A SYMMETRIC TENSOR NORM EXTENDING THE HILBERT-SCHMIDT NORM TO THE CLASS OF BANACH SPACES

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Abstract. *We define a family of tensor norms that extend the Hilbert-Schmidt tensor norm to the class of Banach spaces. In particular, we obtain a symmetric tensor norm extending the Hilbert-Schmidt tensor norm. As a consequence, we give new characterizations of Hilbert-Schmidt operators.*

The interpolated operator ideals $\mathcal{P}_{p,\sigma}$ of (p,σ)-absolutely continuous operators - where $1 \leq p \leq \infty$ and $0 \leq \sigma \leq 1$ - were obtained by Matter [4] by applying a certain interpolative procedure to the ideal $\mathcal{P}_p$ of p-absolutely summing operators [2]. We defined and characterized the new class of operator ideals $\mathcal{D}_{p,\sigma,q,\nu}$ obtained as products of $\mathcal{P}_{p,\sigma}$ and the dual operator ideals $\mathcal{P}_{q,\nu}^{dual}$, and the associated tensor norms $\alpha'_{q',\nu,p',\sigma}$, in [3]. These new ideals extend in a natural way the ideals of (p,q)-dominated operators, and the family $\sigma_{p,\sigma,q,\nu}$ extends the tensor norms $\alpha_{p,q}$ of Lapresté (see e.g. [1]). The aim of this note is to study the Hilbert space components of the ideals $\mathcal{D}_{2,\theta,2,1-\theta}$.

It is well known that there exist different maximal operator ideal - and then different tensor norms - extending Hilbert-Schmidt operators to the class of Banach spaces (e.g. $\mathcal{P}_2$ and $\mathcal{P}_2^{dual}$) [1]. Moreover, Puhl proved in [6] that there exist different selfadjoint and completely symmetric operator ideals generalizing Hilbert-Schmidt operators to the Banach space setting. The purpose of this paper is to show that every tensor norm $\alpha_{2,\theta,2,1-\theta}$ - where $\theta \in [0,1]$ - extends the Hilbert-Schmidt's norm ϑ to the class Banach spaces. In order to do it, we prove two inequalities involging tensor norms of type $\alpha_{p,\sigma,q,\nu}$ and their dual tensor norms $\alpha'_{p,\sigma,q,\nu}$ on tensor products of Banach spaces. The first one generalizes the well known inequality $d'_p \leq g_{p'}$ between Saphars' tensor norms d_p and g_p (see e.g. [1]). The second one is an interpolation theorem for the class of operator ideals of type $\mathcal{D}_{p,\sigma,q,\nu}$, which can be translated to another inequality between dual tensor norms $\alpha'_{p,\sigma,q,\nu}$. We apply these results to obtain the equality $\alpha_{2,\theta,2,1-\theta} = \vartheta$ on the class of Hilbert spaces for each $\theta \in [0,1]$. The extremes of this family given by $\theta = 1$ and $\theta = 0$ are the Saphar tensor norms d_2 and g_2. "In the middle" of this family we find $\alpha_{2,1/2,2,1/2}$ - a symmetric tensor norm - and we prove that it satisfies the inequality

$$\alpha'_{2,1/2,2,1/2} \leq \alpha_{2,1/2,2,1/2}$$

on the class of Banach spaces. As a consequence of $\vartheta = \alpha_{2,\theta,2,1-\theta}$ we also give new characterizations of Hilbert-Schmidt operators.

1. PRELIMINARIES

Throughout this paper we employ standard Banach space notation. We denote by BAN and by HILB the classes of Banach and Hilbert spaces, respectively. We denote by FIN the class of finite dimensional normed spaces. If $E \in$ BAN, B_E is the unit ball of E. If $1 \leq p \leq \infty$, p'

is the conjugate exponent of p. Let$(x_i) \in E^{\mathbb{N}}$ and $0 \leq \sigma < 1$. We define

$$\pi_p((x_i)) := \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{1/p}, \quad w_p((x_i)) := \sup_{x' \in B_{E'}} \left(\sum_{i=1}^{\infty} |<x_i, x'>|^p\right)^{\frac{1}{p}} \text{ and}$$

$$\delta_{p,\sigma}((x_i)) := \sup_{x' \in B_{E'}} \left(\sum_{i=1}^{\infty} \left(|<x_i, x'>|^{1-\sigma} \|x_i\|^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}.$$

We define also $\delta_{p,1}((x_i)) := \sup\{\|x_i\| \mid 1 \leq i < \infty\}$.

If μ is a probability measure on $B_{E'}$ and $p = \infty$ or $\sigma = 1$, the expression

$$\left(\int_{B_{E'}} \left(|<x, x'>|^{1-\sigma} \|x\|^{\sigma}\right)^{\frac{p}{1-\sigma}} d\mu(x')\right)^{\frac{1-\sigma}{p}}$$

must be understood as $\|x\|$.

If α is a tensor norm, we denote by α' the dual tensor norm.

We denote by $\mathcal{SG}$ the operator ideal on HILB of Hilbert-Schmidt operators and by ϑ the Hilbert-Schmidt tensor norm on HILB.

$\vartheta(z) = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} |\lambda_{ij}|^2\right)^{\frac{1}{2}}$ if $z = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} e_i \otimes u_j \in H_1 \otimes H_2$ for any orthonormal systems $(e_i)_{i=1}^{n}$ in $H_1 \in$ HILB and $(u_j)_{i=1}^{m}$ in $H_2 \in$ HILB. $\mathcal{G}_p$ denotes the ideal of p-approximable operators (see e.g. section 15 [5]).

If $1 \leq p \leq \infty$ and $0 \leq \sigma \leq 1$, we denote by $(\mathcal{P}_{p,\sigma}, \Pi_{p,\sigma})$ the injective normed ideal of (p, σ)-absolutely continuous operators in BAN [4]. $(\mathcal{P}_{p,\sigma}, \Pi_{p,\sigma}$ can be obtained using the interpolative procedure given in [2]. $\mathcal{P}_{p,0}$ coincides with the ideal $\mathcal{P}_p$ of p-absolutely summing operators. The cases $p = \infty$ and $\sigma = 1$ are not included in the original definition of Matter. We put $\mathcal{P}_{\infty,\sigma} := \mathcal{L}$ and $\mathcal{P}_{p,1} = \mathcal{L}$.

The following characterization holds.

Theorem 1.1. *(Matter [4]) For every operator $T : E \to F$, the following are equivalent:*

(i) $T \in \mathcal{P}_{p,\sigma}(E, F)$.

(ii) There is a constant $c > 0$ and a regular probability measure μ on $B_{E'}$, such that

$$\|Tx\| \leq C\left(\int_{B_{E'}} \left(|<x, x'>|^{1-\sigma} \|x\|^{\sigma}\right)^{\frac{p}{1-\sigma}} d\mu(x')\right)^{\frac{1-\sigma}{p}} \qquad \forall x \in E.$$

(iii) There exist a constant $C > 0$ such that for every finite sequence $x_1, \ldots, x_n$ in E, $\pi_{\frac{p}{1-\sigma}}((Tx_i)) \leq C\delta_{p,\sigma}((x_i))$.

In addition, $\Pi_{p,\sigma}(T)$ is the smallest number C for which (ii) and (iii) hold.

Definition 1.2. *[3] Let $1 \leq p, q \leq \infty$ and $0 \leq \sigma, \nu \leq 1$. We define the operator ideal $(\mathcal{D}_{p,\sigma,1\nu}, D_{p,\sigma,q,\nu})$ of (p, σ, q, ν)-dominated operators on BAN as the ideal product*

$$\left(\mathcal{P}_{q,\nu}^{dual}, \Pi_{q,\nu}^{dual}\right) \circ \left(\mathcal{P}_{p,\sigma}, \Pi_{p,\sigma}\right).$$

Theorem 1.3. *[3] Let $E,F \in BAN$, $T \in \mathcal{L}(E,F)$, $1 \leq r,p,q \leq \infty$ and $0 \leq \sigma,\nu \leq 1$ such that $\frac{1}{r}+\frac{1-\sigma}{p}+\frac{1-\nu}{q}=1$. The following assertions are equivalent.*

(i) $T \in \mathcal{D}_{p,\sigma,q,\nu}(E,F)$.

(ii) There exist a constant $c>0$ and regular probabilities μ and τ on $B_{E'}$, and $B_{F''}$ respectively, such that for every $x \in E$ and $y' \in F'$ the following inequality holds

$$|<Tx,y'>| \leq C\left(\int_{B_{E'}}\left(|<x,x'>|^{1-\sigma}\|x\|^{\sigma}\right)^{\frac{p}{1-\sigma}}d\mu(x')\right)^{\frac{1-\sigma}{p}}$$

$$\left(\int_{B_{F''}}\left(|<y',y''>|^{1-\nu}\|y'\|^{\nu}\right)^{\frac{q}{1-\nu}}d\tau(y'')\right)^{\frac{1-\nu}{q}}$$

(iii) There exists a constant $C>0$ such that for every $(x_i)_{i=1}^n \subset E$ and $(y_i')_{i=1}^n \subset F'$ the inequality

$$\pi_{r'}((<Tx_i,y_i'>)_{i=1}^n) \leq C\delta_{p,\sigma}((x_i)_{i=1}^n\delta_{q,\nu}((y_i')_{i=1}^n)$$

holds.

Moreover, the norm on $\mathcal{D}_{p,\sigma,q,\nu}$ is $D_{p,\sigma,q,\nu}(T)=\inf C$, where the infimum is taken over all constants C either in (ii) or in (iii).

Definition 1.4. *[3] Let $1 \leq p,q,r \leq \infty$ and $0 \leq \sigma,\nu \leq 1$ satisfying $\frac{1}{r}+\frac{1-\sigma}{p'}+\frac{1-\nu}{q'}=1$ and $E,F \in BAN$. We define on $E \otimes F$ the function*

$$\alpha_{p,\sigma,q,\nu}(z) := \inf\left\{\pi_r((\lambda_i))\delta_{q',\nu}((x_i))\delta_{p',\sigma}((y_i)) \quad | \quad z=\sum_{i=1}^n \lambda_i x_i \otimes y_i\right\}$$

We have proved in [3] that this expression defines a finitely generated tensor norm and that $\alpha'_{p,\sigma,q,\nu}$ is the tensor norm associated to the maximal operator ideal $\mathcal{D}_{q',\nu,p',\sigma}$.

The associated tensor norm for the particular case of (p',σ)-absolutely continuous operators $\mathcal{P}_{p',\sigma}$ is $d'_{p,\sigma} := \alpha'_{1,0,p,\sigma}$, and $g'_{p,\sigma} := \alpha'_{p,\sigma,1,0}$ is the associated tensor norm to $\mathcal{P}^{dual}_{p',\sigma}$. Moreover, for every $1 \leq q \leq \infty$ the equalities

$$\alpha_{p,\sigma,q,1} = g_{p,\sigma} \text{and} \alpha_{q,1,p,\sigma} = d_{p,\sigma}$$

also hold. Thus $\mathcal{D}_{p,1,q,\nu} = \mathcal{P}^{dual}_{q,\nu}$ for each $1 \leq p < \infty$ and $\mathcal{D}_{p,\sigma,q,1} = \mathcal{P}_{p,\sigma}$ for each $1 \leq q < \infty$.

2. THE FAMILY B OF TENSOR NORMS

Definition 1.4. *Let $E,F \in BAN$, $1 \leq p,q,s,t \leq \infty$ and $0 \leq \sigma,\nu \leq 1$ such that*

$$\frac{\nu}{s'}+\frac{\sigma}{t'}+\frac{1-\nu}{q'}+\frac{1-\sigma}{p'}=1 \text{ Then for every } z \in E \otimes F,$$

$$\alpha'_{p,\sigma,q,\nu}(z) \leq \alpha_{t,1-\sigma,s,1-\nu}(z).$$

Proof. The cases $\sigma = 1$ and $\nu = 0$, $\sigma = 0$ and $\nu = 1$ are the well known inequalities $d_p' \leq g_{p'}$ and $g_p' \leq d_{p'}$. For the other cases, it is enough to prove it for finite dimensional Banach spaces M ant N.

Let $1 \leq r \leq \infty$ such that $\frac{1}{r} + \frac{1-\nu}{q'} + \frac{1-\sigma}{p'} = 1$. Then $r'\left(\frac{1-\nu}{äq'}\right) + r'\left(\frac{1-\sigma}{p'}\right) = 1$ and $\frac{1}{r} = \frac{\nu}{s'} + \frac{\sigma}{t'}$.

If $z = \sum_{i=1}^{n} \mu_i x_i \otimes y_i \in M \otimes N$ and $u = \sum_{j=1}^{m} \lambda_j x_j' \otimes y_j' \in M' \otimes N'$.

$$|<z,u>| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |\mu_i|\,|\lambda_j|\,|<x_i,x_j'>|\,|<y_i,y_j'>| =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} |\lambda_j|(\|x_i\|^{1-\nu}|<x_i,\frac{x_j'}{\|x_j'\|}>|^{\nu})$$

$$(\|x_j'\|^{\nu}|<\frac{x_i}{\|x_i\|},x_j'>|^{1-\nu})(|\mu_i|^{r'\left(\frac{1-\nu}{q'}\right)}\|y_i\|^{1-\sigma}|<y_i,\frac{y_j'}{\|y_j'\|>}|^{\sigma})$$

$$(|\mu_i|^{r'\left(\frac{1-\sigma}{p'}\right)}\|y_j'\|^{\sigma}|<\frac{y_i}{\|y_i\|},y_j'>|^{1-\sigma}) \leq$$

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left(|\lambda_j|(\|x_i\|^{1-\nu}|<x_i,\frac{x_j'}{\|x_j'\|}>|^{\nu})(\|y_i\|^{1-\sigma}|<y_i,\frac{y_j'}{\|y_j'\|}>|^{\sigma})\right)^{r}\right)^{\frac{1}{r}}$$

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left(|\mu_i|^{r'\left(\frac{1-\nu}{q'}\right)}\|x_j'\|^{\nu}|<\frac{x_i}{\|x_i\|},x_j'>|^{1-\nu}\right)^{\frac{q'}{ä1-\nu}}\right)^{\frac{1-\nu}{q'}}$$

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left(|\mu_i|^{r'\left(\frac{1-\sigma}{p'}\right)}\|y_j'\|^{\sigma}|<\frac{y_i}{\|y_i\|},y_j'>|^{1-\sigma}\right)^{\frac{p'}{1-\sigma}}\right)^{\frac{1-\sigma}{p'}} \leq$$

$$\leq \left(\sum_{i}|\mu_i|^{r'}\delta_{q',\nu}((x_j'))^{\frac{q'}{1-\nu}}\right)^{\frac{1-\nu}{q'}}\left(\sum_{i=1}^{n}|\mu_i|^{r'}\delta_{p',\sigma}((y_j'))^{\frac{p'}{1-\sigma}}\right)^{\frac{1-\sigma}{p'}}$$

$$\left(\sum_{j=1}^{m}|\lambda_j|^{r}\left(\sum_{i=1}^{n}(\|x_i\|^{1-\nu}|<x_i,\frac{x_j'}{\|x_j'\|}>|^{\nu})^{\frac{s'}{\nu}}\right)^{\frac{r\nu}{s'}}\right.$$

$$\left.\left(\sum_{i=1}^{n}(\|y_i\|^{1-\sigma}|<y_i,\frac{y_j'}{\|y_j'\|}>|^{\sigma})^{\frac{t'}{\sigma}}\right)^{\frac{r\sigma}{t'}}\right)^{\frac{1}{r}} \leq$$

$$\leq \delta_{q',\nu}((x_j'))\delta_{p',\sigma}((y_j'))\left(\sum_{i=1}^{n}|\mu_i|^{r'}\right)^{\frac{1-\nu}{q'}}\left(\sum_{i=1}^{n}|\mu_i|^{r'}\right)^{\frac{1-\sigma}{p'}}\pi_r((\lambda_j))\delta_{s',1-\nu}((x_j))\delta_{t',1-\sigma}((y_j)).$$

This means that $\alpha'_{p,\sigma,q,\nu} \leq \alpha_{t,1-\sigma,s,1-\nu}$.

Proposition 2.2. *Let $E, F \in BAN$ and $T \in \mathcal{L}(E, F)$. The following are equivalent.*
(i) $T \in \mathcal{P}_{p,\sigma}(E, F)$ and $T' \in \mathcal{P}_{q,\nu}(F', E')$.
(ii) For every $\theta \in [0, 1]$ $T \in \mathcal{D}_{p,\omega,q,\tau}(E, F)$ where $\omega = (1-\theta)\sigma + \theta$ and $\tau = (1-\theta) + \theta\nu$. Moreover, $D_{p,\omega,q,\tau}(T) \leq \prod_{p,\sigma}^{1-\theta}(T) \prod_{q,\nu}^{\theta}(T')$.

Proof.
ii)→i) holds taking $\theta = 1$ and $\theta = 0$ since $\mathcal{D}_{p,1,q,\nu} = \mathcal{P}^{dual}_{q,\nu}$ and $\mathcal{D}_{p,\sigma,q,1} = \mathcal{P}_{p,\sigma}$.
i)→ii) By theorem 1.1 for every $x \in E$ and $y' \in F'$ there exist probabilities μ and η such that

$$|<Tx, y'>| \leq \|Tx\|\|y'\| \leq \Pi_{p,\sigma}(T) \left(\int_{B_{E'}} \left(|<x, x'>|^{1-\sigma} \|x\|^{\sigma} \right)^{\frac{p}{1-\sigma}} d\mu(x') \right)^{\frac{1-\sigma}{p}} \|y'\|$$

$$|<x, T'y'>| \leq \|x\|\|T'y'\| \leq \Pi^{dual}_{p,\nu}(T) \left(\int_{B_{F''}} (|<y', y''>|^{1-\nu} \|y'\|^{\nu})^{\frac{q}{1-\nu}} d\eta(y'') \right)^{\frac{1-\nu}{q}} \|x\|$$

Consider $0 \leq \theta \leq 1$. Then

$$|<Tx, y'>| \leq |<Tx, y'>|^{1-\theta} |<x, T'y'>|^{\theta} \leq$$

$$\leq \Pi_{p,\sigma}^{1-\theta}(T) \left(\int_{B_{E'}} (|<x, x'>|^{1-\sigma} \|x\|^{\sigma})^{\frac{p}{1-\sigma}} d\mu(x') \right)^{\frac{1-\sigma}{p}(1-\theta)} \|y'\|^{1-\theta}$$

$$\Pi_{q,\nu}^{\theta}(T') \left(\int_{B_{F''}} (|<y', y''>|^{1-\nu} \|y'\|^{\nu})^{\frac{q}{1-\nu}} d\eta(y'') \right)^{\frac{1-\nu}{q}} \|x\|^{\theta}$$

Now we define $\omega = (1-\theta)\sigma + \theta$ and $\tau = (1-\theta) + \theta\nu$. Then we can write

$$|<Tx, y'>| \leq \Pi_{p,\sigma}^{1-\theta}(T) \Pi_{q,\nu}^{\theta}(T') \left(\int_{B_{E'}} |<x, x'>|^{p} d\mu(x') \right)^{\frac{1-\omega}{p}} \|x\|^{\omega}$$

$$\left(\int_{B_{F''}} |<y', y''>|^{q} d\eta(y'') \right)^{\frac{1-\tau}{q}} \|y'\|^{\tau}$$

This means by theorem 1.3 that $T \in \mathcal{D}_{p,\omega,q,\tau}(E, F)$ and that

$$D_{p,\omega,q,\tau}(T) \leq \Pi_{p,\sigma}^{1-\theta}(T) \Pi_{q,\nu}^{dual^{\theta}}(T)$$

Corollary 2.3. *Let $E, F \in BAN$, $1 \leq p, q \leq \infty$ and $0 \leq \omega, \sigma, \nu, \tau, \theta \leq 1$ satisfying $\omega = (1-\theta)\sigma + \theta$ and $\tau = (1-\theta) + \theta\nu$. Then for every $x \in E \otimes F$*

$$\alpha'_{p,\tau,q,\omega}(z) \leq d'^{1-\theta}_{q,\sigma}(z) g'^{\theta}_{p,\nu}(z).$$

Proof. The result clearly holds for $E, F \in FIN$, just by applying the representation theorem for maximal operator ideals to the finite dimensional case (see e.g. 17.5 [1]). For $E, F \in BAN$ the inequality holds by the fact that all involved tensor norms are finitely generated.

From now on, we apply these results to the particular case given by $p = q = 2$ and $\sigma = \nu = 0$. In order to do this we define the following family of tensor norms.

Definition 2.4. *For every $\theta \in [0,1]$, let $\beta_\theta := \alpha_{2,1-\theta,2,\theta}$. We denote by B the family*

$$B = \{\beta_\theta \mid \theta \in [0,1]\}$$

Note that $\beta_{1/2}$ is a symmetric tensor norm.

Remark 2.5. *If we consider the family of tensor norms B, corollary 2.3 can be read as*

$$\beta'_\theta(z) \leq d_2'^{1-\theta}(z) g_2'^{\theta}(z) = g_2^{1-\theta}(z) d_2^{\theta}(z)$$

for every $z \in E \otimes F$ and for every $E, F \in BAN$.

On the other hand proposition 2.1 gives the inequality

$$\beta'_\theta \leq \beta_{1-\theta} \qquad \text{for each } \theta \in [0,1].$$

Next theorem improves these results on the class of Hilbert spaces.

Theorem 2.6. *$\beta_\theta = \vartheta = \beta'_\theta$ on HILB for each $\theta \in [0,1]$.*

Proof. The first inequality of remark 2.5 and the fact that $g_2 = d_2 = \vartheta = \vartheta'$ on tensor products of Hilbert spaces give $\beta_\theta \geq \vartheta$.

For the other inequality it is enough to prove that $\beta_\theta \leq \vartheta$ on $\ell_2^n \otimes \ell_2^n$ for each $n \in N$, since this is equivalent to $\beta_\theta \leq \vartheta$ on HILB. Fix $n \in \mathbb{N}$ and let $z = \sum_{i=1}^n \sum_{j=1}^n \alpha_{i,j} e_i \otimes e_j \in \ell_2^n \otimes_{\beta_\theta} \ell_2^n$, where $(e_i)_{i=1}^n$ is the canonical basis of ℓ_2^n. Then we can find two isometries $U_1, U_2 : \ell_2^n \to \ell_2^n$ such that $U_1 \otimes U_2(z) = \sum_{i=1}^n \lambda_i e_i \otimes e_i$ and $\vartheta(z) = \left(\sum_{i=1}^n |\lambda_i|^2\right)^{1/2}$ (see e.g. 26.11 [1], and 15.2.4, 15.2.5, 15.5.2 in [5]). Using the fact that for each $\theta \in [0,1]$

$$\delta_{2,\theta}((e_i)_{i=m}^n) = \sup_{x' \in B_{\ell_2}} \left(\sum_{i=m}^n \left(|< e_i, x' >|^{1-\theta} \|e_i\|^\theta \right)^{\frac{2}{1-\theta}} \right)^{\frac{1-\theta}{2}} =$$

$$= \sup_{x' \in B_{\ell_2}} \left(\sum_{i=m}^n \left(|< e_i, x' >|^{1-\theta} \right)^{\frac{2}{1-\theta}} \right)^{\frac{1-\theta}{2}} = w_2^{1-\theta}((e_i)_{i=m}^n) = 1,$$

and the fact that isometries onto preverse tensor norms (26.11 [1]), the following inequalities hold

$$\beta_\theta(z) = \beta_\theta\left(\sum_{i=1}^n \lambda_i e_i \otimes e_i\right) \leq \left(\sum_{i=1}^n |\lambda_i|^2\right)^{1/2} \delta_{2,\theta}((e_i)_{i=1}^n) \delta_{2,1-\theta}((e_i)_{i=1}^n) = \vartheta(z).$$

which concludes the proof.

We can use this result, proposition 5.1 in [4] and theorem 1.3 in order to obtain new characterizations of Hilbert-Schmidt operators.

Corollary 2.7. *Let $H_1, H_2 \in HILB$ and $T \in \mathcal{L}(H_1, H_2)$. For each $\theta \in (0,1)$, the following assertions are equivalent.*

i)$T \in \mathcal{SG}(H_1, H, 2)$.

ii)$T \in \mathcal{D}_{2,\theta,2,1-\theta}(H_1, H_2)$.

iii)$T \in \mathcal{B}_\theta(H_1, H_2)$.

iv) There exist a Hilbert space H and operators $T_0 \in \mathcal{P}_{2,\theta}(H_1, H)$ and $T_1 \in \mathcal{P}^{dual}_{2,1-\theta}(H, H_2)$ such that $T = T_1 \circ T_0$.

v) There exists a constant $C > 0$ and regular probabilities μ and τ on $B_{H_1'}$ and B_{H_2} respectively, such that for every $x \in H_1$ and $y' \in H_2'$ the following inequality holds

$$|<Tx, y'>| \leq$$

$$\leq C \left(\int_{B_{H_1'}} \left(|<x, x'>|^{1-\theta} \|x\|^{\theta} \right)^{\frac{2}{1-\theta}} d\mu(x) \right)^{\frac{1-\theta}{2}} \left(\int_{B_{H_2}} \left(|<y', y>|^{\theta} \|y'\|^{1-\theta} \right)^{\frac{2}{\theta}} d\tau(y) \right)^{\frac{\theta}{2}}.$$

vi) There exists a constant $C > 0$ such that for every $(x_i)_{i=1}^n \subset H_1$ and $(y_i')_{i=1}^n \subset H_2'$ the inequality

$$\pi_2((<Tx_i, y_i'>)_{i=1}^n) \leq C \delta_{2,\theta}((x_i)_{i=1}^n) \delta_{2,1-\theta}((y_i')_{i=1}^n)$$

holds.

Moreover, $\vartheta(T) = B_\theta(T) = D_{2,\theta,2,1-\theta}(T) = \inf C$, where the infimun is taken over all constants C either in v) or in vi).

Proof. *i)* $\leftrightarrow$ *ii)* $\leftrightarrow$ *iii)* holds by theorem 2.7, the representation theorem for $\mathcal{D}_{2,\theta,2,1-\theta}$ and the definition of $\mathcal{B}_\theta$. The implications *iv)* $\rightarrow$ *ii)* $\leftrightarrow$ *v)* $\leftrightarrow$ *vi)* hold by theorem 1.3.

For *i)* $\rightarrow$ *iv)*, let $T \in \mathcal{G}_2(H_1, H_2)$. Since $\frac{1}{2} = \frac{\theta}{2} + \frac{1-\theta}{2}$ for each $\theta \in (0,1)$, there exist by 15.5.9[5] a Hilbert space H and operators $T_1 \in \mathcal{G}_{2/\theta}(H, H_2)$ and $T_0 \in \mathcal{G}_{2/(1-\theta)}(H_1, H)$ such that $T = T_1 \circ T_0$. Using the isometry between $\mathcal{P}_{2,\theta}$ and $T_0 \in \mathcal{G}_{2/(1-\theta)}$ on HILB (see 5.1[4]) and the fact that every operator ideal on Hilbert spaces is completely symmetric (see e.g. 15.1.1[5]), we have $T_1 \in \mathcal{P}^{dual}_{2,1-\theta}$ and $T_0 \in \mathcal{P}_{2,\theta}$. This concludes the proof.

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