_n). If $t_n > 0$ and $x_n \in \Omega$ then

$$D_t v_n(t_n, v_n) \ge 0, \quad A v_n(t_n, x_n) + V(x_n) v_n(t_n, x_n) \le 0,$$

and consequently, using the equation

$$(\lambda_0 + V(x_n)) v_n(t_n, x_n) \le (\lambda_0 + D_t - A) v_n(t_n, x_n) \le 0.$$

Since $\lambda_0 > 0$ this implies that $v_n(t_n, x_n) \le 0$.

If $t_n = 0$ we immediately have $v_n(t_n, x_n) \le 0$. Finally, it is not possible that $t_n > 0$ and $x_n \in \partial\Omega$, without any interior maximum point because of the strong maximum principle ([24, Theorem 2.14]).

Therefore we have proved that $v(t,x) \le n^{-1}\varphi(x)$ for all $0 \le t \le T$ and $x \in \overline{\Omega}$. Thus letting $n \to +\infty$ we conclude that $v \le 0$, as claimed. □

A similar maximum principle holds in the elliptic case. However, we point out that the involved solutions are only of class $W^{2,p}$ and not C^2 in general. To prove such a result we need a maximum principle for operators with bounded coefficients, which is due to Bony (see [9]).

Lemma A.0.6 *Let* Ω *be an open subset of* $\mathbb{R}^N$ *and let* $F : \Omega \to \mathbb{R}^N$ *be a function of class* $W^{1,p}$, *with* $p > N$. *Then the image through* F *of a set with measure zero has still measure zero.*

PROOF. Let Q_1 be a unitary cube of $\mathbb{R}^N$. By Morrey's inequality (see [10, Teorema IX.12]), if $\varphi \in W^{1,p}(Q_1)$ then

$$|\varphi(x) - \varphi(y)| \le C|x-y|^{1-\frac{N}{p}} \left(\int_{Q_1} |D\varphi|^p \right)^{\frac{1}{p}}, \qquad x, y \in Q_1, \tag{A.0.2}$$

where C is a positive constant depending on p and N. In the sequel, we keep the same notation to denote a constant which has such a dependence. If Q_α is a cube with side l_α and ψ is a function in $W^{1,p}(Q_\alpha)$, then $\varphi(x) = \psi(l_\alpha x)$ belongs to $W^{1,p}(Q_1)$ and (A.0.2) applied to φ yields

$$|\psi(l_\alpha x) - \psi(l_\alpha y)| \le C|x-y|^{1-\frac{N}{p}} \left(\int_{Q_1} l_\alpha^p |D\psi(l_\alpha z)|^p dz \right)^{\frac{1}{p}}, \qquad x, y \in Q_1.$$

By changing variables in the integral we get

$$\begin{aligned}|\psi(l_\alpha x)-\psi(l_\alpha y)| &\leq C|x-y|^{1-\frac{N}{p}}\Big(\int_{Q_\alpha} l_\alpha^{p-N}|D\psi(z)|^p dz\Big)^{\frac{1}{p}}\\ &= C\, l_\alpha^{1-\frac{N}{p}}|x-y|^{1-\frac{N}{p}}\Big(\int_{Q_\alpha}|D\psi(z)|^p dz\Big)^{\frac{1}{p}}\\ &\leq C\, l_\alpha^{1-\frac{N}{p}}\Big(\int_{Q_\alpha}|D\psi(z)|^p dz\Big)^{\frac{1}{p}}, \qquad x,y\in Q_1.\end{aligned}$$

Therefore

$$|\psi(\xi)-\psi(\eta)|\leq C\, l_\alpha^{1-\frac{N}{p}}\Big(\int_{Q_\alpha}|D\psi(x)|^p dx\Big)^{\frac{1}{p}}, \qquad \xi,\eta\in Q_\alpha. \tag{A.0.3}$$

Let M be a subset of Ω with $|M|=0$, where $|\cdot|$ denotes the Lebesgue measure. Then, for every $\varepsilon>0$ there exists a family $\{Q_\alpha\}_\alpha$ of disjoint cubes such that $M\subseteq\cup_\alpha Q_\alpha\subseteq\Omega$ and $\sum_\alpha l_\alpha^N\leq\varepsilon$, where l_α denotes the side of Q_α. By applying (A.0.3) to the scalar components $F_1,...,F_N$ of the function F, we obtain for every α and every $x,y\in Q_\alpha$

$$\begin{aligned}|F(x)-F(y)| &\leq \sum_{i=1}^N |F_i(x)-F_i(y)|\leq C\, l_\alpha^{1-\frac{N}{p}}\sum_{i=1}^N\Big(\int_{Q_\alpha}|DF_i(z)|^p dz\Big)^{\frac{1}{p}}\\ &\leq C\, l_\alpha^{1-\frac{N}{p}}\Big(\int_{Q_\alpha}\Big(\sum_{i,j=1}^N|D_jF_i|\Big)^p\Big)^{\frac{1}{p}} =: \lambda_\alpha.\end{aligned}$$

This means that $F(Q_\alpha)$ is contained in the cube $\widetilde{Q}_\alpha$ with side λ_α. It follows that

$$F(M)\subseteq F\Big(\bigcup_\alpha Q_\alpha\Big)\subseteq\bigcup_\alpha F(Q_\alpha)\subseteq\bigcup_\alpha\widetilde{Q}_\alpha$$

and consequently

$$|F(M)|\leq\sum_\alpha|\widetilde{Q}_\alpha|=\sum_\alpha\lambda_\alpha^N=C^N\sum_\alpha\Big[l_\alpha^{N(1-\frac{N}{p})}\Big(\int_{Q_\alpha}\Big(\sum_{i,j=1}^N|D_jF_i|\Big)^p\Big)^{\frac{N}{p}}\Big].$$

Applying Hölder's inequality with exponents $r=p/N$ and $r'=(1-N/p)^{-1}$, we get

$$\begin{aligned}|F(M)| &\leq C^N\Big(\sum_\alpha l_\alpha^N\Big)^{1-\frac{N}{p}}\Big(\sum_\alpha\int_{Q_\alpha}\Big(\sum_{i,j=1}^N|D_jF_i|\Big)^p\Big)^{\frac{N}{p}}\\ &\leq C^N\varepsilon^{1-\frac{N}{p}}\Big(\int_\Omega\Big(\sum_{i,j=1}^N|D_jF_i|\Big)^p\Big)^{\frac{N}{p}}.\end{aligned}$$

Since ε was arbitrary, the thesis follows. □

Proposition A.0.7 *Let Ω be a bounded open set of $\mathbb{R}^N$ with C^1 boundary and let $u\in W^{2,p}(\Omega)$, with $p>N$. Assume that u attains its maximum M at $x_0\in\Omega$ and that $u(x)<M$, for every $x\in\overline{\Omega}\setminus\{x_0\}$. Then, for each closed neighborhood V of x_0 there exists $E\subseteq V$ with $|E|>0$, such that for almost all $x\in E$ the Hessian matrix of u, $(D^2u(x))$, is nonpositive, i.e. $\langle D^2u(x)\xi,\xi\rangle\leq 0$, for all $\xi\in\mathbb{R}^N$.*

Proof. Let S be the hypersurface of $\mathbb{R}^{N+1}$ given by the equation $y = u(x)$, $x \in \Omega, y \in \mathbb{R}$. Since $p > N$, by the Sobolev embeddings the function u belongs to $C^1(\overline{\Omega})$, hence S is of class C^1. This ensures that the tangent hyperplane is well defined at each point of S. Let V be a closed neighborhood of x_0 contained in Ω and let us denote by E the set of points x in V with the property that S lies locally under the tangent hyperplane at $(x, u(x))$. We observe that E is nonempty since it contains x_0. Now, we claim that E has positive measure. Let us first show that there exists $\delta > 0$ such that if $h \in \mathbb{R}^N$ and $|h| < \delta$, then there are a point $\xi \in E$ and a real number α such that the hyperplane of equation $y = \langle h, x\rangle + \alpha$ is tangent to S at the point $(\xi, u(\xi))$. To this aim, we observe that $\inf_{\overline{\Omega}\setminus V}(M - u(x)) > 0$. Otherwise, there exists a sequence $(x_n) \subseteq \overline{\Omega} \setminus V$ such that $u(x_n)$ converges to M. By compactness, we can find $y \in \overline{\Omega} \setminus \{x_0\}$ and a subsequence (x_{n_k}) such that $x_{n_k} \to y$ and therefore, by continuity, $u(x_{n_k}) \to u(y) = M$. But this is impossible since x_0 was, by the assumption, the unique maximum point of u in $\overline{\Omega}$. Now consider $\lambda = \inf\limits_{\overline{\Omega}\setminus V}(M - u(x))\Big(\sup\limits_{\overline{\Omega}\setminus V}|x - x_0|\Big)^{-1} > 0$ and choose $0 < \delta < \lambda$. Then, for every $h \in \mathbb{R}^N$ with $|h| < \delta$ and every $x \in \overline{\Omega} \setminus V$ we have

$$
\begin{aligned}
u(x) - M - \langle h, x - x_0\rangle &< u(x) - M + \inf_{\overline{\Omega}\setminus V}(M - u(x))\Big(\sup_{\overline{\Omega}\setminus V}|x - x_0|\Big)^{-1}|x - x_0| \\
&\le \inf_{\overline{\Omega}\setminus V}(M - u(x)) - (M - u(x)) \le 0,
\end{aligned}
$$

hence

$$
u(x) < \langle h, x\rangle + M - \langle h, x_0\rangle, \qquad \text{for all } x \in \overline{\Omega} \setminus V. \tag{A.0.4}
$$

Since V is compact and $u(x) - \langle h, x\rangle$ is a continuous function in V, there exists $\xi \in V$ such that

$$
\max_{x\in V}(u(x) - \langle h, x\rangle) = u(\xi) - \langle h, \xi\rangle =: \alpha.
$$

In particular, $\alpha \ge u(x_0) - \langle h, x_0\rangle = M - \langle h, x_0\rangle$ and therefore from (A.0.4) it follows that

$$
u(x) < \langle h, x\rangle + \alpha, \qquad \text{for all } x \in \overline{\Omega} \setminus V.
$$

On the other hand, by construction,

$$
u(x) \le \langle h, x\rangle + \alpha, \qquad \text{for all } x \in V,
$$

then $u(x) \le \langle h, x\rangle + \alpha$, for every $x \in \overline{\Omega}$. Since $u(\xi) = \langle h, \xi\rangle + \alpha$, we deduce also that $Du(\xi) = h$ and therefore the hyperplane $y = \langle h, x\rangle + \alpha$ is in fact the tangent hyperplane to S at $(\xi, u(\xi))$. Since it lies over S, we have that $\xi \in E$. Now, define $F : \Omega \to \mathbb{R}^N$ as $F(x) = Du(x)$. From the previous step, if $h \in \mathbb{R}^N$ and $|h| < \delta$, then there exists $\xi \in E$ such that $h = Du(\xi) = F(\xi)$. This means that $B_\delta \subseteq F(E)$ and, as a consequence, $|F(E)| > 0$. Since F is of class $W^{1,p}(\Omega)$, from the previous lemma it follows that E has positive measure, too.

Now, the regularity of u implies that u is almost everywhere twice differentiable in the classical sense. Let $x \in E$ be such that u is twice differentiable at x in the classical sense and assume, by contradiction, that there exists $y \in \mathbb{R}^N$ such that $\sum_{i,j=1}^N D_{ij}u(x)\, y_i y_j > 0$. Without loss of generality we can suppose that $|y| = 1$. Set $f(t) = u(x + ty) - t\langle Du(x), y\rangle$, for $|t| < \varepsilon$, for some $\varepsilon > 0$. Then f is differentiable in $(-\varepsilon, \varepsilon)$ with $f'(0) = 0$ and f'' exists at $t = 0$ with $f''(0) = \sum_{i,j=1}^N D_{ij}u(x)\, y_i y_j > 0$. This implies that $t = 0$ is a strict relative minimum point for f, hence $f(t) > f(0)$ for $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$, which means $u(x + ty) > u(x) + t\langle Du(x), y\rangle$, for $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$. On the other hand, since $x \in E$, for every z sufficiently close to x we have

$$
u(z) \le u(x) + \langle Du(x), z - x\rangle.
$$

Choosing $z = x + ty$ we find

$$u(x+ty) \leq u(x) + t\langle Du(x), y\rangle,$$

which is a contradiction. Thus, we have established that at each point $x \in E$ where u is twice differentiable in the classical sense, $(D^2u(x))$ is nonpositive. This concludes the proof. $\square$

At this point, we are ready to prove the announced maximum principle for $W^{2,p}$ functions involving operators with bounded coefficients. More precisely, let

$$L = \sum_{i,j=1}^{N} \alpha_{ij} D_{ij} + \sum_{i=1}^{N} \beta_i \, D_i + \gamma.$$

Assume that all the coefficients are real-valued functions in $L^\infty(\Omega)$ and that the matrix (α_{ij}) is symmetric and nonnegative and that $\gamma \leq 0$.

Theorem A.0.8 *Let Ω be a bounded open set with C^1 boundary and let $u \in W^{2,p}(\Omega)$, with $p > N$. Assume that u attains a nonnegative maximum at $x_0 \in \Omega$. Then*

$$\liminf_{x\to x_0} \operatorname{ess}\,(Lu)(x) \leq 0,$$

where $\liminf_{x\to x_0} \operatorname{ess}\,(Lu)(x) = \sup_{\rho>0} \inf_{x \in \overline{B_\rho(x_0)}} \operatorname{ess}\, Lu(x)$.

PROOF. Let $\varepsilon > 0$ and set $v(x) = u(x) - \varepsilon|x - x_0|^2$. It is readily seen that $v \in W^{2,p}(\Omega)$ and that x_0 is a strict maximum point for v. Then, from Proposition A.0.7 for each $\rho > 0$, there exists a set $E_\rho \subset \overline{B_\rho(x_0)}$ such that $|E_\rho| > 0$ and $(D^2v(x))$ is nonpositive for almost all $x \in E_\rho$. Since (α_{ij}) is nonnegative a.e., we deduce that

$$\sum_{i,j=1}^{N} \alpha_{ij}(x) D_{ij} v(x) \leq 0, \quad \text{for almost all } x \in E_\rho.$$

On the other hand, since $v \in C^1(\Omega)$, we have that $\lim_{x\to x_0} D_i v(x) = D_i v(x_0) = 0$ and hence, using the boundedness of β_i

$$\lim_{x\to x_0} \sum_{i=1}^{N} \beta_i(x) D_i v(x) = 0.$$

Finally, since $\gamma(x) \leq 0$ and $v(x_0) = u(x_0) \geq 0$ we have that $\lim_{x\to x_0} \gamma(x)v(x) = 0$, if $v(x_0) = 0$. If $v(x_0) > 0$ then, by continuity, $v(x) > 0$ for x close to x_0, hence $\gamma(x)v(x) \leq 0$. Therefore we have

$$\begin{aligned}
\liminf_{x\to x_0} \operatorname{ess}\,(Lv)(x) &= \sup_{\rho>0} \inf_{x\in \overline{B_\rho(x_0)}} \operatorname{ess}\,(Lv)(x) \\
&\leq \sup_{\rho>0} \inf_{x\in E_\rho} \operatorname{ess} \left(\sum_{i,j=1}^{N} \alpha_{ij}(x) D_{ij} v(x) + \sum_{i=1}^{N} \beta_i(x) D_i v(x) + \gamma(x) v(x) \right) \\
&\leq 0.
\end{aligned}$$

Thus we have established that $\liminf_{x\to x_0} \operatorname{ess}\,(Lv)(x) \leq 0$. Since

$$Lv(x) = Lu(x) - 2\varepsilon \sum_{i=1}^{N} \alpha_{ii}(x) - 2\varepsilon \sum_{i=1}^{N} \beta_i(x)(x_i - x_0^i) - \varepsilon\gamma(x)|x - x_0|^2,$$

we obtain that

$$\liminf_{x\to x_0} \operatorname{ess} Lu(x) \leq 2\varepsilon \sum_{i=1}^{N} \|\alpha_{ii}\|_\infty.$$

Letting $\varepsilon \to 0$, we get the statement. $\square$

In the sequel, we use the previous result to derive an elliptic maximum principle for the operator A defined in (A.0.1). First we state an easy corollary of Theorem A.0.8, which is more useful for our aims.

Corollary A.0.9 *Let u belong to $W^{2,p}_{\rm loc}(\mathbb{R}^N)$ for any $p < \infty$ and suppose that $Au \in C(\mathbb{R}^N)$. If u has a relative nonnegative maximum at the point x_0, then $Au(x_0) \leq 0$.*

Proposition A.0.10 *Let Ω be an open set in $\mathbb{R}^N$ with C^2 boundary. Let $u \in C_b(\overline{\Omega}) \cap W^{2,p}(\Omega \cap B_R)$ for all $R > 0$ and $p < \infty$, such that $Au \in C_b(\overline{\Omega})$ and*

$$\lambda u(x) - Au(x) \leq 0, \qquad x \in \Omega,$$

for some $\lambda > 0$. Let $x_0 \in \partial\Omega$ such that $u(x_0) > 0$ and $u(x) < u(x_0)$ for all $x \in \Omega$. Then

$$\frac{\partial u}{\partial \eta}(x_0) > 0. \tag{A.0.5}$$

Proof. We follow the proof of the classical Hopf maximum principle (see e.g. [26, Lemma 3.4]). By the regularity assumption on $\partial\Omega$, we can consider a ball $B(y,r) \subset \Omega$ such that $\overline{B}(y,r) \cap \partial\Omega = \{x_0\}$. Assume that $u > 0$ in $B(y,r)$. It is readily seen that there exists $\alpha > 0$ such that the function $z(x) = e^{-\alpha|x-y|^2} - e^{-\alpha\, r^2}$ satisfies $Az > 0$ in $D = B(y,r) \setminus \overline{B}(y,r/2)$. Set $w = u + \varepsilon z$, where $\varepsilon > 0$ is chosen in such a way that $w(x) < u(x_0)$ for all $x \in \partial B(y,r/2)$. Then $w(x) \leq u(x_0)$ in ∂D and

$$Aw(x) = Au(x) + \varepsilon Az(x) > \lambda u(x) > 0, \qquad x \in D. \tag{A.0.6}$$

Let $\overline{x} \in \overline{D}$ the maximum point of w in $\overline{D}$. It is not possible that $\overline{x} \in D$, otherwise from Corollary A.0.9 we should have $Aw(\overline{x}) \leq 0$, which is in contradiction with (A.0.6). Then $\overline{x} \in \partial D$ and necessarily $\overline{x} = x_0$. It follows that

$$\frac{\partial w}{\partial \eta}(x_0) = \frac{\partial u}{\partial \eta}(x_0) + \varepsilon \frac{\partial z}{\partial \eta}(x_0) \geq 0.$$

Since $\partial z/\partial\eta(x_0) < 0$, this implies (A.0.5). □

Proposition A.0.11 *Let Ω be an open set in $\mathbb{R}^N$ with C^2 boundary. Assume (H) and in addition suppose that $\dfrac{\partial \varphi}{\partial \eta} \geq 0$ on $\partial\Omega$, where η is the outward unit normal vector to $\partial\Omega$. Let $u \in C_b(\overline{\Omega}) \cap W^{2,p}(\Omega \cap B_R)$ for all $R > 0$ and $p < \infty$, such that $Au \in C_b(\overline{\Omega})$ and*

$$\begin{cases} \lambda u(x) - Au(x) \leq 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \eta}(x) \leq 0, & x \in \partial\Omega, \end{cases} \tag{A.0.7}$$

for some $\lambda \geq \lambda_0$. Then $u \leq 0$.

Proof. As in Proposition A.0.5, we introduce the sequence

$$u_n(x) = u(x) - \frac{1}{n}\varphi(x), \qquad x \in \Omega$$

and we note that

$$\begin{cases} \lambda u_n(x) - Au_n(x) \leq 0, & x \in \Omega, \\ \dfrac{\partial u_n}{\partial \eta}(x) \leq 0, & x \in \partial\Omega. \end{cases} \tag{A.0.8}$$

We prove that $u_n \le 0$, for all $n \in \mathbb{N}$; then the conclusion follows letting $n \to \infty$. Each u_n has a maximum point $x_n \in \overline{\Omega}$. If $x_n \in \Omega$ then $u_n(x_n) \le 0$. Indeed, if $u_n(x_n) > 0$, then from Corollary A.0.9 it follows that $Au_n(x_n) \le 0$ and, using (A.0.8), $u_n(x_n) \le 0$, which is a contradiction. Now assume that $x_n \in \partial\Omega$ and $u_n(x) < u_n(x_n)$ for all $x \in \Omega$ (otherwise there would exist an interior maximum point and we could apply the previous step). Then from Proposition A.0.10 and (A.0.8) it follows that $u_n(x_n) \le 0$ and this completes the proof. □

Next, we deal with Dirichlet parabolic problems. We skip the proof of the following proposition, since it is exactly the same as that of Proposition A.0.5.

Proposition A.0.12 *Let Ω be an open set of $\mathbb{R}^N$ and assume hypothesis (H). Let $u \in C([0,T] \times \overline{\Omega}) \cap C^{1,2}(]0,T[\times\Omega)$ be a bounded function satisfying*

$$\text{(A.0.9)} \qquad \begin{cases} u_t(t,x) \le Au(t,x), & 0 < t \le T,\ x \in \Omega, \\ u(t,x) \le 0, & 0 < t \le T,\ x \in \partial\Omega, \\ u(0,x) \le 0 & x \in \Omega, \end{cases}$$

Then $u \le 0$.

Now we present a maximum principle for discontinuous solutions to the Dirichlet parabolic problem (A.0.9). The result is suggested in [29] and involves special domains.

Theorem A.0.13 *Assume hypothesis (H). Let Ω be an open subset of $\mathbb{R}^N$, $g_i : \overline{\Omega} \to \mathbb{R}$, $i = 1,\dots,n$, be C^2-functions. Suppose that*

$$\Omega = \{x : g_i(x) > 0,\ \ i = 1,\dots,n\},\ \ |Dg_i| \ge 1 \text{ on } \Gamma_i = \partial\Omega \cap \{g_i = 0\}.$$

Define $Q = (0,T) \times \Omega$, $\partial' Q = (0,T) \times \partial\Omega \cup \{0\} \times \overline{\Omega}$ and $\partial_{tx} Q = \{0\} \times \partial\Omega$. Let $u \in C^{1,2}(Q)$, u continuous on $\overline{Q} \setminus \partial_{tx}Q$, bounded on Q. If $u_t \le Au$ in Q and $u \le 0$ in $\partial' Q \setminus \partial_{tx} Q$, then $u \le 0$ in Q.

Finally, if $u_t = Au$, $|u(t,\xi)| \le K$ for $t > 0$, $\xi \in \partial\Omega$ and $|u(0,x)| \le K$, $x \in \Omega$, then $\|u\|_\infty \le K$.

Proof. The proof is given into two steps.
Step 1. We assume in addition that Ω is bounded.

In this case the functions g_i are bounded in Ω together with their derivatives up to the second order. A long but straightforward computation shows that the functions

$$\text{(A.0.10)} \qquad \psi_i(t,x) = \frac{1}{t^{\varepsilon\nu}} \exp\left(\lambda t - \frac{\varepsilon g_i^2(x)}{t}\right)$$

verify, for $\varepsilon > 0$ small enough and λ large enough, $(D_t - A)\psi_i \ge 0$, $i = 1,\dots,n$, in $(0,\infty) \times \Omega$.

Let $M = \sup_Q u = \sup_{\overline{Q}\setminus\partial_{tx}Q} u > 0$ (otherwise the proof is finished). Let $\gamma > 0$ and define

$$u_\gamma(t,x) = u(t,x) - M\gamma^{\varepsilon\nu} \sum_{i=1}^n \frac{1}{(t+\gamma)^{\varepsilon\nu}} \exp\left(\lambda(t+\gamma) - \frac{\varepsilon g_i^2(x)}{t+\gamma}\right),$$

where ε and λ are given in (A.0.10). Clearly $(D_t - A)u_\gamma \le 0$. Take $\eta > 0$ such that $\lambda\gamma - \frac{\varepsilon\eta}{\gamma} > 0$ and consider

$$I_\eta = \{x \in \overline{\Omega} :\ \exists i = i(x) = 1,\dots,n\ :\ g_i^2(x) \le \eta\}.$$

For each $x \in I_\eta$, one has

$$\gamma^{\varepsilon\nu} \sum_{i=1}^n \frac{1}{\gamma^{\varepsilon\nu}} \exp\left(\lambda\gamma - \frac{\varepsilon g_i^2(x)}{\gamma}\right) \ge \exp\left(\lambda\gamma - \frac{\varepsilon\eta}{\gamma}\right) > 1.$$

By continuity, there exists $\delta > 0$ such that for any $(t, x) \in [0, \delta] \times I_\eta$,

$$\gamma^{\varepsilon\nu} \sum_{i=1}^{n} \frac{1}{(t+\gamma)^{\varepsilon\nu}} \exp\left(\lambda(t+\gamma) - \frac{\varepsilon g_i^2(x)}{t+\gamma}\right) > 1.$$

It follows that $u_\gamma \leq M - M = 0$ in $([0, \delta] \times I_\eta) \setminus \partial_{tx} Q$.

Since $u(0, x) \leq 0$, $x \in \Omega \setminus I_\eta$, we have $u_\gamma(0, x) < 0$, $x \in \Omega \setminus I_\eta$ as well. Because Ω is bounded, by continuity we obtain $u_\gamma(t, x) \leq 0$, $(t, x) \in [0, \delta] \times \Omega \setminus I_\eta$, for some $\delta > 0$.

We have obtained that $u_\gamma \leq 0$ in $([0, \delta] \times \overline{\Omega}) \setminus \partial_{tx} Q$. Applying the classical maximum principle in $[\delta, T] \times \overline{\Omega}$, we get that $u_\gamma \leq 0$ in $([0, T] \times \overline{\Omega}) \setminus \partial_{tx} Q$. Letting $\gamma \to 0^+$, we infer the claim.

Step 2. We consider a possibly unbounded Ω.

Here we will use the Lyapunov function φ. Set $v = e^{-\lambda_0 t} u$ and observe that $v_t - Av + \lambda_0 v \leq 0$. We prove that $v \leq 0$ in Q. Fix $R > 1$ and consider

$$\Omega_R = \Omega \cap B_R = \{g_i > 0\} \cap \{R^2 - |x|^2 > 0\}, \qquad Q_R = (0, T) \times \Omega_R.$$

Note that Ω_R satisfies the same geometric assumptions of Ω if one adds to the set $\{g_1, \ldots, g_n\}$ the function $g_0(x) = R^2 - |x|^2$. Let $C_R = \inf_{\partial B_R \cap \Omega} \varphi$. Remark that $C_R \to \infty$ as $R \to \infty$. Define

$$v_R(t, x) = v(t, x) - \|v\|_\infty \frac{\varphi(x)}{C_R}, \quad (t, x) \in Q_R.$$

It is easy to see that $(D_t - A + \lambda_0) v_R \leq 0$ in Q_R. Moreover $v_R(0, x) \leq 0$, $x \in \Omega_R$.

If $t \in (0, T)$, then $v_R(t, x) \leq 0$ for $x \in \partial B_R \cap \Omega$, since $\frac{\varphi}{C_R} \geq 1$. Moreover $v_R(t, x) \leq 0$ for $x \in \partial\Omega$, $t \in (0, T)$. This shows that $v_R \leq 0$ on the parabolic boundary of Q_R.

Applying Step 1 to the operator $\tilde{A} = A - \lambda_0$ in Ω_R, we get $v_R \leq 0$, in Q_R, that is

$$v(t, x) \leq \|v\|_\infty \frac{\varphi(x)}{C_R}.$$

Letting $R \to \infty$, we get the claim.

The last statement easily follows considering the functions $\pm u - K$. □

Observe that the above theorem covers also the case of certain non smooth domains, whose boundaries can be described by a finite number of functions g_i as in the statement, see e.g. Example 3.6.1.

Let us show that uniformly C^2 domains are covered by Theorem A.0.13.

Corollary A.0.14 . *Theorem A.0.13 holds for uniformly C^2-domains.*

Proof. It suffices to show that there exists a C^2-function $g : \overline{\Omega} \to \mathbb{R}$ such that $g > 0$ in Ω, $|Dg| \geq 1$ in $\partial\Omega = \{g = 0\}$. Let r be the distance function from $\partial\Omega$. Then $r \in C^2(\Omega_\delta)$ for some $\delta > 0$ and $|Dr| = 1$ on $\partial\Omega$. Let moreover θ be a smooth function such that $0 \leq \theta \leq 1$, $\theta = 1$ in $\Omega_{\delta/2}$, $\theta = 0$ outside Ω_δ. It is easy to check that $g = \theta r + 1 - \theta$ satisfies the claim. □

Appendix B

Smooth domains and regularity properties of the distance function

In this Appendix we collect some regularity results of the distance function $r(x) = \mathrm{dist}(x, \partial\Omega)$, when $\partial\Omega$ is the boundary of a smooth open subset Ω of $\mathbb{R}^N$. These results are well-known in the case where Ω is bounded (see e.g. [26, section 14.6]), but most of them may be extended, without much effort, to the unbounded case, as it is shown below.

First we define open sets with uniformly $C^{2+\alpha}$ boundaries, for $0 \le \alpha < 1$.

Definition B.0.15 *Let Ω be an open subset of $\mathbb{R}^N$. We say that $\partial\Omega$ is uniformly of class $C^{2+\alpha}$ if there exist a covering of $\partial\Omega$, at most countable, $\{U_j\}_{j\in\mathbb{N}}$, and a sequence of diffeomorphisms $\varphi_j : \overline{U}_j \to \overline{B}_1$ of class $C^{2+\alpha}$ such that*

$$\varphi_j(U_j \cap \Omega) = \{y \in B_1 \mid y_N > 0\}$$
$$\varphi_j(U_j \cap \partial\Omega) = \{y \in B_1 \mid y_N = 0\}$$

and the following properties are satisfied:

(i) there exists $k \in \mathbb{N}$ such that $\bigcap_{j\in J} U_j = \emptyset$, if $|J| > k$;

(ii) there exists $0 < \varepsilon < 1$ such that $\{x \in \Omega \mid r(x) < \varepsilon\} \subseteq \bigcup_{j\in\mathbb{N}} V_j$, where $V_j = \varphi_j^{-1}(B_{1/2})$;

(iii) there exists $C > 0$ such that

$$\sup_{j\in\mathbb{N}} \sum_{0\le|\beta|\le 2+\alpha} \|D^\beta \varphi_j\|_\infty + \|D^\beta \varphi_j^{-1}\|_\infty \le C.$$

Now we show that such a set Ω satisfies a *uniform interior sphere condition*, i.e. at each point $y_0 \in \partial\Omega$ there exists a ball B_{y_0} depending on y_0, contained in Ω and such that $\overline{B}_{y_0} \cap \partial\Omega = \{y_0\}$; moreover the radii of these balls are bounded from below by a positive constant.

Proposition B.0.16 *If $\partial\Omega$ is uniformly of class C^2, then it satisfies a uniform interior sphere condition.*

Proof. Using condition (iii) and taking into account that φ_j is a diffeomorphism from $\overline{U}_j$ into $\overline{B}_1$, it is easy to see that if $y \in V_j$ and $|x - y| < 1/(2C)$, then $x \in U_j$.

Let $y_0 \in \partial\Omega$ and let $\eta(y_0)$ denote the unit inward normal vector to $\partial\Omega$ at y_0. For $0 \le t < 1/(2C)$ the point $x = y_0 + t\eta(y_0)$ belongs to U_j and ($\varphi_j^{(N)}$ denotes the N-th component of φ_j)

$$\varphi_j^{(N)}(x) = tD\varphi_j^{(N)}(y_0) \cdot \eta(y_0) + R(t)$$

with $|R(t)| \leq Ct^2/2$. Since $\varphi_j^{(N)} = 0$ on $U_j \cap \partial\Omega$, then $D\varphi_j^{(N)}(y_0) = k\eta(y_0)$, with $k \geq C^{-1}$, by (iii). This yields $\varphi_j^{(N)}(x) \geq tC^{-1} - Ct^2/2 > 0$ for $0 < t < 2/C^3 := \delta$.

Thus, we have proved that

$$y + t\eta(y) \in \Omega, \qquad y \in \partial\Omega,\ t \in]0, \delta[.$$

Now, let $y \in \partial\Omega$ and set $B = B(z, \delta/2)$, where $z = y + \eta(y)\delta/2$. Then, it is easy to see that $B \subset \Omega$ and $y \in \partial B$. If y is not the unique point in $\partial\Omega \cap \partial B$, then it suffices to replace the above ball with that of radius $\delta/4$, centered at $z = y + \eta(y)\delta/4$. □

We are now ready to prove the properties of the distance function used in this paper.

Proposition B.0.17 *Assume that $\partial\Omega$ is uniformly of class C^2 and let δ be a positive constant such that at each point of $\partial\Omega$ there exists a ball which satisfies the interior sphere condition at y_0 with radius greater or equal to δ. Then*

(a) for every $x \in \Omega_\delta = \{y \in \overline{\Omega} \,|\, r(y) < \delta\}$ there exists a unique $\xi = \xi(x) \in \partial\Omega$ such that $|x - \xi| = r(x)$;

(b) $r \in C_b^2(\Omega_\delta)$;

(c) $Dr(x) = \eta(\xi(x))$, for every $x \in \Omega_\delta$.

Proof. (a) The existence part is obvious. For the uniqueness assertion, let $x \in \Omega_\delta$ and $y \in \partial\Omega$ such that $r(x) = |x - y|$. From Proposition B.0.16 there exists a ball $B = B(z, \rho)$ such that $B \subset \Omega$ and $\overline{B} \cap \partial\Omega = \{y\}$. Moreover from the definition of δ, $x \in B$. It is easy to see that x and z lie on the normal direction $\eta(y)$ and that the balls $B(x, r(x))$ and $B(z, \rho)$ are tangent at y. Then $B(x, r(x))$ still verifies the interior sphere condition at y. It follows that for every $\overline{y} \in \partial\Omega \setminus \{y\}$, one has $\overline{y} \notin B(x, r(x))$, so that y is actually the unique point such that $|x - y| = r(x)$.

The proof of the last two assertions relies on the first statement and the implicit function theorem and it is completely similar to that of the case Ω bounded. We refer to [26, section 14.6]. □

Appendix C

Some a priori estimates

The present appendix is devoted to the proof of some a priori estimates involving uniformly elliptic operators. More precisely, we derive a Schauder type parabolic estimate and an L^p elliptic estimate, by making use of classical methods suitably adapted for our purposes. Even though such estimates are well known, we have not found a proof for them exactly in the form we need.

C.1 A Schauder type parabolic estimate

Suppose we are given a second order differential operator

$$\Gamma = \sum_{i,j=1}^{N} a_{ij} D_{ij} + \sum_{i=1}^{N} b_i D_i + c, \tag{C.1.1}$$

whose coefficients $a_{ij} = a_{ji}, b_i, c$ belong to $C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)$, where $\alpha \in]0,1[$, Ω is a bounded open subset of $\mathbb{R}^N$ with $C^{2+\alpha}$ boundary and $T < +\infty$. Assume also that

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2, \tag{C.1.2}$$

for some $\nu > 0$. Then the operator $L = D_t - \Gamma$ is uniformly parabolic in $]0,T[\times\Omega$. Set

$$K = \max\left\{\|a_{ij}\|_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}, \|b_i\|_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}, \|c\|_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}\right\},$$

where we recall that

$$\|v\|_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)} = \|v\|_\infty + [v]_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}$$

$$[v]_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)} = \sup_{t\in]0,T[,\, x,y\in\Omega,\, x\neq y} \frac{|v(t,x)-v(t,y)|}{|x-y|^\alpha} + \sup_{t,s\,\in]0,T[,\, t\neq s,\, x\in\Omega} \frac{|v(t,x)-v(s,x)|}{|t-s|^{\frac{\alpha}{2}}}.$$

Classical parabolic interior Schauder estimates, (see [29, Section 8.11]), say that for every $\varepsilon > 0$ and $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ with $\mathrm{dist}(\Omega_1, \Omega\setminus\Omega_2) > 0$, there exists a constant $C > 0$, depending on $N, \alpha, \nu, K, \varepsilon, \mathrm{dist}(\Omega_1, \Omega\setminus\Omega_2)$, such that for every function $u \in C^{1+\frac{\alpha}{2},2+\alpha}(]0,T[\times\Omega_2)$ one has

$$\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(]\varepsilon,T[\times\Omega_1)} \leq C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega_2)} + \|u\|_{C(]0,T[\times\Omega_2)}\Big),$$

where (we do not write explicitly the domain)

$$\|u\|_{1+\frac{\alpha}{2},2+\alpha} = \|u\|_{1,2} + [u]_{1+\frac{\alpha}{2},2+\alpha}$$

$$\|u\|_{1,2} = \|u\|_\infty + \|u_t\|_\infty + \|Du\|_\infty + \|D^2u\|_\infty,$$

$$[u]_{1+\frac{\alpha}{2},2+\alpha} = [u_t]_{\frac{\alpha}{2},\alpha} + [D^2u]_{\frac{\alpha}{2},\alpha}$$

(see [30, Theorem IV.10.1]). Here, we derive interior estimates only with respect to the time variable. More precisely, we set

$$\begin{aligned} Q &= (-\infty, T)\times\Omega,\\ Q_\varepsilon &= (\varepsilon, T)\times\Omega,\\ S_\varepsilon &= (\varepsilon, T)\times\partial\Omega. \end{aligned}$$

Then, under the stated assumptions on Ω and Γ, the following theorem holds.

Theorem C.1.1 *There exists $C>0$ depending on $N,\alpha,\nu,K,\varepsilon,\Omega$ such that for every $u\in C^{1+\frac{\alpha}{2},2+\alpha}(Q_\varepsilon)$ with normal derivative $\frac{\partial u}{\partial\eta}$ equal to 0 on $\partial\Omega$, one has*

$$\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q_{2\varepsilon})}\le C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)}+\|u\|_{C(Q_\varepsilon)}\Big).$$

The proof of the above theorem relies on the classical technique used to prove interior estimates, namely, the introduction of a sequence of suitable cut-off functions. In this case, we choose such functions depending only on t.

Proof. We recall that, given a function $v\in C^{1+\frac{\alpha}{2},2+\alpha}(Q)$, the following interpolatory estimate holds (see [29, Lemma 10.2.1])

$$\|v_t\|_\infty+\|Dv\|_\infty+\|D^2v\|_\infty+[Dv]_{\frac{\alpha}{2},\alpha}+[v]_{\frac{\alpha}{2},\alpha}\le\theta\|v\|_{1+\frac{\alpha}{2},2+\alpha}+M\theta^{-\gamma}\|v\|_\infty, \tag{C.1.3}$$

where γ and M are positive constants and $\theta>0$ is arbitrarily small. Such an estimate can be deduced from the analogous one in $\mathbb{R}^{N+1}$ by using suitable extension operators (which do exist thanks to the regularity of Ω). Moreover if v has normal derivative equal to zero on $\partial\Omega$ then

$$\|v\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)}\le C\Big(\|Lv\|_{C^{\frac{\alpha}{2},\alpha}(Q)}+\|v\|_{C(Q)}\Big), \tag{C.1.4}$$

with $C=C(\alpha,\nu,N,K,\Omega)>0$. Let us introduce the sequences

$$t_n=\sum_{j=0}^{n}2^{-j},\qquad s_n=\varepsilon(3-t_n).$$

We observe that (s_n) is decreasing with $s_0=2\varepsilon$, $s_\infty=\varepsilon$ and $s_n-s_{n+1}=\varepsilon 2^{-n-1}$. Moreover, let ψ_n be a sequence of functions in $C^\infty(\mathbb{R})$ such that $\psi_n(t)=1$ for $t\in(s_n,T)$, $\operatorname{supp}\psi_n\subset(s_{n+1},2T)$, $0\le\psi\le1$ and

$$\|\psi_n'\|_\infty\le L2^n,\quad \|\psi_n''\|\le L4^n, \tag{C.1.5}$$

for some constant $L>0$ depending also on ε. Hence, the function $\psi_n u$ is in $C^{1+\frac{\alpha}{2},2+\alpha}(Q)$ and

$$\frac{\partial(\psi_n u)}{\partial\eta}=\psi_n\frac{\partial u}{\partial\eta}=0,\qquad \text{on }\partial\Omega.$$

Applying estimate (C.1.4) we obtain

$$\|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)}\le C\Big(\|L(\psi_n u)\|_{C^{\frac{\alpha}{2},\alpha}(Q)}+\|\psi_n u\|_{C(Q)}\Big), \tag{C.1.6}$$

with $C>0$ independent of n. One has $L(\psi_n u)=\psi_n Lu+\psi_n' u$. Then, from (C.1.5) it follows that

$$\begin{aligned} \|\psi_n Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q)} &\le \|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)}+\|Lu\|_{C(Q_{n+1})}\|\psi_n\|_{C^{\frac{\alpha}{2}}(I_{n+1})}\\ &\le \|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)}+2^n c(\varepsilon,K)\|u\|_{C^{1,2}(Q_{n+1})},\\ &\le \|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)}+4^n c(\varepsilon,K)\|u\|_{C^{1,2}(Q_{n+1})}, \end{aligned} \tag{C.1.7}$$

where $I_{n+1} = (s_{n+1}, T)$ and $Q_{n+1} = I_{n+1} \times \Omega$. Analogously,

$$\text{(C.1.8)} \qquad \begin{aligned} \|\psi'_n u\|_{C^{\frac{\alpha}{2},\alpha}(Q)} &\le \|\psi'_n\|_{C(I_{n+1})}\|u\|_{C^{\frac{\alpha}{2},\alpha}(Q_{n+1})} + \|\psi'_n\|_{C^{\frac{\alpha}{2}}(I_{n+1})}\|u\|_{C(Q_{n+1})} \\ &\le 2^n L\|u\|_{C^{\frac{\alpha}{2},\alpha}(Q_{n+1})} + 4^n L\|u\|_{C(Q_{n+1})} \\ &\le 4^n L\|u\|_{C^{\frac{\alpha}{2},\alpha}(Q_{n+1})}. \end{aligned}$$

Taking (C.1.7) and (C.1.8) into account, from (C.1.6) we infer (for a possibly different C)

$$\begin{aligned} \|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} &\le C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big) \\ &\quad +4^n c(K,\varepsilon)\Big(\|u\|_{C^{1,2}(Q_{n+1})} + \|u\|_{C^{\frac{\alpha}{2},\alpha}(Q_{n+1})}\Big) \\ &\le C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big) \\ &\quad +4^n c(K,\varepsilon)\Big(\|\psi_{n+1}u\|_{C^{1,2}(Q)} + \|\psi_{n+1}u\|_{C^{\frac{\alpha}{2},\alpha}(Q)}\Big), \end{aligned}$$

where in the last inequality we have used the fact that $\psi_{n+1} = 1$ in Q_{n+1}. Using the interpolatory estimate (C.1.3) we find that for every $\theta > 0$

$$\begin{aligned} \|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} &\le C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big) + 4^n c(K,\varepsilon)\theta\|\psi_{n+1}u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} \\ &\quad +4^n C(K,\varepsilon)\theta^{-\gamma}\|\psi_n u\|_{C(Q)}. \end{aligned}$$

Let us consider $\xi = 4^n c(K,\varepsilon)\theta$, with ξ independent of n. Choosing a small θ we may assume that $\xi < 1$. Since $\theta^{-\gamma} = \left(\frac{\xi}{C(K,\varepsilon)}\right)^{-\gamma} 4^{n\gamma}$, the last estimate becomes

$$\begin{aligned} \|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} &\le C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big) \\ &\quad +\xi\|\psi_{n+1}u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} + c_1(K,\varepsilon,M,\gamma)4^{(\gamma+1)n}\|u\|_{C(Q)}. \end{aligned}$$

Taking, if necessary, a smaller ξ in order to have $4^{\gamma+1}\xi < 1$, by multiplying by ξ^n and summing from 0 to ∞ we obtain

$$\begin{aligned} \sum_{n=0}^{\infty}\xi^n\|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} &\le \frac{C}{1-\xi}\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big) \\ &\quad +\sum_{n=1}^{\infty}\xi^n\|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} + C_2\|u\|_{C(Q)}. \end{aligned}$$

Hence

$$\|\psi_0 u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} \le \overline{C}(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}),$$

with $\overline{C} = \overline{C}(\varepsilon, K, N, \nu, \alpha, \Omega)$. Since $\psi_0 = 1$ in $Q_{2\varepsilon}$, the statement follows. □

C.2 An L^p elliptic estimate

Let Γ be the operator defined in (C.1.1). Unlike the previous section, here it is sufficient to assume that the coefficients a_{ij} are uniformly continuous and bounded in Ω and that b_i, c belong to $L^\infty(\Omega)$, with Ω bounded open subset of $\mathbb{R}^N$ of class C^2. We also assume the ellipticity condition (C.1.2).

We present interior elliptic estimates, where the involved subdomains are not assumed to have compact closure in Ω, but are allowed to have a part of the boundary overlapped on $\partial\Omega$. Neumann boundary conditions are prescribed only on this part.

Theorem C.2.1 *Let $1 < p < \infty$ and let Ω_0 and Ω_1 be open subsets contained in Ω such that $\partial\Omega_0 \cap \partial\Omega \neq \emptyset$, $\partial\Omega_1 \cap \partial\Omega \neq \emptyset$ and $\operatorname{dist}(\Omega_0, \Omega \setminus \Omega_1) > 0$. Assume also that Ω_1 is of class C^2. Then there exists a constant $C > 0$, depending on $p, N, \nu, \Omega_0, \Omega_1$, the L^∞ norms of all the coefficients and the modulus of continuity of a_{ij}, such that for every function $u \in W^{2,p}(\Omega_1)$ with $\frac{\partial u}{\partial \eta} = 0$ on $\partial\Omega_1 \cap \partial\Omega$, the estimate*

$$\|u\|_{W^{2,p}(\Omega_0)} \leq C(\|\Gamma u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)})$$

holds.

Proof. Let us consider an increasing sequence of domains Ω_n such that $\Omega_\infty = \Omega_1$ and $\operatorname{dist}(\Omega_n, \Omega \setminus \Omega_{n+1}) = O(2^{-n})$. Let θ_n be a function in $C^\infty(\mathbb{R}^N)$ such that $\theta_n = 1$ in Ω_n, $\theta_n = 0$ in an open set containing $\Omega \setminus \Omega_{n+1}$, $0 \leq \theta \leq 1$, $\frac{\partial \theta}{\partial \eta} = 0$ on $\partial\Omega$. We note that in the case where Ω is the halfspace $\{x_N > 0\}$, it is sufficient to take θ_n as an even reflection with respect to x_N in order to have $\frac{\theta_n}{\partial \eta} = 0$ when $x_N = 0$. For a regular bounded set, one can constructed such a function using the first step and local coordinates. Moreover, the first and second order derivatives of the functions θ_n satisfy the estimates

$$\|D\theta_n\|_\infty \leq L2^n, \qquad \|D^2\theta_n\|_\infty \leq L4^n.$$

Since $\theta_n u \in W^{2,p}(\Omega_1)$ and

$$\frac{\partial(\theta_n u)}{\partial \eta} = \frac{\partial \theta_n}{\partial \eta} u + \frac{\partial u}{\partial \eta}\theta_n = 0, \quad \text{on } \partial\Omega_1$$

we may apply the classical global L^p estimate (see [32, Theorem 3.11(iii)]) and we find that

$$\|\theta_n u\|_{W^{2,p}(\Omega_1)} \leq C(\|\Gamma(\theta_n u)\|_{L^p(\Omega_1)} + \|\theta_n u\|_{L^p(\Omega_1)}). \tag{C.2.1}$$

Now, it is readily seen that $\Gamma(\theta_n u) = \theta_n \Gamma u + B_n u$, where B_n is a first order differential operator, whose coefficients involve the coefficients of Γ, θ_n, $D\theta_n$ and $D^2\theta_n$. Therefore

$$\begin{aligned}\|B_n u\|_{L^p(\Omega_1)} &\leq 4^n C\|u\|_{W^{1,p}(\Omega_{n+1})} \leq 4^n C\|\theta_{n+1}u\|_{W^{1,p}(\Omega_1)} \\ &\leq 4^n C(\varepsilon\|\theta_{n+1}u\|_{W^{2,p}(\Omega_1)} + \varepsilon^{-1}\|\theta_{n+1}u\|_{L^p(\Omega_1)}),\end{aligned}$$

where we have used the interpolatory estimate $\|v\|_{W^{1,p}(\Omega_1)} \leq \varepsilon\|v\|_{W^{2,p}(\Omega_1)} + c\varepsilon^{-1}\|v\|_{L^p(\Omega_1)}$, which holds for every function $v \in W^{2,p}(\Omega_1)$ and every $\varepsilon > 0$.
Besides, we have $\|\theta_n \Gamma u\|_{L^p(\Omega_1)} \leq \|\Gamma u\|_{L^p(\Omega_1)}$. From (C.2.1) it follows that

$$\begin{aligned}\|\theta_n u\|_{W^{2,p}(\Omega_1)} \leq\ & C(\|\Gamma u\|_{L^p(\Omega_1)} + 4^n\varepsilon\|\theta_{n+1}u\|_{W^{2,p}(\Omega_1)} \\ & +4^n\varepsilon^{-1}\|\theta_{n+1}u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}).\end{aligned}$$

Set $\xi = C4^n\varepsilon$. We need ξ independent of n. Then $\varepsilon^{-1} = (\xi/C)^{-1}4^n$ and the last inequality becomes

$$\|\theta_n u\|_{W^{2,p}(\Omega_1)} \leq C(\|\Gamma u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}) + \xi\|\theta_{n+1}u\|_{W^{2,p}(\Omega_1)} + C_1 4^{2n}\|\theta_{n+1}u\|_{L^p(\Omega_1)}.$$

Choose ε in such a way that $\xi < 1$ and $\xi 4^2 < 1$. Then multiplying by ξ^n and summing on n from 0 to $+\infty$ we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \xi^n \|\theta_n u\|_{W^{2,p}(\Omega_1)} \leq\ & \frac{C}{1-\xi}(\|\Gamma u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}) \\ & + \sum_{n=1}^{\infty} \xi^n \|\theta_n u\|_{W^{2,p}(\Omega_1)} + C_2\|u\|_{L^p(\Omega_1)},\end{aligned}$$

which yields

$$\|\theta_0 u\|_{W^{2,p}(\Omega_1)} \leq C(\|\Gamma u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}).$$

Since $\theta_0 = 1$ in Ω_0 we get

$$\|u\|_{W^{2,p}(\Omega_0)} \leq C(\|\Gamma u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}),$$

and the proof is concluded. □

Notation

Let Ω be an open set of $\mathbb{R}^N$, $1 \le p < +\infty$, $k, N \in \mathbb{N}$, $0 < \alpha < 1$, $T > 0$, $a < b$.

$\lvert x \rvert$	euclidean norm of $x \in \mathbb{R}^N$;
$\langle x, y \rangle$	euclidean inner product in $\mathbb{R}^N$;
$B(x,r)$	open ball in $\mathbb{R}^N$ centered in x with radius $r > 0$;
B_r	$B(0,r)$;
Q	$(0,T) \times \Omega$;
$\partial' Q$	$(0,T) \times \partial\Omega \cup \{0\} \times \overline{\Omega}$;
$\partial_{tx} Q$	$\{0\} \times \partial\Omega$;
$\operatorname{card} J$	cardinality of a given set J;
$\lvert J \rvert$	Lebesgue measure of a given set J;
J^c	complementary set of J;
χ_{J}	characteristic function of a set J, that is the function defined as $\chi_J(x) = 1$ if $x \in J$ and $\chi_J(x) = 0$ if $x \notin J$;
$\mathbb{1}$	characteristic function of $\mathbb{R}^N$;
$\operatorname{supp} u$	support of a given function u;
D_t	partial derivative with respect to the variable t;
D_i	partial derivative with respect to x_i;
D_{ij}	$D_{x_i x_j}$;
Du	space gradient of a real-valued function u with norm $\lvert Du \rvert^2 = \sum_{i=1}^{N} (D_i u)^2$;
$D^2 u$	Hessian matrix of a real-valued function u with respect to the space variables with norm $\lvert D^2 u \rvert^2 = \sum_{i,j=1}^{N} (D_{ij} u)^2$;
$C_c^\infty(\Omega)$	space of real-valued C^∞ functions with compact support in Ω;
$C_b(\overline{\Omega})$	space of bounded continuous functions in $\overline{\Omega}$;

$C_b^k(\overline{\Omega})$	space of real-valued functions with derivatives up to order k in $C_b(\overline{\Omega})$;
$C_0(\Omega)$	space of functions in $C_b(\overline{\Omega})$ vanishing at $\partial\Omega$ and at infinity;
$C_0(\mathbb{R}^N)$	space of functions in $C(\mathbb{R}^N)$ vanishing at infinity;
$C^1(\mathbb{R}^N;\mathbb{R}^N)$	space of functions $F=(F_1,...,F_N)$ such that $F_i \in C^1(\mathbb{R}^N)$, for every i;
$C^{1,2}((a,b)\times\Omega)$	space of functions $u(t,x)$ which are continuous in $(a,b)\times\Omega$ with their indicated derivatives (not necessarily bounded);
$C^{k+\alpha}(\Omega)=C^{k+\alpha}(\overline{\Omega})$	space of functions such that the derivatives of order k are α-Hölder continuous in Ω;
$C^{1+\alpha/2,2+\alpha}((a,b)\times\Omega)$ $=C^{1+\alpha/2,2+\alpha}([a,b]\times\overline{\Omega})$	space of functions $u=u(t,x)$ such that $D_t u$ and $D_{x_ix_j}u$ are α-Hölder continuous in $(a,b)\times\Omega$ with respect to the parabolic distance $d\big((t,x),(s,y)\big)=\lvert t-s\rvert^{1/2}+\lvert x-y\rvert$;
$C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((0,+\infty)\times\overline{\Omega})$	space of functions u such that $u\in C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Omega}')$, for all $0<\varepsilon<T$ and bounded open $\Omega'\subseteq\Omega$;
$C^{1+\alpha}_{\mathrm{loc}}(\overline{\Omega})$	space of the functions which belong to $C^{1+\alpha}(\overline{\Omega}')$, for all bounded open set $\Omega'\subseteq\Omega$;
$C^k(\overline{\mathbb{R}})$	space of continuous functions with finite limits at $\pm\infty$ together with their derivatives up to order k;
$\lVert\cdot\rVert_\infty$	sup-norm;
$\lVert u\rVert_{[a,b]}$	$\sup\limits_{x\in[a,b]}\lvert u(x)\rvert$;
$\lVert u\rVert_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}$	$\lVert u\rVert_\infty+[u]_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}$;
$[u]_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}$	$\sup\limits_{\substack{t\in]0,T[,\\ x,y\in\Omega,\\ x\neq y}}\dfrac{\lvert u(t,x)-u(t,y)\rvert}{\lvert x-y\rvert^{\alpha}}+\sup\limits_{\substack{t,s\in]0,T[,\\ t\neq s,\\ x\in\Omega}}\dfrac{\lvert u(t,x)-u(s,x)\rvert}{\lvert t-s\rvert^{\frac{\alpha}{2}}}$;
$\lVert u\rVert_{1,2}$	$\lVert u\rVert_\infty+\lVert u_t\rVert_\infty+\lVert Du\rVert_\infty+\lVert D^2u\rVert_\infty$;
$[u]_{1+\frac{\alpha}{2},2+\alpha}$	$[u_t]_{\frac{\alpha}{2},\alpha}+[D^2u]_{\frac{\alpha}{2},\alpha}$;
$\lVert u\rVert_{1+\frac{\alpha}{2},2+\alpha}$	$\lVert u\rVert_{1,2}+[u]_{1+\frac{\alpha}{2},2+\alpha}$;
$(L^p(\Omega),\lVert\cdot\rVert_p)$	usual Lebesgue space;
$(W^{k,p}(\Omega),\lVert\cdot\rVert_{k,p})$	usual Sobolev space;
$W^{k,p}_{\mathrm{loc}}(\Omega)$	space of functions belonging to $W^{k,p}(\Omega')$ for all bounded open set Ω' such that $\overline{\Omega'}\subset\Omega$;
$W^{k,p}_0(\Omega)$	closure of $C^\infty_c(\Omega)$ in $W^{k,p}(\Omega)$;
$\mathcal{M}(\mathbb{R}^N)$	set of all Borel probability measures in $\mathbb{R}^N$.

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