ON VECTOR-VALUED SEQUENCE SPACES

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Abstract. *In tlzis paper we investigate tlze topological properties of vector valuecl sequence spnces. After an introduction of normal Banach sequence spaces* A *we consider the vector valuecl sequence spaces* $\lambda(E)$, E *a locally convex Hausdorff space and we prove some bcisic facts concerning tlzis spaces. We give complete characterizations for barrelled vector valued DF spaces and distinguished vector valued Fréchet spaces. At tlie end ive give sufficient conclitioiis guaranteeing thcit* $\lambda(E)$ *is bornological.*

1. INTRODUCTION

Given a locally convex Hausdorff space E and denote with $cs(E)$ the system of all continuous seminorms on E. The vector space /,(E) of all bounded sequences $(x_n)_{n\in N}$ in E is naturally equipped with the locally convex Hausdorff topology induced by the seminorms

$$(x_n)_{n\in N} \longmapsto \sup_{n\in N} p(x_n), \ p \in cs(E).$$

If E is metrizable (or norined), then the same holds for $l_\infty(E)$. The space $c_0(E)$ of all zero sequences in E is a (closed) subspace of $l_\infty(E)$. Both spaces are part of a more genera1 concept of constructing vector valued sequence spaces:

Consider a normal Banach sequence space $(\lambda, \|\ .\ \|_\lambda)$, i.e. a Banach space coiitaining φ which is included in ω such that its closed unit ball is normal. For every locally convex Hausdorff space E this condition required for A guarantees that

$$\lambda(E) := \{(x_n)_{n\in N} \text{ E } E^N \ : \ (p(x_n))_{n\in N} \text{ E A for all } p \text{ E } cs(E)\}$$

is a vector space. The seminorms

$$(x_n)_{n\in N} \longmapsto \|(p(x_n))_{n\in N}\|_\lambda, \ p \in cs(E),$$

define a (quite natural) Hausdorff locally convex topology on $\lambda(E)$. Using this principle of construction, one obtains a large class of vector valiied sequence spaces containing (besides the above mentioned examples) e.g. the spaces $l_p(E)$ of the p-summable sequences, $p \in [1, \infty)$.

In [17] we investigated completeness properties of this vector valued sequence spaces.

These notes are mainly devoted to the examination of the (topological) properties of the spaces $\lambda(E)$ in dependence on properties of the spaces λ and E. Since $\lambda(E)$ contains E as a complemented subspace, it is clear that the behavior of $\lambda(E)$ is i.g. not "better" than the behavior of E.

At the end of this introduction we recall some notations and the terminology which will be used in the chapters 2-6.

The second chapter is devoted to the study of normal Banach sequence spaces. We introduce some additional properties of normal Banach sequence spaces (see[4],§1), which will be important in the following chapters. Moreover, a connection to perfect sequence spaces is obtained: A norinal Banach sequence space, whose closed unit ball is even closed in ω, is already a perfect space.

In the third chapter first we prove some fundainental properties of vector valued sequeiice spaces $\lambda(E)$, using to some extent the results given in the previous chapter. At the end we show (analogously as it is done in [15]) that $\lambda(E)$ contains a coniplemented copy of $l_\infty(E)$ whenever φ is not dense in λ and E is an arbitrary locally convex Hausdorff space.

Chaptei- four comprises a description of the bounded subsets of Frechet- aiid gDF-space-valued sequence spaces. In this context we prove the following characterization of gDF spaces:

A locally convex Hausdorff space E is a gDF space if and only if it contains a sequence $(B_n)_{n\in N}$ of bounded and absolutely convex subsets satisfying $2B_n \subset B_{n+1}$,$n \in N$, such that for every sequeiice $(U_n)_{n\in N}$ of zero neighbourhoods in E the set $\bigcap_{n\in N}(U_n + B_n)$ is again a zero neiglibourhood. In this case, the closures of the B_n's define a fundamental sequence of bounded sets in $\boldsymbol{E}$.

Using this result, one obtains that $\lambda(E)$ is a gDF space (or a DF space), if so is E. Moreover, we characterize the strong dual of $\lambda(E)$ in the case that φ is dense in λ aiid E is either metrizable or a quasibarrelled DF space. This result is essentially contained in [34] (with a different approach). In the last part of chapter four we prove some permanence properties of vector vaiued sequence spaces. For exainple, we show that the bounded subsets of $\lambda(E)$ are inetrizable, whenever E is a gDF space satisfying the same property.

Barrelledness conditions concerning vector valued sequence spaces are investigated in the fifth chapter. It is well-known (see [16], that $\lambda(E)$ is a (quasi)barrelled DF space, whenever E is a (quasi)barrelled DF space and φ is dense in the normal Banach sequence space λ. It is also well-known that for a given DF space E the space $l_\infty(E)$ is quasibarrelled if and only if E satisfies the dual density condition, cf. [2]. If, in addition, E is barrelled then $l_\infty(E)$ is also barrelled, cf. [2]. We can show that this remains true if one replaces l_∞ by any normal Banach sequence space in which φ is not dense. Furtherinore, a characterization of the distinguishedness of Frechet valued sequence spaces is obtained:

Let E be a metrizable locally convex space and let λ be a norinal Banach sequence. If φ is dense in λ and dense in the α-dual $\lambda^\times$ of λ, then $\lambda(E)$ is distinguished if and only if E is distinguished (cf. [16]).

If φ is not dense in λ or $\lambda^\times$ the space $\lambda(E)$ is distinguished if and only if E satisfies the density condition.

This last part of the described result contains clearly the cases $\lambda = l_1$, cf. [1], and $\lambda = l_\infty$.

Sufficient conditions guaranteeing that $\lambda(E)$ is (ultra)bornological are given in the last chapter. For example we prove that $\lambda(E)$ is bornological for every DFM space E whenever the normal Banach sequence space $(\lambda, \|\ .\|_\lambda)$ satisfies the following property:

$$\gamma) \qquad \|(\alpha_k)_{k\in N}\|_\lambda = \sup_{n\in N} \|((\alpha_k)_{k\leq n}, (0)_{k>n})\|_\lambda \text{ for every } (\alpha_k)_{k\in N} \in \lambda.$$

In particular, $c_0(E)$ is bornological for every DFM space E (a result of S. Dierolf, unpublishecl).

It is a great pleasure for me to express here my gratitude to Prof.Dr. S. Dierolf, my PhD supervisor, for the great support during the last years. Especially, I have to thank her for many helpful suggestions, comments and i-emarks.

I also do thank warmly Prof.Dr. J. Bonet and Dr. A. Peris for fruitful conversations and providing the best possible atmosphere for research during my stay at the Universidad Politecnica de Valencia in February 1993. In fact, the ideas of a large part of these notes were born with their support.

My gratitude also is due to my colleagues in Trier who showed continuous interest to my work. In particular, J. Wengenroth has helped me with useful and inspiring comments all the tiine.

Last, but not least, I do thank Prof.Dr. P. Dierolf, my co-PhD supervisor, for his attention and the "Studienstiftung des Deutschen Volkes" for its financial support.

Notations. For a locally convex Hausdorff space $E := (E,T)$ over the scalar field R or C we denote by $U_0(E)$ the zero neighbourhood filter of E or, equivalently,

$$U_0(E) := \{U \subset E : U \supset p^{-1}([0, i]) \text{ for some } p \in cs(E)\}.$$

We write E' for the topological dual of E and we write E'' for the bidual of E, i.e. $E'' = (E')\beta(E', E))'$ where we denote for a dual pair (E,F) by $\sigma(E,F)$ the weak and by $\beta(E,F)$ the strong topology defined on E, respectively. A closed linear subspace L of E defines a quotient E/L which we equip with the quotient topology. If $\|\cdot\|$ is a norm on E, we denote this normed space with $(E,\|\cdot\|)$. Complete normed spaces are called Banach spaces and complete metrizable locally convex spaces are called Frechet spaces.

Let B and C be subsets of the locally convex Hausdorff space E and let ρ be a scalar. Then

$$B + C := \{x + y : x \in B, y \in C\} \text{ and}$$
$$\rho B := \{\rho x : x \in B\}.$$

We denote by $[B]$ the linear span of B and in a fixed dual system $\langle E,F\rangle$ by B° its polar:

$$B^\circ := \{f \in F : \sup_{x\in B} |f(x)| \le 1\}.$$

Moreover, we write $\Gamma(B)$ for the absolutely convex hull of B. If B is already absolutely convex, we may equip $[B]$ with the Minkowski functional p_B of B, which is defined by

$$p_B(x) := \inf\{\rho > 0 : x \in \rho B\}.$$

$([B], p_B)$ is then a seminormed space. If B is bounded and absolutely convex and $([B], p_B)$ is a Banach space, we will call B a Banach disc. B is called a barrel if it is an absorbent, absolutely convex and closed subset of E. We call B bornivorous if it absorbs every bounded subset of E.

For a given sequence $(E_n)_{n\in N}$ of locally convex Hausdorff spaces we denote by $\prod_{n\in N} E_n$ its cartesian product equipped with the product topology and by $\bigoplus_{n\in N} E_n$ its locally convex direct sum. We write $E^N := \prod_{n\in N} E$ and $E^{(N)} := \bigoplus_{n\in N} E$. For $K \in \{R, C\}$ let $\omega := K^N$ and $\varphi := K^{(N)}$.

Now let us introduce some well-known topological properties for locally convex Hausdorff spaces E

E is called barrelled, if every barrel in E is a zero neighbourhood. It is called quasibarrelled, if every bornivorous barrel in E is a zero neighbourhood. We call E bornological if every absolutely convex and bornivorous subset **of** E is a zero neighbourhood. E is ultrabornological if and only if every absolutely convex subset of E which absorbs the Banach discs is already a zero neighbourhood.

2. NORMAL BANACH SEQUENCE SPACES

For sequences $a = (\alpha_n)_{n\in N}$ and $\beta = (\beta_n)_{n\in N}$ of real numbers we define:

$$a \leq \beta \text{ if and only if } \alpha_n \leq \beta_n \text{ for every } n \in N,$$

$$a < \beta \text{ if and only if } a < \beta_n \text{ for every } n \in N,$$

and for $a = (\alpha_n)_{n\in N} \in w$ let

$$|\alpha| := (|\alpha_n|)_{n\in N}.$$

Definition 2.1. *We call n set* **A** $\subset$ w normal *if*

$$A = \bigcup_{\alpha\in A} \{\beta \in \omega : |\beta| \leq |\alpha|\}.$$

Every normal set is clearly circled and the span of an (absolutely) convex and normal set is again normal. **A** linear subspace of w is normal if and only if it is normal in the sense of G. Kothe ([24], §30, 1.). Now we introduce the notion of normal Banach sequences spaces (cf. J. Bonet & S. Dierolf [4], §1):

Definition 2.2 *We call n Banach space* $(A, \|\,.\|_\lambda)$ normal Banach sequence space *if it satisfies tiie following two properties:*

a) $\varphi \subset A \subset w$.

(3) *The closed unit ball* B_λ *of* $(\lambda, \|\cdot\|_\lambda)$ *is normal.*

Property $\beta)$ is equivalent to:
For all $a \in \lambda, \beta \in \omega$ satisfying $|\beta| \leq |\alpha|$ it follows $\beta \in A$ and $\|\beta\|_\lambda \leq \|\alpha\|_\lambda$.
For $p \in [1,\infty)$ let

$$l_p := \{a \in \omega : \sum_{n=1}^{\infty} |\alpha_n|^p < \infty\}$$

be the space of ali p-summable sequences equipped with the usual norm

$$\|\cdot\|_p : l_p \to R, \alpha \longmapsto \left(\sum_{n=1}^{\infty} |\alpha_n|^p\right)^{\frac{1}{p}}.$$

Moreover, let

$$l_\infty := \{\alpha \in \omega : \sup_{n\in N} |\alpha_n| < \infty\}$$

be the space of all bounded sequences equipped with the norm

$$\|\cdot\|_\infty : l_\infty \to R, \alpha \longmapsto \sup_{n\in N} |\alpha_n|$$

and

$$c_0 := \{ a \in w : \lim_{n\to\infty} a_n = 0$$

be the space of all sequences converging to 0 normed by $\|\ .\ \|_\infty$. Then c_0 and l_p are normal Banach sequence spaces ($p \in [1,\infty]$).

Remark 2.3.

i) *Let A be a normal Bnnnch sequence space.* ***Then the*** *projection*

$$P_n : \omega \to \lambda, \alpha \longmapsto ((\alpha_k)_{k\leq n}, (0)_{k>n})$$

is *for every* $n \in N$ *welldefined ancl satisfies (because of property β)* $P_n(B_\lambda) \subset P_{n+1}(B_\lambda) \subset B_\lambda, n \in N$. *For every* $a \in$ A *and* $k \in N$ *we have* $\|P_k(\alpha)\|_\lambda \leq \lim_{n\to\infty} |P_n(\alpha)\|_\lambda = \sup_{n\in N} \|P_n(\alpha)\|_\lambda \leq \|\alpha\|_\lambda$.

ii) *If A is a normal Bnnnch, sequence spnce then the inclusions* $\varphi \hookrightarrow$ A $\hookrightarrow w$ ***nre*** *continuous. Indeed, the first one is continuous, because φ cnrries the finest locally convex topology on φ. Property β) implies* $|\alpha_n| = \|(\delta_{nk})_{k\in N}\|_\lambda^{-1} \|(\alpha_n\delta_{nk})_{k\in N}\|_\lambda \leq \|(\delta_{nk})_{k\in N}\|_\lambda^{-1} \|\alpha\|_\lambda$ *for every* $a \in$ A *ancl every* $n \in N$. *So the second inclusion* A $\hookrightarrow w$ *is also continuous.*

Fora normal Banach sequence space (A, $\|\ .\ \|_\lambda$) and $0 < a = (\alpha_n)_{n\in N} \in w$ let

$$\alpha \cdot \lambda := \{(\alpha_n\beta_n)_{n\in N} : \beta \in \lambda\},$$

$$\|(\alpha_n\beta_n)_{n\in N}\|_{\alpha\cdot\lambda} := \|\beta\|_\lambda, \beta \in \lambda,$$

be the diagonal transform of A with respect to α. Than ($\alpha \cdot$ A, $\|\ .\ \|_{\alpha\cdot\lambda}$) is also a normal Banach sequence space. The next result, due to S. Dierolf & C. Fernández [13], is a sharper version of remark 2.3 ii):

Proposition 2.4. *Let λ be a normal Banach sequence space with closed unit ball* B_λ *and let* $\alpha := (\|(\delta_{nk})_{k\in N}\|_\alpha^{-1})_{n\in N}$. *Then*

$$B_{\alpha\cdot l_1} \subset B_\lambda \subset B_{\alpha\cdot l_\infty}$$

and hence all inclusions

$$\varphi \hookrightarrow \alpha\cdot l_1 \hookrightarrow \lambda \hookrightarrow \alpha\cdot l_\infty \hookrightarrow \omega$$

are continuous.

Proof. If we define $\alpha^{-1} := (\alpha_n^{-1})_{n\in N}$ then the assertion is equivalent to

$$B_{l_1} \subset B_{\alpha^{-1}\cdot\lambda} \subset B_{l_\infty} \qquad (*).$$

α^{-1} .A is a normal Banach sequence space with $\|(\delta_{nk})_{k\in N}\|_{\alpha^{-1}\cdot\lambda} = 1$ for every $n \in N$.

To prove the first part of (*) let $x = (x_n)_{n\in N} \in B_{l_1}$ be given. Then for every natural numbers $m > k$ the triangle inequality yields

$$\|P_m(x) - P_k(x)\|_{\alpha^{-1}\cdot\lambda} \leq \sum_{n=k+1}^{m} |x_n| = \|P_m(x) - P_k(x)\|_1 \text{ and}$$

$$\|P_m(x)\|_{\alpha^{-1}\cdot\lambda} \leq \sum_{n=1}^{m} |x_n| = \|P_m(x)\|_1 \leq 1.$$

Therefore $(P_m(x))_{m\in N}$ is a Cauchy sequence in $\alpha^{-1}\cdot\lambda$ which converges to some $y \in B_{\alpha^{-1}\cdot\lambda}$. Because the spaces l_1 and $\alpha^{-1}\cdot\lambda$ are both continuously inculded in w we get that y coincides with x, the limit of $(P_m(x))_{m\in N}$ in l_1. This implies $x \in B_{\alpha^{-1}\cdot\lambda}$.

It remains to show $B_{\alpha^{-1}\cdot\lambda} \subset B_{l_\infty}$. But this is clear because property β) guarantees for every $x = (x_n)_{n\in N} \in B_{\alpha^{-1}\cdot\lambda}$ and every $n \in N$ that

$$|x_n| = \|(x_n\delta_{nk})_{k\in N}\|_{\alpha^{-1}\cdot\lambda} \leq \|(x_n)_{n\in N}\|_{\alpha^{-1}\cdot\lambda} \leq 1.$$

The proof is complete.

Of interest are the following properties which can be satisfied by normal Banach sequence spaces λ.

Definition 2.5. *We say that a normal Banach sequence space $(\lambda, \|\cdot\|_\lambda)$ satisfies*

γ) *if* $\|\alpha\|_\lambda = \sup_{n\in N} \|P_n(\alpha)\|_\lambda$ *for every* $\alpha \in \lambda$.

γ_w) *if there exists* $\rho \geq 1$ *such that* $\|\alpha\|_\lambda \leq \rho \sup_{n\in N} \|P_n(\alpha)\|_\lambda$ *for every* $\alpha \in \lambda$.

δ) *if for every* $\alpha \in w$ *with* $\sup_{n\in N} \|P_n(\alpha)\|_\lambda < \infty$ *it follows that* $\alpha \in \lambda$ *and* $\|\alpha\|_\lambda = \sup_{n\in N} \|P_n(\alpha)\|_\lambda$.

δ_w) *if there is* $\rho \geq$ *such that for every* $\alpha \in w$ *with* $\sup_{n\in N} \|P_n(\alpha)\|_\lambda < \infty$ *it follows that* $\alpha \in \lambda$ *and* $\|\alpha\|_\lambda \leq \rho \sup_{n\in N} \|P_n(\alpha)\|_\lambda$.

ε) *if* $\lim_{n\to\infty} \|P_n(\alpha) - \alpha\|_\lambda = 0$ *for every* $\alpha \in \lambda$.

We may remark that according to remark 2.3 i) one can replace in the above definition "$\sup_{n\in N} \|P_n(\alpha)\|_\lambda$" always by "$\lim_{n\to\infty} \|P_n(\alpha)\|_\lambda$".

Every normal Banach sequence space with property ε) satisfies also property γ). For $p \in [1,\infty)$ the space l_p satisfies the properties δ) and ε), the space l_∞ satisfies δ) but not ε), and the space c_0 satisfies ε) but not δ). Later we give an example of a normal Banach sequence space, which does not satisfy even condition γ_w).

Connected with the properties introduced in the previous definition are the following spaces associated to a normal Banach sequence space:

Definition 2.6. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space and let*

$$\lambda^{(\delta)} := \{\alpha \in w : \sup_{n\in N} \|P_n(\alpha)\|_\lambda < \infty\}$$

equipped with the norm

$$\|\cdot\|_{\lambda^{(\delta)}} : \lambda^{(\delta)} \to R, \alpha \longmapsto \sup_{n\in N} \|P_n(\alpha)\|_\lambda.$$

Moreover, define $\lambda^{(\gamma)}$ *to be the closure of* A *in* $(\lambda^{(\delta)}, \|\ .\ \|_{\lambda^{(\delta)}})$.

First we prove that these associated spaces are also normal Banach sequence spaces:

Proposition 2.7. *Let* (A, $\|\ .\ \|_\lambda$) *be* a *normal Banach sequence space.*

i)Then $(\lambda^{(\delta)}, \|\ .\ \|_{\lambda^{(\delta)}})$ *is also* a *normal Banaclz sequence space.*

ii)If $\varphi \subset [L] = L \subset$ A *is normal, then the closure* $\bar{L}^\lambda$ *equipped with the induced norma* $\|\ .\ \|_\lambda$ *is* a *normal Banach sequence space.*

Especially $(\bar{\varphi}^\lambda, \|\ .\ \|_\lambda)$ *and* $(\lambda^{(\gamma)}, \|\ .\ \|_{\lambda^{(\delta)}})$ *are normal Banach sequence spaces.*

Proof. i) Property ***a)***is clearly satisfied and it is easy to see that $B_{\lambda^{(\delta)}}$ is bounded in *w*. Let

$$f_n : \omega \to R, \alpha \longmapsto \|P_n(\alpha)\|_\lambda, n \in N.$$

Then the mappings f_n are continuous for every n E N and $B_{\lambda^{(\delta)}} = \bigcap_{n\in N} f_n^{-1}([0, \text{i}])$. Hence, $B_{\lambda^{(\delta)}}$ is a closed and bounded subset of *w* and therefore $\lambda^{(\delta)}$ is a Banach space. To prove property β) let *a* E $B_{\lambda^{(\delta)}}$ and β *E w* with $|\beta| \leq |\alpha|$ be given. Because A satisfies property β) we get $\sup_{n\in N} \|P_n(\beta)\|_\lambda \leq \sup_{n\in N} \|P_n(\alpha)\|_\lambda$ and this implies β E $B_{\lambda^{(\delta)}}$.

ii) First we remark, that $(\bar{L}^\lambda, \|\cdot\|_\lambda)$ is a Banach space, which satisfies property ***a).***

To prove that $B_{\bar{L}^\lambda} = B_\lambda \cap \bar{L}^\lambda$ is normal we note that intersections of normal sets are also normal. So it remains to show that $\bar{L}^\lambda$ is a normal subset of *w*. Therefore let α E $\bar{L}^\lambda$ and β E *w* with $|\beta| \leq |\alpha|$ be given. Then there exists a sequence $(\alpha^{(k)})_{k\in N} = ((\alpha_n^{(k)}))_{n\in N})_{k\in N}$ E L with $\|\alpha^{(k)} - \alpha\|_\lambda \to O(k \to \infty)$. We define for n, k E N: $(\beta^{(k)})_{k\in N} = ((\beta_n^{(k)})_{n\in N})_{k\in N}$ by

$$\beta_n^{(k)} := \begin{cases} \beta_n & \text{if } |\beta_n| \leq |\alpha_n^{(k)}| \\ \dfrac{|\alpha_n^{(k)}|}{|\beta_n|}\beta_n & \text{if } |\beta_n| > |\alpha_n^{(k)}| \end{cases}$$

Since L is normal and $|\beta^{(k)}| \leq |\alpha^{(k)}|$ we have $\beta^{(k)} \in L$ for all $k \in N$.

Let $n, k \in N$.

If $|\beta_n| \leq |\alpha_n^{(k)}|$ then $|\beta_n^{(k)} - \beta_n| = 0 \leq |\alpha_n^{(k)} - \alpha_n|$.

If $|\beta_n| > |\alpha_n^{(k)}|$ then $|\beta_n^{(k)} - \beta_n| = (1 - \frac{|\alpha_n^{(k)}|}{|\beta_n|})|\beta_n| = |\beta_n| - |\alpha_n^{(k)}| \leq |\alpha_n| - |\alpha_n^{(k)}| \leq |\alpha_n - \alpha_n^{(k)}|$.

This implies $|\beta^{(k)} - \beta| \leq |\alpha^{(k)} - \alpha|$ for every $k \in N$. Using that λ is a normal Banach sequence space we obtain:

$$\|\beta^{(k)} - \beta\|_\lambda \leq \|\alpha^{(k)} - \alpha\|_\lambda \to 0\ (k \to \infty).$$

Remembering $\beta^{(k)}$ E L, this yields $\beta \in \bar{L}^\lambda$ and we are done.

Remark 2.8. *The proof of* i) *shows that for* **every** *normal Banach sequeizce space* (A, $\|\ .\ \|_\lambda$) *the closed unir ball* $B_{\lambda^{(\delta)}}$ *is even closed* **in** *w.*

Theorem 2.9. *(cf. [4], 1.1)Let* $(\lambda, \|\ .\ \|_\lambda$ *be* a *normal Banaclz sequence space.*

i) *T.f.a.e. (The following are equivalent)*

1) (A, $\|\ .\ \|_\lambda$) *satisfies property* γ*).*

2) $(\lambda, \|\ .\ \|_\lambda) = (\lambda^{(\gamma)}, \|\cdot\|_{\lambda^{(\delta)}})$.

3) B_λ *is closed in* A *equipped with the relative topology induced by w.*

ii) T.f.a.e.

1) $(\Lambda, \|\cdot\|_\lambda)$ satisfies property γ_w).

2) $\Lambda = \lambda^{(\gamma)}$ algebraically and the corresponding norms are equivalent.

3) There exists $\rho \geq 1$ such that tlie closure of B_λ in Λ equipped with tlte relative topology induced by w is contained in ρB_λ.

iii) T.f.a.e.

1) $(\Lambda, \|.\|_\lambda)$ satisfies property δ).

2) $(\lambda, \|.\|_\lambda) = (\lambda^{(\delta)}, \|.\|_{\lambda^{(\delta)}})$.

3) B_λ is closed in w.

iv) T.f.a.e.

1) $(\lambda, \|.\|_\lambda)$ satisfies property δ_w).

2) $\Lambda = \lambda^{(\delta)}$ algebraically and the corresponding norms are equivalent.

3) There exists $\rho \geq 1$ such that the closure of B_λ in w is contained in ρB_λ.

v) T.f.a.e.

1) $(\Lambda, \|.\|_\lambda)$ satisfies property ε).

2) φ is dense in $(\Lambda, \|.\|_\lambda)$.

Proof. We only prove iii), iv) and v). i) and ii) can be proved siinilarly as iii) and iv).

iii) If $(\Lambda, \|.\|_\lambda)$ satisfies property 6) then clearly $(\Lambda, \|.\|_\lambda) = (\lambda^{(\delta)}, \|.\|_{\lambda^{(\delta)}})$.

If $(\Lambda, \|.\|_\lambda) = (\lambda^{(\delta)}), \|.\|_{\lambda^{(\delta)}})$ we have $\beta_\lambda = B_{\lambda^{(\delta)}}$ and according to remark 2.8 this set is closed in w.

If B_λ is closed in ω, we have for $\alpha \in w$ with $\lim_{n\to\infty} \|P_n(\alpha)\|_\lambda = a < \infty$ that $a \in \overline{aB_\lambda}^\omega = aB_\lambda$, therefore $\alpha \in \Lambda$ and $a = \lim_{n\to\infty} \|P_n(\alpha)\|_\lambda \leq \|\alpha\|_\lambda \leq a$. So $\Lambda, \|.\|_\lambda)$ satisfies property δ).

iv) If $(\Lambda, \|.\|_\lambda)$ satisfies property δ_w) then clearly $\Lambda = \lambda^{(\delta)}$ algebraically and there exists $\rho \geq 1$ such that for every $\alpha \in \Lambda$

$$\sup_{n\in N} \|P_n(\alpha)\|_\lambda \leq \|\alpha\|_\lambda \leq \rho \sup_{n\in N} \|P_n(\alpha)\|_\lambda.$$

Hence $\|.\|_\lambda$ and $\|.\|_{\lambda^{(\delta)}}$ are equivalent.

If $\Lambda = \lambda^{(\delta)}$ algebraically and the correspondinp norms are equivalent then there is $\rho \geq 1$ such that $\overline{B_\lambda}^\omega \subset B_{\lambda^{(\delta)}} = \overline{B_{\lambda^{(\delta)}}}^\omega \subset \rho B_\lambda$.

Hence, for $\alpha \in w$ with $\lim_{n\to\infty} \|P_n(\alpha)\|_\lambda = a < \infty$ we have $a \in \overline{aB_\lambda}^\omega \subset a\rho B_\lambda$, therefore $\alpha \in \Lambda$ and $\|\alpha\|_\lambda \leq a\rho$. So $(\Lambda, \|.\|_\lambda)$ satisfies property δ_w).

v) If $(\Lambda, \|.\|_\lambda)$ satisfies property ε), then φ is dense in $(\Lambda, \|.\|_\lambda)$, because $P_n(\alpha) \in \varphi$ for every $a \in \Lambda$.

For the converse, let $a \in \Lambda$ and $\varepsilon > 0$. Then there is $\beta \in \varphi$ with $\|\alpha - \beta\|_\lambda < \varepsilon$. Choose $m \in N$ siich that $P_m(\beta) = \beta$. Then $|P_n(\alpha) - \alpha| \leq |P_n(\beta) - \alpha| = |\beta - \alpha|$ implies

$$\|P_n(\alpha) - \alpha\|_\lambda \leq \|\beta - \alpha\|_\lambda < \varepsilon, m \leq n \in N.$$

Remark 2.10. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space. Since $\lambda^{(\gamma)} = (\lambda^{(\gamma)})^{(\gamma)}$ and $\lambda^{(\delta)} = (\lambda^{(\delta)})^{(\delta)}$ always we have that $\lambda^{(\gamma)}$ satisfies property γ), $\lambda^{(\delta)}$ satisfies property δ) and $(\bar{\varphi}^\lambda, \|\cdot\|_\lambda)$ satisfies property ε).*

For a given normal Banach sequence space A we have shown that $\lambda^{(\delta)}, \lambda^{(\gamma)}$ and $\bar{\varphi}^{\lambda}$ are again normal Banach sequence spaces. Now we prove that $\lambda \cap \mu$ and A $+ \mu$ are normal Banach sequence spaces, if both A and μ are of that type.

Proposition 2.11. *Let* A *and* μ *be normal Banach sequence spaces with closed unit balls* B_λ *and* $B_{,,}$ *respectively. Let* $\| \, . \|_{\lambda \cap \mu}$ *be the Minkowski functional of* $B_\lambda \cap B_\mu$ *and* $\| \, . \|_{\lambda + \mu}$ *be the Minkowski functional of* $B_\lambda + B_\mu$. *Then* $(\lambda \cap \mu, \| \, . \|_{\lambda \cap \mu})$ *and* $(\lambda + \mu, \| \, . \|_{\lambda + \mu})$ *are normal Banach sequence spaces.*

Proof. In both cases property *a)* is clearly satisfied.

It is easy to see that $B_\lambda \cap B_{,,}$ is normal and that its Minkowski functional defines a complete norm on A $\cap \mu$. Hence $\lambda \cap \mu$ is a normal Banach sequence space.

We define

$$q : \lambda \times \mu \longrightarrow \lambda + \mu, (\alpha, \beta) \longmapsto \alpha + \beta.$$

This map q is linear, continuous and open. Hence (A $+ \mu, \| \, . \|_{\lambda + \mu}$) is a separated quotient of a Banach space and therefore itself a Banach space.

It remains to show that $B_{\lambda + \mu}$ is normal.

Let $\alpha^{(1)}$ E B_λ, $\alpha^{(2)}$ E $B_{,,}$ and β E ω with $|\beta| \leq |\alpha^{(1)} + \alpha^{(2)}|$ be given. For all $n \in N$ there is a scalar $z_{,,}$, $|z_n| \leq$ i, such that $\beta_n = z_n |\alpha_n^{(1)}| + z_n |\alpha_n^{(2)}|$.

Then $(z_n |\alpha_n^{(1)}|)_{n \in N}$ E B_λ and $(z_n |\alpha_n^{(2)}|)_{n \in N} \in B_\mu$, because B_λ and $B_{,,}$ are normal. Therefore

$$\beta = (z_n |\alpha_n^{(1)}|)_{n \in N} + (z_n |\alpha_n^{(2)}|)_{n \in N} \in B_\lambda + B_\mu.$$

and this implies that $B_\lambda + B_\mu$ is norinal.

Since the intersection of normal sets is also norinal, we get that

$$B_{\lambda + \mu} - \bigcap_{\rho > 1} \rho (B_\lambda + B_\mu)$$

is normal and we are done.

Example 2.12. *Let* $\alpha_{2n-1} := 1, \alpha_{2n} := n, n$ E N, *and* $a := (\alpha_n)_{n \in N}$. *Then* $(\alpha \cdot c_0) \cap l_\infty$ *is* R *normal Banach sequence space satisfying property* y *) but neither property* **6)** *nor property* ε).

Let for k E N be I_{k+1} C I_k C N such that $|I_k| = |I_k \setminus I_{k+1}| = \infty$.

We define

$$B_k := \{ (\alpha_n)_{n \in N} \in l_\infty : \| (\alpha_n)_{n \in N} \|_\infty + k \limsup_{n \to \infty, n \in I_k} |\alpha_n| \leq 1 \}, k \in N.$$

B_k is normal and it is the closed unit ball of a norm on l_∞ which is equivalent to $\| \, . \|_\infty$. Hence B $:= \bigcap_{k \in N} B_k$ is normal and closed in l If we denote with λ the linear span of B (which contains φ) and with $\| \, . \|_\lambda$ the Minkowski functional of B, then (A, $\| \, . \|_\lambda$) is a normal Banach sequence space.

Example 2.13. (A, $\| \, . \|_\lambda$) *does not satisfy property* γ_w).

Proof. For $k \in N$ let $\alpha^{(k)} = (\alpha_n^{(k)})_{n\in N}$ be defined by:

$$\alpha_n^{(k)} := \begin{cases} 1 & : \quad n \in I_k \setminus I_{k+1} \\ 0 & : \quad n \notin I_k \setminus I_{k+1}. \end{cases}$$

For $m, k \in N$ we have

$$\|\alpha^{(k)}\|_\infty + m \limsup_{n\to\infty, n\in I_m} |\alpha_n^{(k)}| = \begin{cases} m+1 & : \quad m \leq k \\ 1 & : \quad m > k. \end{cases}$$

This implies $\alpha^{(k)} \in \lambda$ and $\|\alpha^{(k)}\|_\lambda = k+1, k \in N$. On the other side we compute $\|P_n(\alpha^{(k)})\|_\lambda = 1$ for every $k, n \in N$ and therefore $(\lambda, \|\ .\|_\lambda)$ does not satisfy property γ_w).

We remark that the first example of such type was constructed by S. Dineen, see [13].

In the remaining part of this chapter we examine the connection between normal Banach sequence spaces and perfect spaces.

Definitioii 2.14. *For a linear space $\lambda \subset w$ we define (tlie so-called α-dual)*

$$\lambda^\times := \{\beta \in \omega : \sum_{n\in N} |\alpha_n\beta_n| < \infty \text{ for all } \alpha \in \lambda\}.$$

λ is called perfect *if* $\lambda = \lambda^{\times\times}$.

The following elementary properties are taken froin [24], §30, 2.

Remark 2.15. *Let $\lambda \subset w$ be a linear space. Then $\varphi \subset \lambda^\times = \lambda^{\times\times\times}$ and $\lambda^\times$ is a normal subset of w.*

If one assumes, in addition, that λ contains φ then λ and $\lambda^\times$ form in a canonical way a dual system $\langle\lambda, \lambda^\times\rangle$ with bilinear forni $(\alpha, \beta) \longmapsto \sum_{n=1}^\infty \alpha_n\beta_n$. So, for example, the strong topology $\beta(\lambda, \lambda^\times)$ and tlie weak topology $\sigma(\lambda, \lambda^\times)$ on λ are uniquely determined.

Now we prove a connection between normal Banach sequence spaces and perfect spaces:

Theorem 2.16. *Ler $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space satisfying property δ_w). Then λ is perfect and $\|\ .\|_\lambda$ induces tiie strong topology $\beta(\lambda, \lambda^\times)$ on λ.*

Proof.

1) For $A \subset \lambda$ let

$$A^\circ := \{\beta \in \lambda^\times : |\sum_{n\in N} \alpha_n\beta_n| \leq 1 \text{ for all } \alpha \in A\}.$$

If B_λ denotes (as usual) the closed unit ball of λ, we obtain (since B_λ is normal)

$$B_\lambda^\circ = \{\beta \in \omega : \sum_{n\in N} |\alpha_n\beta_n| \leq \text{ for all } \alpha \in B_\lambda\}.$$

B_λ° is an absolutely convex, bounded and normal subset of w and its span $[B_\lambda^\circ]$ contains φ. Moreover, B_λ° is closed in ω:

Let $\beta = (\beta_n)_{n\in N} \in \overline{B_\lambda^\circ}^\omega$. Then there is a sequence $((\beta_n^{(k)})_{n\in N})_{k\in N} = (\beta^{(k)})_{k\in N}$ in B_λ° such that $\lim_{k\to\infty} \beta_n^{(k)} = \beta_n$ for all $n \in N$.

Because B_λ is normal, we have for every $N \in \mathbf{N}, \alpha \in B_\lambda$ that $P_N(\alpha) \in B_\lambda$ and hence for every $k \in N$: $\sum_{n=1}^N |\alpha_n\beta_n^{(k)}| \leq 1$. This implies for every $\alpha \in B_\lambda$:

$$\sum_{n\in N} |\alpha_n\beta_n| = \sup_{N\in N}\sum_{n=1}^{n} |\alpha_n\beta_n| = \sup_{n\in N}\lim_{k\to\infty}\sum_{n=1}^{N} |\alpha_n\beta_n^{(k)}| \leq 1$$

. and it follows $\beta \in B_\lambda^\circ$.

If we denote by $\|\cdot\|_\times$ the Minkowski functional of B_λ°, we get that $([B_\lambda^\circ], \|.\|_\times)$ is a normal Banach sequence space. Theorem 2.9 implies that this space satisfies even property 6).

It is clear that $[B_\lambda^\circ] \subset \lambda^\times$. In the next step we prove that even $[B_\lambda^\circ] = \lambda^\times$ is true. Assume this is wrong. Then there exist $(\beta_n)_{n\in N} = \beta \in \lambda^\times$ and a sequence $((\alpha_n^{(k)})_{n\in N})_{k\in N} = (\alpha^{(k)})_{k\in N} = (|\alpha^{(k)}|)_{k\in N}$ in B_λ with $\sum_{n\in N} \alpha_n^{(k)} |\beta_n| \geq 2^k$ for every $k \in N$. The completeness of $(\lambda, \|.\|_\lambda)$ guarantees that $(\alpha_n)_{n\in N} = \alpha := \sum_{k=1}^\infty 2^{-k}\alpha^{(k)} \in B_\lambda$. Then

$$\sum_{n=1}^{\infty} |\alpha_n\beta_n| = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} 2^{-k}\alpha_n^{(k)}|\beta_n| = \sum_{k=1}^{\infty} 2^{-k}\sum_{n=1}^{\infty} \alpha_n^{(k)}|\beta_n| \geq \sum_{k=1}^{\infty} 1 = \infty,$$

a contradiction to $\beta \in \lambda^\times$.

So we proved up to now that $(\lambda^\times, \|.\|_\times)$ is a normal Banach sequence space with closed unit ball B_λ° which satisfies property 6). If we denote with $\|.\|_{\times\times}$ the Minkowski functional of $B_\lambda^{\circ\circ}$ (the second polar taken in $\lambda^{\times\times}$) we get that $(\lambda^{\times\times}, \|\cdot\|_{\times\times})$ is also a normal Banach sequence space with closed unit ball $B_\lambda^{\circ\circ}$ which satisfies property 6).

2) For $N \in N$ let $L_n := \{\alpha \in \omega : \alpha_n = 0$ for all $n > N\}$ and define for $A \subset L_N$:

$$A^{\circ L_N} := \{\beta \in L_N : |\sum_{n=1}^{N} \alpha_n\beta_n| \leq 1 \text{ for all } \alpha \in A\}.$$

Since B_λ° is normal, we have $B_\lambda^\circ \cap L_N = (B_\lambda \cap L_N)^{\circ L_N}$ and the same argument also yields $B_\lambda^{\circ\circ} \cap L_N = (B_\lambda \cap L_N)^{\circ L_N \circ L_N}$. The bipolar theorem implies $(B_\lambda \cap L_N)^{\circ L_N \circ L_N} = B_\lambda \cap L_N$ and therefore the norms $\|\cdot\|_\lambda$ and $\|.\|_{\times\times}$ coincide on L_N. This is true for every $N \in N$ and hence both norms coincide on $\varphi(= \{P_n(\alpha) : \alpha \in \omega, n \in N\})$. With 1) we conclude that the normal Banach sequence spaces $(\lambda, \|.\|_\lambda)$ and $(\lambda^{\times\times}, \|.\|_{\times\times}$ both satisfy property δ_w). It follows

$$\lambda = \{\alpha \in \omega : \sup_{n\in N} \|P_n(\alpha)\|_\lambda < \infty\} = \{\alpha \in \omega : \sup_{n\in N} \|P_n(\alpha)\|_{\times\times} < \infty\} = \lambda^{\times\times}$$

and with theorem 2.9 we obtain that the norms $\|.\|_\lambda$ and $\|.\|_{\times\times}$ are equivalent.

To prove that $\|.\|_\lambda$ generates the strong topology $\beta(\lambda, \lambda^\times)$ on λ it suffices to show that $B_\lambda^{\circ\circ}$ (the closed unit ball of $\|.\|_{\times\times}$ is a bounded barrel in $(\lambda, \sigma(\lambda, \lambda^\times))$.

$B_\lambda^{\circ\circ}$ is clearly absolutely convex and absorbant. In 1) we have shown that $[B_\lambda^\circ] = \lambda^\times$ and therefore $\{\sum_{n=1}^\infty \alpha_n\beta_n : \alpha \in B_\lambda\}$ is bounded for every $\beta \in \lambda^\times$. Hence $B^{\circ\circ\lambda}$ is $\sigma(\lambda, \lambda^\times)$ - bounded. Again from 1) we deduce that $B_\lambda^{\circ\circ}$ is closed in ω and this implies that it is $\sigma(\lambda, \lambda^\times)$ - closed.

Remark **2.17.** *i) The completeness of* $(\lambda, \|\cdot\|_\lambda)$ *is essential for the proof of theorem 2.16, it is not sufficient to take* λ *as a normal and dense subspace of a normal Banach sequence space:*
Consider φ *equipped whith* $\|\cdot\|_1$. *Then this space is dense in* $(l_1, \|\cdot\|_1)$ *and* φ *is normal but* $\beta(\varphi, \varphi^\times) = \beta(\varphi, w)$ *is the finest locally convex topology on* φ.
ii) From the proof of theorem 2.16 follows:
For every normal Banach sequence space λ *the* α*-dual* $\lambda^\times$ *is a normal Banach sequence space satisfying property* δ)*, if we equip* $\lambda^\times$ *with the Minkowski functional of* $B_\lambda^{\circ\lambda^\times}$.
iii) Analysing the proof of the above theorem one can give the following sharper result:
Let $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space. Then* $\lambda^{\times\times} = \lambda^{(\delta)}$ *and* $\|\cdot\|_{\lambda^{(\delta)}}$ *generates the strong topology* $\beta(\lambda^{\times\times}, \lambda^\times)$ *on* $\lambda^{\times\times}$.
iv) Let $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space satisfying property* ε). *With theorem 2.9 and the previous remark we obtain that* λ *is the regular subspace (in the sense of T. Kōmura & Y. Kōmura [23]) of the perfect space* $\lambda^{(\delta)}$. *Confer also [36], p. 195.*

3. VECTOR VALUED SEQUENCE SPACES

Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space with closed unit ball B_λ. For an absolutely convex subset C of a locally convex Hausdorff space $E = (E, \mathcal{T})$ we denote by p_C its Minkowski functional (defined on the linear span $[C]$ of C). So we have that the system $cs(E)$ of all continuous seminorms on E is equal to $\{p_U : U \in \mathcal{U}_0(E)$ absolutely convex$\}$. We define the vector valued sequence space

$$\lambda(E) := \lambda((E, \mathcal{T})) := \{x^N : (p(x_n))_{n\in N} \in \lambda \text{ for every } p \in cs(E)\}$$

and for an absolutely convex subset C of E

$$\lambda(C) := \lambda(C, (E, \mathcal{T})) := \{x \in \lambda(E) \cap [C]^N \; : \; (p_C(x_n))_{n\in N} \in B_\lambda\}.$$

Then $\lambda(E)$ is for every normal Banach sequence space λ and every locally convex Hausdorff space $(E, \mathcal{T})$ a vector space which we equip with the locally convex Hausdorff topology admitting

$$\{\lambda(U) \;:\; U \in \mathcal{U}_0(E) \text{ absolutely convex}\}$$

as a zero basis. Equivalently, the topology on $\lambda(E)$ is induced by the seminorms $x = (x_n)_{n\in N} \longmapsto \|(p(x_n))_{n\in N}\|_\lambda, p \in cs(E)$.
We note that $\lambda(E)$ contains E as a complemented subspace.
If E is metrizable, then so is $\lambda(E)$. Moreover, if $(E, \|\cdot\|_E)$ is a normed space, then the topology of $\lambda(E)$ is induced by the norm

$$x = (x_n)_{n\in N} \|(\|x_n\|)_{n\in N}\|_\lambda.$$

We will denote this normed space by $\lambda((E, \|\cdot\|_E))$ or by $(\lambda(E), \|\cdot\|_{\lambda(E)})$.
Let the locally convex space $(E, \mathcal{T})$ be continuously included in the locally convex Hausdorff space $(F, \mathcal{S})$. Then $\lambda((E, \mathcal{T}))$ is continuously included in $\lambda((F, \mathcal{S}))$. Hence, if B is a bounded and absolutely convex subset of $(E, \mathcal{T})$ we get $\lambda(([B], p_B)) \subset \lambda((E, \mathcal{T}))$ and therefore (algebraically) $\lambda(B, ([B], p_B)) = \lambda(B, (E, \mathcal{T}))$.

We now prove that for every absoliitely coiivex $C \subset E$ the set $\lambda(C)$ depends only on the relative topology on C indiiced by $(E,\mathcal{T})$. We denote by $\mathcal{T}_C$ **the** finest locally coiivex topology on the spari $[C]$ of C which coincides with $\mathcal{T}$ on C.

Proposition 3.1. ***Ler*** $(\lambda, \|\,.\,\|_\lambda)$ *be a normal Banach sequence space, $(E,\mathcal{T})$ be a locally convex Hausdorff space and $C \subset E$ be absolutely convex. Then*

$$\lambda(C,(E,\mathcal{T})) = \lambda(C,([C],\mathcal{T}_C))$$

and the relative topologie induced by $\lambda((E,\mathcal{T}))$ and $\lambda(([C],\mathcal{T}_C))$ coincide on this set.

Proof. 1) Let $V \in \mathcal{U}_0(([C],\mathcal{T}_C))$ be absoliitely convex.

Then there is $U \in \mathcal{U}_0((E,\mathcal{T}))$ absolutely convex siich that $U \cap C \subset \frac{1}{4}V$.

For $s = (x_n)_{n\in N} \in [C]^N$ with $(p_U(x_n))_{n\in N} \in B_\lambda$ and $(p_C(x_n))_{n\in N} \in B_\lambda$ **we** get for $n \in N$:

$$\begin{aligned}
x_n &\in (2p_U(x_n)U) \cap (2p_C(x_n)C) \subset 2(p_U(x_n) + p_C(x_n))U \cap C \\
&\subset 2(p_U(x_n) + p_C(x_n))\frac{1}{4}V \qquad \text{if } p_U(x_n), p_C(x_n) > 0, \\
x_n &\in p_C(x_n)\frac{1}{4}V \qquad \text{if } p_U(x_n) = 0 < p_C(x_n), \\
x_n &\in p_U(x_n)\frac{1}{4}V \qquad \text{if } p_C(x_n) = 0 < p_U(x_n), \\
x_n &\in \rho\frac{1}{4}V \text{ for every } \rho > 0 \qquad \text{if } p_C(x_n) = p_U(x_n) = 0.
\end{aligned}$$

For every $n \in N$ this implies $0 \leq p_V(x_n) \leq 2(p_U(x_n)+p_C(x_n))\frac{1}{4}$ and therefore $(p_V(x_n))_{n\in N} \in B_\lambda$ since B_λ is normal. Therefore we obtain

$$\begin{aligned}
&\{x \in [C]^N \ : \ (p_U(x_n))_{n\in N}, (p_C(x_n))_{n\in N} \in B_\lambda\} \\
\subset \ &\{x \in [C]^N \ : \ (p_V(x_n))_{n\in N} \in B_\lambda\}.
\end{aligned}$$

2) Clearly one has $\lambda(([C],\mathcal{T}_C)) \subset \lambda((E,\mathcal{T})) \cap [C]^N$, this shows $\lambda(C,([C],\mathcal{T}_C)) \subset \lambda\,(C,(E,\mathcal{T}))$.

Let $V \in \mathcal{U}_0(([C],\mathcal{T}_C))$ be absolutely convex. Because of 1) there is an absolutely convex $U \in \mathcal{U}_0((E,\mathcal{T}))$ such that

$$\begin{aligned}
\lambda(C,(E,\mathcal{T})) &\subset \{x \in [C]^N : (p_U(x_n))_{n\in N}, (p_C(x_n))_{n\in N} \in \lambda\} \\
&\subset \{x \in [C]^N : (p_V(x_n))_{n\in N} \in \lambda\}.
\end{aligned}$$

Since V is arbitrary we get $\lambda(C,(E,\mathcal{T})) \subset \lambda(([C],\ \mathcal{T}_C))$ and therefore $\lambda(C,(E,\ \mathcal{T})) \subset \lambda(C,([C],\mathcal{T}))$. To prove the remaining topological identity we first note that $\lambda((E,\mathcal{T}))$ induces on $\lambda(C,(E,\mathcal{T}))$ a coarser topology than $\lambda(([C],\mathcal{T}_C))$. For the converse let an absolutely convex $V \in \mathcal{U}_0(([C],\mathcal{T}_C))$ be given. With 1) and $\lambda(([C],\mathcal{T}_C)) \supset \lambda(C,(E,\mathcal{T}))$ we get the existence of an absolutely convex $U \in \mathcal{U}_0((E,\mathcal{T}))$ such that

$$\begin{aligned}
&\lambda(C,(E,\mathcal{T})) \cap \{x \in \lambda((E,\mathcal{T})) \ : \ (p_U(x_n))_{n\in N} \in B_\lambda\} \\
\subset \ &\{x \in \lambda(([C],\mathcal{T}_C)) \ : \ (p_U(x_n))_{n\in N}, (p_C(x_n))_{n\in N} \in B_\lambda\} \\
&\subset \{x \in \lambda(([C],\mathcal{T}_C)) \ : \ (p_V(x_n))_{n\in N} \in B_\lambda\}
\end{aligned}$$

and this yields the conclusion.

Corollary 3.2. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space, E be a vector space and $C \subset E$ be absolutely convex. Moreover, let F_1, F_2 be algebraic subspaces of E containing C, equipped with locally convex Hausdorff topologie $\mathcal{T}_1 (\mathcal{T}_2$, resp.). If both topologies coincide on C we have*

$$\lambda(C, (F_1, \mathcal{T}_1)) = \lambda(C, (F_2, \mathcal{T}_2))$$

and the topologies induced by $\lambda((F_1, \mathcal{T}_1))$ and $\lambda((F_2, \mathcal{T}_2))$ coincide on this set.

In the next result we consider properties of the sets $\lambda(B)$ if B is a bounded, closed and absolutely convex subset of a locally convex space E. It is clear that then $\lambda(B)$ is again bounded in $\lambda(E)$. We prove implications of additional properties required for λ.

Theorem 3.3. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space and let E be a locally convex Hausdorff space. Moreover, let $B \subset E$ be absolutely convex, bounded and closed.*

i) Then $\overline{\lambda(B)}^{E^N} \subset [B]^N$ and for every $x = (x_n)_{n\in N} \in \overline{\lambda(B)}^{E^N}$ and every $m \in N$ we have $((p_B(x_n))_{n\le m}, (0)_{n>m}) = P_m((p_B(x_n))_{n\in N}) \in B_\lambda$.

ii) If $(\lambda, \|\cdot\|_\lambda)$ satisfies property γ) then $\lambda(B)$ is closed in its span $[\lambda(B)]$ equipped with the relative topology induced by E^N.

iii) If $(\lambda, \|\cdot\|_\lambda)$ satisfies proeprty γ_w) then there is $\rho > 0$ such that the closure of $\lambda(B)$ taken in its span $[\lambda(B)]$ equipped with the relative topology induced by E^N is contained in $\rho\lambda(B)$.

iv) If $(\lambda, \|\cdot\|_\lambda)$ satisfies property δ) then $\lambda(B)$ is closed in E^N.

v) If $(\lambda, \|\cdot\|_\lambda)$ satisfies property δ_w) then there is $\rho > 0$ such that the closure of $\lambda(B)$ taken in E^N is contained in $\rho\lambda(B)$.

vi) If $(\lambda, \|\cdot\|_\lambda)$ satisfies property ε) then $E^{(N)}$ is large in $\lambda(E)$, i.e. for every bounded set B_1 in $\lambda(E)$ there is another bounded set B_2 in $\lambda(E)$ such that B_1 is contained in the closure of $B_2 \cap E^{(N)}$. In particular $E^{(N)}$ is dense in $\lambda(E)$.

Proof. i) Since B is closed, we get $\overline{\lambda(B)}^{E^N} \subset \prod_{n\in N} \|(\delta_{nk})_{k\in N}\|_\lambda B \subset [B]^N$. Let $x = (x_n)_{n\in N} \in \overline{\lambda(B)}^{E^N}$ and $m \in N$ be given. For arbitrary $\varepsilon > 0$ there is $x^{(\varepsilon)} = (x_n^{(\varepsilon)})_{n\in N} \in \lambda(B)$ such that $p_B(x_n) \le p_B(x_n^{(\varepsilon)}) + \varepsilon$ for every $n \in \{1, 2, \ldots, m\}$. Hence

$$\|P_m((p_B(x_n))_{n\in N})\|_\lambda \le \|P_m((p_B(x_n^{(\varepsilon)}) + \varepsilon)_{n\in N})\|_\lambda \le 1 + \varepsilon \sum_{n=1}^{m} \|(\delta_{nk})_{k\in N}\|_\lambda.$$

ii) Let $x = (x_n)_{n\in N} \in \overline{\lambda(B)}^{E^N} \cap [\lambda(B)]$ be given. This implies $x \in \lambda(E)$ and $(p_B(x_n))_{n\in N} \in \lambda$. Using i) we get

$$\sup_{m\in N} \|P_m((p_B(x_n))_{n\in N}\|_\lambda \le 1$$

and property γ) implies $(p_B(x_n))_{n\in N} \in B_\lambda$.

iii) If $(\lambda, \|\cdot\|_\lambda)$ satisfies property γ_w) theorem 2.9 guarantees the existence of an equivalent norm on λ which satisfies property γ). The assertion follows with ii).

iv) Let $x = (x_n)_{n\in N} \in \overline{\lambda(B)}^{E^N}$. With i) we get

$$\sup_{m\in N} \|P_m((p_B(x_n))_{n\in N}\|_\lambda \le 1$$

and property *6)* implies $(p_B(x_n))_{n\in N} \in B_\lambda$. It remains to show that $x \in \lambda(E)$. But this is implied by tlie boundedness of B.

v) Can be proved by the same arguinent used for iii).

vi) Let B_1 be a bounded subset of $\lambda(E)$. Theii we have for every $p \in cs(E)$ that $\rho_p := \max\{1, \sup_{x\in B_1} \|(p(x_n))_{n\in N}\|_\lambda\}$ is finite. Thus,

$$B_2 := \{x \in \lambda(E) \ : \ (p(x_n))_{n\in N} \in \rho_p B_\lambda \text{ for every } p \in cs(E)\}$$

is bounded in $\lambda(E)$ and contains B_1.

Let $x = (x_n)_{n\in N} \in B_1$ be given. The construction of B_2 guarantees that it contains the sections $((x_n)_{n\leq m}, (0)_{n>m})$ for every $m \in N$. Since $(\lambda, \|\cdot\|_\lambda)$ satisfies property ε) we have

$$\lim_{m\to\infty} \|((0)_{n\leq m}, (p(x_n))_{n>m})\|_\lambda = \lim_{m\to\infty} \|P_m((p(x_n))_{n\in N} - (p(x_n))_{n\in N}\|_\lambda = 0$$

for everyp $\in cs(E)$. This implies that the sections $((x_n)_{n\leq m}, (0)_{n>m})$ converge to x $(m \to \infty)$ and we are done.

Remark 3.4. *In tlie previous theorem tlie boundedness of the set* **B** *is essential as the trivial example* $E := \mathbf{R}, B := \mathbf{R}$ *shows. In this case we have* $\overline{\lambda(B)}^{E^N} = E^N \not\subset \lambda(E)$ *for every normal Banach sequence space* λ.

In contrast to part ii) of the previous theorem we present a simple example of a normal Banach sequence space $(\lambda, \|\cdot\|_\lambda)$ satisfying property γ), **a** locally convex space E containing an absolutely convex, bounded and closed **set** B such that $\lambda(B)$ is not closed in $\lambda(E)$ (hence not closed in $\lambda(E)$ equipped with tlie relative topology induced by E^N).

Example 3.5. B_{l_∞} *(the closed unit ball of* l_∞*) is an absolutely convex, bounded and closed* **subset** *of* ω*, but* $c_0(B_{l_\infty})$ *is not closed in* $c_0(\omega)$.

Iiideed, for $n \in N$ let $\alpha_n := ((0)_{\nu\leq n}, (1)_{\nu>n}) \in B_{l_\infty}$. Then $\alpha := (\alpha_n)_{n\in N} \in c_0(\omega)\backslash c_0(B_{l_\infty})$. But for every $m \in N$ the sections $\alpha^{(m)} := ((\alpha_n)_{n\leq m}, (0)_{n>m})$ are contained in $c_0(B_{l_\infty})$ and $\alpha^{(m)} \to \alpha$ $(m \to \infty)$ in $c_0(\omega)$.

Instead of ω one can choose in this exainple diagonal transforms $\beta \cdot l_\infty$ of l_∞ (if $0 < \beta_n \to \infty$ $(n \to \infty)$).

The following result is an application of proposition 2.4.

Proposition 3.6. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space and let* **E** *be a locally convex Hausdorff space. For* $\alpha := (\|(\delta_{nk})_{k\in N}\|_\lambda^{-1})_{n\in N}$ *tlie inclusions*

$$E^{(N)} \hookrightarrow (\alpha \cdot l_1)(E) \hookrightarrow (\alpha \cdot l_\infty)(E) \hookrightarrow E^N$$

are continuous.

In contrast to the previous result we will prove that for every locally convex Hausdorff space E **the** space $\lambda(E)$ contains $l_\infty(E)$ as **a** coiiipleineiited subspace wheiiever the norinal Banach sequence space λ satisfies not the property ε). **A** siinilar result, assuming in addition that λ satisfies property γ) and E is a Banach space, was ali-eady given by C. Fernández [15]. **Our** proof is mainly adapted from her paper. The technical part of the proof is done in the following lemma.

Lemma 3.7. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space with closed unit ball B_λ and let E be a locally convex Hausdorff space. Ler $\alpha = (\alpha_n)_{n\in N} \in \varphi$ with $\|\alpha\|_\lambda = 1$. Then there is a continuous linear map $T : \lambda(E) \to E$ which satisfies the following three conditions:*

i) $p(T(x)) \leq \|(p(x_n))_{n\in N}\|_\lambda$ for every $x = (x_n)_{n\in N} \in \lambda(E)$ and every $p \in cs(E)$.

ii) $T((\alpha_n x_0)_{n\in N}) = x_0$ for all $x_0 \in E$.

iii) $\{(x_n)_{n\in N} \in \lambda(E) : x_n = 0 \text{ if } \alpha_n \neq 0, n \in N\}$ is contained in the kernel of T.

Proof. Let $\alpha = (\alpha_n)_{n\in N} \in \varphi$ with $\|\alpha\|_\lambda = 1$ be given. We denote $S := \{n \in N : a_n \neq 0\}$. Then the space $L_S := \{(\beta_n)_{n\in N} \in \lambda : \beta_n = 0 \text{ if } n \notin S\}$ coiitains α and it is of finite diinensioii. We equip L_S with the norm induced by $\|\cdot\|_\lambda$. The dual of this space is isomorphic to L_S. Using basic duality theory we get the existeiice of $(\alpha_n)_{n\in N} \in L_S$ with the following two properties:

$$|\sum_{n=1}^{\infty} a_n\beta_n| \leq 1 \text{ for every } (\beta_n)_{n\in N} \in L_S \cap B_\lambda \text{ and } \sum_{n=1}^{\infty} a_n\alpha_n = 1.$$

Since B_λ is normal we even get

$$\sum_{n=1}^{\infty} |a_n\beta_n| \leq 1 \text{ for every } (\beta_n)_{n\in N} \in B_\lambda.$$

We define the linear map $T : \lambda(E) \to E$ by $T((x_n)_{n\in N}) := \sum_{n=1}^{\infty} a_n x_n$.

For every $x_0 \in E$ we have $\sum_{n=1}^{\infty} a_n\alpha_n x_0 = x_0$. This implies ii) and $(a_n)_{n\in N} \in L_S$ guarantees iii). To prove i) let $x = (x_n)_{n\in N} \in \lambda(E)$ and $p \in cs(E)$ be given. Using the properties of $(a_n)_{n\in N}$ proved before we get:

$$p(T(x)) = p\left(\sum_{n=1}^{\infty} a_n x_n\right) \leq \sum_{n=1}^{\infty} |a_n| p(x_n) \leq \|(p(x_n))_{n\in N}\|_\lambda.$$

This shows property i) and hence the continuity of T.

Theorem 3.8. *(cf. [15], 2.3) Ler $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space not satisfying property ε) and let E be a locally convex Hausdorff space. Then $\lambda(E)$ contains a complemented copy of $l_\infty(E)$.*

Proof. λ does not satisfy property ε) and hence there is $\mu \in \lambda$ with $\limsup_{n\to\infty} \|P_n(\mu) - \mu\|_\lambda > 0$. Hence $(P_n(\mu))_{n\in N}$ is not a Cauchy sequence in λ. This implies the existeiice of a strictly increasing cequence $(m_k)_{k\in N}$ of natural numbers and of $\delta > 0$ with $\|P_{m_{k+1}}(\mu) - P_{m_k}(\mu)\|_\lambda > \delta$ for every $k \in N$. We define $(\alpha_n^{(k)})_{n\in N} := \alpha^{(k)} := \|P_{m_{k+1}}(\mu) - P_{m_k}(\mu)\|_\lambda^{-1}(P_{m_{k+1}}(\mu) - P_{m_k}(\mu))$ and $(\alpha_n)_{n\in N} := \alpha := \sum_{k=1}^{\infty} \alpha^{(k)}$, the limit taken in ω. (By construction, $\sum_{k=1}^{\infty} \alpha^{(k)}$ caiinot converge in λ). For every $n \in N$ there is $m \in N$ such that $\alpha_n = \sum_{k=1}^{\infty} \alpha_n^{(k)} = \alpha_n^{(m)}$. $(|\alpha_n|)_{n\in N} \leq \delta^{-1}(|\mu_n|)_{n\in N}$ implies $\alpha \in \lambda$.

For $x = (x_k)_{k\in N} \in l_\infty(E)$ we define

$$(i(x)_n)_{n\in N} := i(x) := \sum_{k=1}^{\infty} (\alpha_n^{(k)} x_k)_{n\in N} \text{ the limit taken in } E^N).$$

For $p \in cs(E)$, $(x_k)_{k\in N} \in l_\infty(E)$ and $n \in N$ we get $p(i(x)_n) \leq \|(p(x_k))_{k\in N}\|_\infty |\alpha_n|$, therefore $(p(i(x)_n))_{n\in N} \in \lambda$ and $\|(p(i(x)_n))_{n\in N}\|_\lambda \leq \|\alpha\|_\lambda \|(p(x_k))_{k\in N}\|_\infty$. This implies that

$$i \ : \ l_\infty(E) \to \lambda(E), x \longmapsto i(x)$$

is a welldefined, linear and continuous map. It is easy to see that i is also injective.

Now we construct a continuous linear mapping $T : \lambda(E) \to l_\infty(E)$ such that $T \circ i$ is the identity on $l_\infty(E)$.

For each $k \in N$ we have that $(\alpha_n^{(k)})_{n\in N} = \alpha^{(k)} \in \varphi$ and $\|\alpha^{(k)}\|_\lambda = 1$. For $k \in N$ let $T_k : \lambda(E) \to E$ be the continuous linear mapping associated to $\alpha^{(k)}$ constructed in lemma 3.7. We define:

$$T : \lambda(E) \to l_\infty(E), x \longmapsto (T_k(x))_{k\in N}.$$

T is welldefined, linear and continuous, because for every $k \in N$, $x = (x_n)_{n\in N} \in \lambda(E)$ and $p \in cs(E)$ we have $p(T_k(x)) \leq \|(p(x_n))_{n\in N}\|_\lambda$ (by condition i) of lemma 3.7).

By condition iii) of lemma 3.7 we have

$T_k(\{(x_n)_{n\in N} \in \lambda(E) : x_n = 0 \text{ if } \alpha_n^{(k)} \neq 0\}) = \{0\}$ and therefore $T_k(\sum_{m=1, m\neq k}^\infty (\alpha_n^{(m)} x_m)_{n\in N}) = 0$ for every $k \in N$. This and condition ii) of lemma 3.7 imply for every $x = (x_k)_{k\in N} \in l_\infty(E)$:

$$\begin{aligned} T(i(x)) &= T(\sum_{m=1}^\infty (\alpha_n^{(m)} x_m)_{n\in N}) = (T_k(\sum_{m=1}^\infty (\alpha_n^{(m)} x_m)_{n\in N}))_{k\in N} \\ &= (T_k((\alpha_n^{(k)} x_k)_{n\in N}))_{k\in N} = (x_k)_{k\in N} = x. \end{aligned}$$

So, $T \circ i$ is the identity on $l_\infty(E)$. Recalling that $i : l_\infty(E) \to \lambda(E)$ is a continuous inclusion and $T : \lambda(E) \to l_\infty(E)$ is linear and continuous, this proves the assertion.

Remark 3.9. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space satisfying not property ε) and let E be a locally convex Hausdorff space. Then $\bar\varphi^\lambda(E)$ contains a complemented copy of $c_0(E)$.*

To see this, we use the maps $i : l_\infty(E) \to \lambda(E)$ and $T : \lambda(E) \to l_\infty(E)$ constructed in the proof of 3.8. From $i(E^{(N)}) \subset E^{(N)}$, $T(E^{(N)}) = E^{(N)}$ and $T \circ i|_{c_0(E)} = id_{c_0(E)}$ we obtain that $i(c_0(E)) \subset \bar\varphi^\lambda(E)$ and $T(\bar\varphi^\lambda(E)) = c_0(E)$. With the same argument used at the end of the proof of 3.8 we get the desired statement.

4. BOUNDED SETS AND DUALITY IN VECTOR VALUED SEQUENCE SPACES

Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space and let E be a locally convex Hausdorff space. In this chapter we will consider descriptions for bounded subsets of the space $\lambda(E)$ especially if E is metrizable or a gDF space. It will turn out that there is a coiiiiectioii betweeii these descriptioiis aiid the topological dual of $\lambda(E)$ provided λ satisfies property ε).

Definition 4.1. *Let E be a locally convex Hausdorff space and let $\mathcal{B}$ be a subset of the powerset $\mathcal{P}(E)$ of E. We call $\mathcal{B}$ a* fundamental system of bounded sets *(in E) if it contains only absolutely convex and bounded sets and if every bounded set is contained in some element of $\mathcal{B}$.*

A *sequence* $(B_n)_{n\in N}$ *of subsets of* E *is called a* fundamental sequence of bounded sets *if* $\{B_n : n \in N\}$ *is a fundamental system of* **bounded sets in** E.

With this definition $\mathcal{B}(E) := \{B \subset E : B$ bounded and absolutely convex$\}$ is a fundamental system of bounded sets. If E possesses a fundamental sequence of bounded sets $(B_n)_{n\in N}$, then $(C_n)_{n\in N} := (2^n \sum_{k=1}^{n} B_k)_{n\in N}$ is a fundamental sequence of bounded sets in E satisfying $2C_n \subset C_{n+1}$ for every $n \in N$.

It is easy to see that for every locally convex Hausdorff space E the sets $\{l_\infty(B) : B \in \mathcal{B}(E)\}$ and $\{B^N : B \in \mathcal{B}(E)\}$ are fundamental systems of bounded sets in $l_\infty(E)$ (cf. M. Florencio & P.J. Paul [16], 2.5). Now we give an example that for general normal Banach sequence spaces λ such a description of bounded sets in $\lambda(E)$ is not possible. We may even choose $\lambda = l_1$.

Example 4.2. *Consider* c_0 *equipped with the weak topology. Then every bounded set in* $E := (c_0, \sigma(c_0, l_1))$ *is absorbed by* **the bounded** *and* **closed** *set* B_{c_0} *(the closed unit ball in* c_0*), hence* $(nB_{c_0})_{n\in N}$ *is a fundamental sequence of bounded sets in* $E (= (c_0, \sigma(c_0, l_1)))$. *We have* $[\overline{l_1(B_{c_0})}^{l_1(E)}] \neq l_1(E)$ *and therefore* $(n\overline{l_1(B_{c_0})}^{l_1(E)})_{n\in N}$ *is not a fundamental sequence of bounded sets in* $l_1(E)$.

Indeed, $x := (x_n)_{n\in N} := (((0)_{k<n}, 1, (0)_{k>n}))_{n\in N}$ is contained in $l_1(E)$, because for every $f \in l_1 = c_0'$ we have $\sum_{n\in N} \leq \|f\|_1$. On the other hand $\sum_{n\in N} \|x_n\|_\infty = \infty$, so x is not contained in the span of $l_1(B_{c_0})$. Using that l_1 satisfies property *6)* we can apply 3.3 to conclude that $l_1(B_{c_0})$ is closed in E^N and therefore closed in $l_1(E)$. Hence $x \notin [\overline{l_1(B_{c_0})}^{l_1(E)}]$.

If E is a metrizable locally convex space, the situation looks much nicer. We prove the following lemma to get a canonical description for a fundamental system of bounded sets in $\lambda(E)$.

Lemma 4.3. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space and let* E *be a locally convex Hausdorff space. If* $(C_k)_{k\in N}$ *is a sequence of absolutely convex and closed subsets of* E *and* $0 < (\alpha_k)_{k\in N} = \alpha \in l_1$, *then*

$$\bigcap_{k\in n} \lambda(C_k) \subset \|\alpha\|_1 \lambda(\bigcap_{k\in N} \alpha_k^{-1} C_k).$$

Proof. Set $V := \bigcap_{k\in N} \alpha_k^{-1} C_k$. Let $(x_n)_{n\in N} \in \bigcap_{k\in N} \lambda(C_k)$.

For every $n \in N$ and every $k \in N$ with $p_{C_k}(x_n) > 0$ we have

$$x_n \in \alpha_k p_{C_k}(x_n)(\alpha_k^{-1} C_k).$$

Hence, for $n \in N$:

$$x_n \in (\sum_{j=1}^{\infty} \alpha_j p_{C_j}(x_n))\alpha_k^{-1} C_k, k \in N, \quad \text{if there is } j \in N \text{ with } p_{C_j}(x_n) > 0$$

and

$$x_n \in \rho \bigcap_{k\in N} \alpha_k^{-1} C_k, \rho > 0, \quad \text{if } p_{C_j}(x_n) = 0 \text{ for every } j \in N.$$

This implies $p_V(x_n) \leq \sum_{j=1}^{\infty} \alpha_j p_{C_j}(x_n)$ for every $n \in N$ and therefore $(p_V\ (x_n))_{n\in N} \leq (\sum_{j=1}^{\infty} \alpha_j p_{C_j}\ (x_n))_{n\in N} = \sum_{j=1}^{\infty} \alpha_j (p_{C_j}(x_n))_{n\in N}$. The completeness of λ and $(p_{C_j}(x_n))_{n\in N} \in B_\lambda, j \in N$, guarantees $\sum_{j=1}^{\infty} \alpha_j\ (p_{C_1}(x_n))_{n\in N} \in \|\alpha\|_1 B_\lambda$ and so we get $(p_V(x_n))_{n\in N} \in \|\alpha\|_1 B_\lambda$.

The following result is due to *C.* Fernandez and can be found in [28], 2.1.

Corollary 4.4. *Let* $(\lambda, \|\ .\ \|_\lambda)$ *be a normal Banach sequence space and let* E *be* ci *metrizable locally convex space. Then* $\{\lambda(B)\ :\ B \in \mathcal{B}(E)\}$ *is a fundamental system of bounded sets in* $\lambda(E)$.

Proof. *E* possesses a zero basis $(U_n)_{n\in N}$ consisting of absolutely convex and closed sets. Let *B* be a bounded subset of $\lambda(E)$. Tlien there exist $\rho_n > 0$ such that p **C** $\rho_n\lambda(U_n)$ for every $n \in N$. If C : $\bigcap_{n\in N} 2^n \rho_n U_n$, then *C* is boiinded in *E* and lemma 4.3 implies $B \subset \lambda(C)$.

Remark 4.5. *For* $\lambda = l_1$ *the above corollary has been proved by A,* Pietsch (see *[32]) and* it *means that the space E satisfies the property (B)of* Pietsch. *If* λ *is perfect, the previous corollary is covered by* ci *result of R.C. Rosier [34] and he called locally convex Hausdorff spaces E* fundamentally λ-bounded *if* $\{\lambda(B)\ :\ B \in \mathcal{B}(E)\}$*is a fundamental system of bounded sets in* $\lambda(E)$.

As the example 4.2 shows, an analogous description as proved in corollary 4.4 for the bounded **sets** *in* $\lambda(E)$ *is not possible in general. Even in tlte case of DFM spaces (the strong duals of Fréchet-Montel spaces) there are* **counter examples:**

For the DFM space E constructed by Köthe [24], §31, 5. we have $c_0(E) \not\subset \bigcap_{B\in\mathcal{B}(E)} c_0(B)$. *(But later we will see that in this case one has to use the closures of the sets* $c_0(B)$ *to get a fundamental system of bounded sets in* $c_0(E)$.*)*

Now we consider **tlie** *situation on the dual side of metrizable locally convex spaces.* **We** *recall some definitions:*

Definition *4.6. Let* $(E,\mathcal{T})$ *be* ci *locally convex Hausdorff space possessing* ci *fundamental sequence of bounded sets* $(B_n)_{n\in N}$. *E is called* ci

i) gDF space *if* $\mathcal{T}$ *is the finest locally convex topology on E whose restriction to each* B_n *coincides with* $\mathcal{T}$.

ii) DF space *f* $(E,\mathcal{T})$*s* $\aleph_0$*-quasibarrelled, i.e. whenever U is the countable intersection of absolutely convex and closed zero neighbourhoods and U is bornivorous then U is again* ci *zero neighbourhood.*

Remark 4.7. *i) Let E be* ci *locally* **con***vex Hausdorff space which possesses* ci *fundamental sequence of bounded sets* $(B_n)_{n\in N}$ *satisfying* $2B_n \subset B_{n+1}$, $n \in N$. **Then** *E (cf. e.g. [29], 8.1.12) is a gDF space if and only if for every sequence* $(U_n)_{n\in N}$ *of zero neighbourhoods the set* $\bigcap_{n\in N}(U_n + B_n)$ *is again a zero neighbourhood.*

ii) Every DF space is ci *gDF space (cf. e.g. [29], 8.3.3) and every gDF space is* quasi-normable *(cf. e.g. [29], 8.3.37), i.e. for every absolutely convex zero neighbourhood U in E* ***there*** *exists an absolutely convex zero neighbourhood V in E such that for every* $\rho > 0$ *there is* ci *bounded subset B* **in** *E with* $V \subset \rho U + B$.

iii) Every gDF space satisfies the countable neighbourhood property, *i.e., given any sequence* $(U_n)_{n\in N}$ *of zero neighbourhoods in E, there are* $\rho_n > 0$ *for every* $n \in N$ *such that* $\bigcap_{n\in N} \rho_n U_n$ *is a zero neighbourhood in E (see e.g. [29] 8.3.5).*

iv) The strong dual of a metrizable locally convex space is a complete DF space (cf. e.g. [29], 8.3.9) and the strong dual of a gDF space is a Fréchet space (cf. e.g. [29], 8.3.7).

Now we improve remark 4.7 i). The idea is due to J. Bonet and A. Peris (personal communication). Compare part ii) with [3], 5.A, p. 579.

Theorem 4.8. *Let E be a locally convex Hausdorff space.*

i) E is a gDF space if and only if E contains a sequence $(B_n)_{n\in N}$ of bounded and absolutely convex subsets satisfying $2B_n \subset B_{n+1}$ for each n such that for every sequence $(U_n)_{n\in N}$ of zero neighbourhoods the set $\bigcap_{n\in N}(U_n + B_n)$ is again a zero neighbourhood. In this case, $(\overline{B_N}^E)_{n\in N}$ is a fundamental sequence of bounded sets in E.

ii) E is a DF space if and only if E contains a sequence $(B_n)_{n\in N}$ of bounded and absolutely convex subsets satisfying $2B_n \subset B_{n+1}$ for each n such that for every sequence $(U_n)_{n\in N}$ of zero neighbourhoods and very $(\rho_n)_{n\in N} \in (0,\infty)^N$ the set $\bigcap_{n\in N}(U_n + \sum_{k=1}^n \rho_k B_k)$ is again a zero neighbourhood.

Proof. i) If E is a gDF space, then remark 4.7 i) implies the existence of the required sequence $(B_n)_{n\in N}$. For the converse, let $(B_n)_{n\in N}$ a sequence of bounded and absolutely convex subsets, satisfying $2B_n \subset B_{n+1}$ for each n, be given such that for every sequence $(U_n)_{n\in N}$ of zero neighbourhoods the set $\bigcap_{n\in N}(U_n + B_n)$ is again a zero neighbourhood. Using again remark 4.7 i) it remains to show that $(\overline{B_n}^E)_{n\in N}$ is a fundamental sequence of bounded sets in E.

Assume the existence of a bounded subset B of E which is not contained in $\overline{B_n}^E$ for every $n \in N$. Since $2B_n \subset B_{n+1}$ there exists a bounded sequence $(x_n)_{n\in N}$ in E with $x_n \notin \overline{nB_n}^E$ for every $n \in N$. Hence, for every $n \in N$ there are zero neighbourhoods U_n in E with $x_n \notin nU_n + nB_n$. This implies $x_k \notin k\bigcap_{n\in N}(U_n + B_n)$ for every $k \in N$, a contradiction, because $(x_k)_{k\in N}$ is bounded and and $\bigcap_{n\in N}(U_n + B_n)$ is a zero neighbourhood.

ii) Let E be a DF space. By definition, E possesses a fundamental sequence of bounded sets $(B_n)_{n\in N}$ (satisfying $2B_n \subset B_{n+1}$, $n \in N$). Let a sequence $(U_n)_{n\in N}$ of zero neighbourhoods in E and $(\rho_n)_{n\in N} \in (0,\infty)^N$ be given. We have to show that $U := \bigcap_{n\in N}(U_n + \sum_{k=1}^n \rho_k B_k)$ is a zero neighbourhood. There exists absolutely convex zero neighbourhoods V_n in E such that $2V_n \subset U_n$ for every $n \in N$. Hence

$$V := \overline{\bigcap_{n\in N}(V_n + \sum_{k=1}^{n} \rho_k B_k)}^E \subset U.$$

Then V absorbs each B_n, and therefore it is bornivorous. By definition, E is $\aleph_0$-quasibarrelled and this yields that V is a zero neighbourhood in E (which is contained in U).

For the converse, let a sequence $(B_n)_{n\in N}$ of bounded and absolutely convex subsets of E, satisfying $2B_n \subset B_{n+1}$ for each n, be given such that for every sequence $(U_n)_{n\in N}$ of zero neighbourhoods and every $(\rho_n)_{n\in N} \in (0,\infty)^N$ the set $\bigcap_{n\in N}(U_n + \sum_{k=1}^n \rho_k B_k)$ is again a zero neighbourhood. If $\rho_n := \frac{1}{2^n}$, $n \in N$, we get with i) that E is a gDF space and that $(\overline{B_n}^E)_{n\in N}$ is a fundamental sequence of bounded sets in E.

It remains to prove that E is $\aleph_0$-quasibarrelled. To see this, let a sequence $(V_n)_{n\in N}$ of closed and absolutely convex zero neighborhoods in E be given such that $V := \bigcap_{n\in N} V_n$ is bornivorous. There are $\rho_n > 0$ with $2^{n+1}\rho_n B_n \subset V$ for every $n \in N$ and therefore

$\sum_{k=1}^{n} \rho_k B_k \subset \frac{1}{2} V_n$ for every $n \in N$. This iinplies that V contains the zero neighbourhood $\bigcap_{n\in N} (\frac{1}{2} V_n + \sum_{k=1}^{n} \rho_k B_k)$ and we are done.

To apply the previous theorein to vector valued sequence spaces we need the following lemina which is based on an idea of R. Hollstein [21], lemma 1. (A similar version is due to F. Mayoral [27], 2.8.1.)

Lemma 4.9. *Ler $(\lambda, \|\,.\,\|_\lambda)$ be a normal Banach sequence space and let E be a locally convex Hausdorff space. Let $B_1, \ldots, B_m \subset E$ be bounded and absolutely convex and let U be an absolutely convex zero neighbourhood in E. Then for every $\varepsilon > 0$ we have*

$$\lambda(U + \sum_{k=1}^{m} B_k) \subset (1 + \varepsilon)\lambda(U) + \sum_{k=1}^{m} \lambda(B_k).$$

Proof. Let $\varepsilon > 0$ be given. By 2.4 there is some $0 < (\alpha_n)_{n\in N} \in \varepsilon B_\lambda$.

Since $\lambda(U)$ is a zero neighbourliood which absorbs the bounded sets $\lambda(B_k)$ we can assume without loss of generality that $m = 1$. The full assertion follows by induction.

Let $(x_n)_{n\in N} \in \lambda(V)$ with $V := U + B_1$. The boundediiess of the set B_1 then implies for every $n \in N$:

$$x_n \in (p_V(x_n) + \alpha_n)U + p_V(x_n)B_1.$$

Hence there exist $y_n \in (p_V(x_n) + \alpha_n)U$ and $z_n \in p_V(x_n)B_1$ such that $x_n = y_n + z_n$ for every $n \in N$. By the normality of the closed unit ball B_λ of λ we get

$$(p_U(y_n))_{n\in N} \in \|(p_V(x_n) + \alpha_n)_{n\in N}\|_\lambda B_\lambda \subset (1 + \varepsilon)B_\lambda$$

and

$$(p_{B_1}(z_n))_{n\in N} \in \|(p_V(x_n))_{n\in N}\| B_\lambda \subset B_\lambda.$$

Thcn $(y_n)_{n\in N} \in (1 + \varepsilon)\lambda(U)$ *and* $(z_n)_{n\in N} \in \lambda(B_1)$ by the boundedness of B_1.

Theorem 4.10. *Ler $(\lambda, \|\,.\,\|_\lambda)$ be a normal Banach sequence space and let E be a DF space (gDF space). Then $\lambda(E)$ is a DF space (gDF space, resp.). If $(B_n)_{n\in N}$ is a fundamental sequence of bounded sets in E satisfying $2B_n \subset B_{n+1}$ for every $n \in N$, then $(\overline{\lambda(B_n)}^{\lambda(E)})_{n\in N}$ is a fundamental sequence of bounded sets in $\lambda(E)$.*

Proof. In both cases there exists a fundamental sequence of bounded sets $(B_n)_{n\in N}$ in E satisfying $2B_n \subset B_{n+1}$ for every $n \in N$.

Let E be a gDF space and let $(W_n)_{n\in N}$ be an arbitrary sequence of zero iieighbourhoods in $\lambda(E)$. Then there are absolutely convex zero tieighbourhoods U_n in E with $2\lambda(U_n) \subset W_n$ for every $n \in N$. Using lemma 4.9 we get

$$\begin{aligned}\lambda(\bigcap_{n\in N}(U_n + B_n)) &\subset \bigcap_{n\in N} \lambda(U_n + B_n) \subset \bigcap_{n\in N}(2\lambda(U_n) + \lambda(B_n)) \\ &\subset \bigcap_{n\in N}(W_n + \lambda(B_n)).\end{aligned}$$

Since E is a gDF space we get that $\bigcap_{n\in N}(U_n + B_n)$ is *a* zero neighbourhood in E and this implies that $\bigcap_{n\in N}(W_n + \lambda(B_n))$ is a zero neighbourhood in $\lambda(E)$. Applying theorem 4.8 i) we see that $\lambda(E)$ is a gDF space with fundamental sequence of bounded sets $(\overline{\lambda(B_n)}^{\lambda(E)})_{n\in N}$.

The proof in the DF case runs along the same lines: Let E be **a** DF space **and** let $(W_n)_{n\in N}$ be a sequence of zero neighbourhoods in $\lambda(E)$ and let $(\rho_n)_{n\in N} \in (0,\infty)^N$ be arbitrary. Then there are absolutely convex zero neighbourhoods U_n in E with $\lambda(2U_n) \subset W_n$ for every $n \in \boldsymbol{N}$. Using lemma 4.9 we get

$$\begin{aligned}\lambda(\bigcap_{n\in N}(U_n + \sum_{k=1}^{n} \rho_k B_k)) &\subset \bigcap_{n\in N} \lambda(U_n + \sum_{k=1}^{n} \rho_k B_k)\\ &\subset \bigcap_{n\in N}(2\lambda(U_n) + \sum_{k=1}^{n} \rho_k \lambda(B_k))\\ &\subset \bigcap_{n\in N}(W_n + \sum_{k=1}^{n} \rho_k \lambda(B_k)).\end{aligned}$$

Since E is **a** DF space we get with theorem 4.8 ii) that $\bigcap_{n\in N}(U_n + \sum_{k=1}^{n} \rho_k B_k)$ is *a* zero neighbourhood in E **and** this implies that $\bigcap_{n\in N}(W_n + \sum_{k=1}^{n} \rho_k \lambda(B_k))$ is *a* zero neighbourhood in $\lambda(E)$. Applying again theorem 5.8 ii) we arrive at the conclusion.

Corollary 4.11. ***Let*** *$(\lambda, \|\,.\,\|_\lambda)$ be a normal Banach sequence space and let E be a locally convex space. If E is a gDF space, then $\{\overline{\lambda(B)}^{\lambda(E)} : B \in \mathcal{B}(E)\}$ is a fundamental system of bounded sets in $\lambda(E)$. If either E is a gDF space and λ satisfies property 6)* **or** *E is metrizable, then even $\{\lambda(B) : B \in \mathcal{B}(E)\}$ is a fundamental system of bounded sets in $\lambda(E)$.*

Proof. Because of 4.10 and 4.4 only **the** second assertion in the case that E is **a** gDF space **and** λ satisfies property 6) still needs a proof.

But this follows from the fact that $\lambda(B)$ is closed in $\lambda(E)$ whenever B is a bounded, closed **and** absolutely convex subset of E **and** λ satisfies property 6) (cf. 3.3).

Remark 4.12. *If E is a gDF space and λ satisfies property γ), then $\lambda(E)$ is a topological subspace of $\lambda^{(\delta)}(E)$ and for a given fundamental sequence of bounded* ***sets*** *$(B_n)_{n\in N}$ satisfying $2B_n \subset B_{n+1}$ for every $n \in \boldsymbol{N}$* **we get** *that $(\lambda^{(\delta)}(B_n) \cap \lambda(E))_{n\in N}$ is a fundamental sequence of bounded sets in $\lambda(E)$.*

Now we consider the topological dual $\lambda(E)'$ of $\lambda(E)$ if λ satisfies property ε). It will turn out that the properties proved in corollary 4.11 are important for a canonical description of the strong dual of $\lambda(E)$. The results concerning this topic were essentially proved by R.C. Rosier [34] in the more general context of $\mathcal{M}$-topologies. We give here short proofs for the case we consider. Let $(\lambda, \|\,.\,\|_\lambda)$ be a normal Banach sequence space satisfying property ε) (i.e. φ is dense in λ) and let E be **a** locally convex Hausdorff space. In this case $E^{(N)}$ is dense in $\lambda(E)$ and therefore we can identify **a** continuous linear form $f \in \lambda(E)'$ via

$$f_n(x) := f(((0)_{k<n}, x, (0)_{k>n}))), n \in N, x \in E,$$

with the sequence $(f_n)_{n\in N}$ in E'.

Lemma 4.13. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space satisfying property* ε) *and let E be a locally convex Hausdorff space. If* $(f_n)_{n\in N} = f \in \lambda(E)'$ *and if* $C \subset E$ *is absolutely convex, then we have for every* $0 \neq (\alpha_n)_{n\in N} = \alpha \in \lambda$:

$$\|(\alpha_n P_{C^\circ}(f_n))_{n\in N}\|_1 \le \|\alpha\|_\lambda \sup_{x\in\lambda(C)} |f(x)|.$$

Proof. Without loss of generality we inay assume that the ripht hand side is finite, $\|\alpha\|_\lambda = 1$ and $|\alpha| = \alpha$ E φ.

First we note $f(x) = \sum_{n\in N} f_n(x_n)$ fors $= (x_n)_{n\in N}$ E $E^{(N)}$ and $\oplus_{n\in N}\alpha_n C$ C $\lambda(C)$. It follows

$$\begin{aligned}\|(\alpha_n p_{C^\circ}(f_n))_{n\in N}\|_1 &= \sum_{n\in N} \alpha_n p_{C^\circ}(f_n) = \sum_{n\in N} \sup_{z\in\alpha_n C} |f(z)| \\ &\le \sup\{\sum_{n\in N} |f_n(x_n)| \ : \ (x_n)_{n\in N} \in \oplus_{n\in N}\alpha_n C\} \\ &= \sup\{|\sum_{n\in N} f_n(x_n)| \ : \ (x_n)_{n\in N} \in \oplus_{n\in N}\alpha_n C\} \\ &\le \sup\{|f(x)| \ : x \text{ E } \lambda(C)\}.\end{aligned}$$

Fora normal Banach sequence space λ we considered in remark 2.17 ii) its a-dual $\lambda^\times$. We had equipped this space with the norin $\|\,.\,\|_{\lambda^\times}$ admitting the polar $B_\lambda^{\circ\lambda^\times}$ of B_λ as its closed unit ball. We could show that $(\lambda^\times, \|\,.\,\|_{\lambda^\times})$ is a normal Baiiach sequence space satisfying property 6). With the help of the previous lemma we caii prove (cf. [34], p. 492-494) the following representation of the dual of $\lambda(E)$.

Theorem 4.14. *Lei* $(\lambda, \|\,.\,\|_\lambda)$ *be ci normal Banach sequence space satisfying property* ε) *and let E be a locally convex Hausdorff space. Then*

i) $\lambda(E)' \subset \lambda^\times((E', \beta(E', E)))$.

iij For every absolutely convex and closed subset C *of E we have*

$$\lambda(C)^{\circ\lambda(E)'} = \lambda^\times(C^\circ) \cap \lambda(E)'.$$

iiij The strong topology **on** *the dual of* $\lambda(E)$ *is finer than the relative topology induced by* $\lambda^\times((E', \beta(E', E)))$. *They coincide if and only if* $\{\overline{\lambda(B)}^{\lambda(E)} : B \text{ E } \mathcal{B}(E)\}$ ***is** a fundamental system of bounded* **sets in** $\lambda(E)$.

Proof. *i) Let* $f = (f_n)_{n\in N}$ *be ci continuous linear form on* $\lambda(E)$. *Then there* **is** *an absolutely convex zero neighbourhood U in E such that* $\sup_{x\in\lambda(U)} |f(x)| \le 1$. *The previous lemma implies* $(p_{U^\circ}(f_n))_{n\in N}$ E $\lambda^\times$. *Hence* f E $\lambda^\times$ $((([U^\circ], p_{U^\circ}))$. U° *is equicontinuous and therefore strongly bounded and we* ***getf*** $\in \lambda^\times(([U^\circ], p_{U^\circ})) \subset \lambda^\times((E', \beta(E', E)))$.

iij ***Let*** *C* **C** *E be absolutely convex and closed. Let* $f = (f_n)_{n\in N}$ E $\lambda(C)^{\circ\lambda(E)'}$. *Using i) we see* $f \in \lambda^\times((E', \beta(E', E)))$*and with lemma* **4.13** *we get* $(P_{C^\circ}(f_n))_{n\in N} \in B_{\lambda^\times}$. *Hence* f E $\lambda^\times(C^\circ)$.

For the converse inclusion let $f = (f_n)_{n\in N}$ E $\lambda^\times(C^\circ) \cap \lambda(E)'$ *be given. Then we have for* $x \in \lambda(C)$ *that*

$$|f(x)| \le \sum |f_n(x_n)| \le \sum p_C(x_n) p_{C^\circ}(f_n) \le 1,$$

because $p_C(x_n) \in B_\lambda$ and $p_{C^\circ}(f_n) \in B_{\lambda^\times} (= B_\lambda^{\circ\lambda^\times})$.

Applying corollary 4.11 we get:

Corollary 4.15. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space satisfying property* E) *and* E *be a gDF space or a metrizable locally convex space. Then the strong dual of* $\lambda(E)$ *is a topological subspace of* $\lambda^\times((E', \beta(E', E)))$.

Remark 4.16. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space satisfying property* E). *Then* $\lambda' = \lambda^\times$ *and it is easy to see that dual norm* $\|\cdot\|_{\lambda'}$ *is* $\|\cdot\|_{\lambda^\times}$.

We now consider the general case (the proof is implicitly contained in [34] and [16] where algebraic and topological identities for more complicated vector valued sequence spaces were proved). The second part is proved (under much stronger assumptions) in [28], 2.3.

Theorem 4.17. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space satisfying property* E) *and let* E *be a locally convex Hausdorff space. Then* $\lambda(E)'$ *coincides algebraically with* $\lambda^\times((E', \beta(E', E)))$ *if and only if for every* $f = (f_n)_{n\in N} \in \lambda^\times((E', \beta(E', E)))$ *there is* $U \in \mathcal{U}_0(E)$ *such that* $f \in \lambda^\times(U^\circ)$.

If one of this is true and if, in addition, $\{\overline{\lambda(B)}^{\lambda(E)} : B \in \mathcal{B}(E)\}$ *is a fundamental system of bounded sets in* $\lambda(E)$, *then the strong dual of* $\lambda(E)$ *is* $\lambda^\times((E', \beta(E', E)))$.

Proof. Let U be an absolutely convex zero neighbourhood in E and $f = (f_n)_{n\in N} \in \lambda^\times(U^\circ)$. Then we have for every $x = (x_n)_{n\in N} \in \lambda(U)$

$$\sum_{n\in N} |f_n(x_n)| \le \sum_{n\in N} p_{U^\circ}(f_n) p_U(x_n) \le 1$$

and hence $f \in \lambda(E)'$.

For the converse, let $f \in \lambda(E)'$ be given. Then there is an absolutely convex and closed zero neighbourhood U in E such that $f \in \lambda(U)^{\circ\lambda(E)'}$. Theorem 4.14 part ii) guarantees $f \in \lambda^\times(U^\circ)$.

To prove the remaining topological identity we apply theorem 4.14 part iii).

Remark 4.18. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normed Banach sequence space satisfying property* E) *and let* $(E, \|\cdot\|_E)$ *be a normal space. If we denote the dual norm on* E' *by* $\|\cdot\|_{E'}$, *then the theorems 7.17 and 4.14 part ii) imply that* $\lambda((E, \|\cdot\|_E))' = \lambda^\times((E', \|\cdot\|_{E'}))$ *(isometrically).*

Corollary 4.19. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space satisfying property* E) *and let* E *be a quasibarrelled locally convex Hausdorff space. If* $\{\lambda^\times(B) : B \in \mathcal{B}((E', \beta(E', E)))\}$ *is a fundamental system of bounded sets in* $\lambda^\times((E', \beta(E', E)))$, *then the dual of* $\lambda(E)$ *is (algebraically)* $\lambda^\times((E', \beta(E', E)))$. *If, in addition,* $\{\overline{\lambda(B)}^{\lambda(E)} : B \in \mathcal{B}(E)\}$ *is a fundamental system of bounded sets in* $\lambda(E)$, *then the strong dual of* $\lambda(E)$ *is* $\lambda^\times((E', \beta(E', E)))$.

Proof. Using theorem 4.14 it remains to show the algebraic identity. Let $f = (f_n)_{n\in N} \in \lambda^\times((E', \beta(E', E)))$ be arbitrary. Then there is an absolutely convex and bounded set B in $(E', \beta(E', E))$ with $f \in \lambda^\times(B)$. E is quasibarrelled, hence $B \subset U^\circ$ for some zero neighbourhood U in E. Applying theorem 4.17 we have $f \in \lambda(E)'$.

Using the corollaries 4.11 and 4.19 and having in mind that $\lambda^\times$ satisfies property 6) (for every normal Banach sequence space λ) we get:

Corollary 4.20. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space satisfying property* &). *If E is a metrizable locally convex space or if E is a quasibarrelled DF space, then tlie strong dual of $\lambda(E)$ is $\lambda^\times((E', \beta(E', E)))$.*

In the last part of this chapter we waiit to consider properties of boundecl subsets of vector valued sequence spaces. We use essentially lemma 4.3 and lemma 4.9 to get the results. But first let us piove tlie following lemma:

Lemma 4.21. *Let E be a locally convex Hausdorff space. Let B a bounded and absolutely convex subset of E and let U be an absolutely convex zero neighbourhood in E. For $\varepsilon > 0$ tlie fowing liolds:*

i) for every $x \in \bar{B}^E$ we have $\bar{B}^E \cap (x + U) \subset \overline{2B \cap (1 + \varepsilon)U}^E + x$,

ii) $\bar{B}^E \cap U \subset \overline{B \cap (1 + \varepsilon)U}^E$.

Proof. i) Let $x \in \bar{B}^E$ be given. If V is an absolutely convex zero iieighbourhood which is contained in $\frac{\varepsilon}{2}U$, theii

$$\begin{aligned}\bar{B}^E \cap (x + U) &\subset (B + V) \cap (x + U) \subset ((2B + 2V) \cap U) + x \\ &\subset ((2B) \cap (2V + U)) + 2V + x \subset 2B \cap (1 + \varepsilon)U + 2V + x.\end{aligned}$$

This implies i).

ii) Let V be as in the pioof of part i). Then

$$\bar{B}^E \cap U \subset (B + V) \cap U \subset (B \cap (V + U)) + V \subset (B \cap (1 + \varepsilon)U) + V.$$

This proves ii).

Proposition 4.22. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space and let $(E, \mathcal{T})$ be a locally convex Hausdorff space such that $\{\overline{\lambda(B)}^{\lambda(E)} : B \in \mathcal{B}(E)\}$ is a fundamental system of bounded sets in $\lambda(E)$ (e.g. let E be a gDF space). If every bounded subset of E is metrizable, then tlie same holds for every bounded subset of $\lambda(E)$.*

Proof. Every boiinded subset of $\lambda(E)$ is contained in $\overline{\lambda(B)}$ for some $B \in \mathcal{B}(E)$, so we may restrict ourself to such sets. Let $B \in \mathcal{B}(E)$ be given and consider a metrizable locally convex topology $\mathcal{S}$ ori tlie span of B which coincides on B with $\mathcal{T}$. Theii proposition 3.1 yields that on $\lambda(B)$ tlie topologies induced by $\lambda((E, \mathcal{T}))$ and $\lambda(([B], \mathcal{S}))$ coincide. $\lambda(([B], \mathcal{S}))$ is metrizable and we apply lemma 4.21 to get tlie assertion.

The followiiig result is of tlie same type.

Proposition 4.23. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space and suppose $(E, \mathcal{T})$ is a locally convex Hausdorff space such that $\{\overline{\lambda(B)}^{\lambda(E)} : B \in \mathcal{B}(E)\}$ is a fundamental system of bounded sets in $\lambda(E)$ (e.g. E is a gDF space). If E satisfies tlie strict Mackey condition (i.e. for every $B \in \mathcal{B}(E)$ there is $C \supset B$ in $\mathcal{B}(E)$ such that the topologies on B induced by $\mathcal{T}$ and p_C coincide), then $\lambda(E)$ satisfies the strict Mackey condition.*

Proof. Every bounded set of $\lambda(E)$ is contained in $\overline{\lambda(B)}$ for some $B \in \mathcal{B}(E)$, so we may again restrict ourself to such sets. Fix $B \in \mathcal{B}(E)$. Then it exists some $C \in \mathcal{B}(E)$ such that foi every $\varepsilon > 0$ there is an absolutely convex zero neighbourhood U with $B \cap U \subset \varepsilon C$. Using the lemmata 4.21 and 4.3 we get:

$$\begin{aligned}\overline{\lambda(B)}^{\lambda(E)} \cap \lambda(U) &\subset \overline{2\lambda(B) \cap \lambda(U)}^{\lambda(E)} \\ &\subset \overline{4\lambda(B \cap U}^{\lambda(E)} \subset 4\varepsilon \overline{\lambda(C)}^{\lambda(E)}\end{aligned}$$

and it follows that the topologies induced on $\overline{\lambda(B)}^{\lambda(E)}$ by $\lambda(E)$ and $p_{\overline{\lambda(C)}}$ coincide.

We noted already that for every normed (metrizable locally convex) space E the space $\lambda(E)$ is again normed (metrizable, resp.). Now, we consider quasinorniable spaces E, i.e. locally convex Hausdorff spaces satisfying the following property:

For every absolutely convex zero neighbourhood U in E there exists an absolutely convex zero neighbourhood V in E such that for every $\rho > 0$ there is a bounded subset B in E with $V \subset \rho U + B$.

Proposition 4.24. *Let* $(\lambda, \|\ .\|_\lambda)$ *be a normal Banach sequence space and assume* E *is a quasinormable space. Then* $\lambda(E)$ *is quasinormable.*

Proof. Let W be an absolutely convex zero neighbourhood in $\lambda(E)$. Then there is an absolutely convex zero neighbourhood U in E with $2\lambda(U) \subset W$. By assumption is E quasinormable. Hence, there exists another absolutely convex zero neighbourhood $V \subset U$ in E such that for every $\rho > 0$ there is a bounded subset B in E with $V \subset \rho U + B$. Applying lemma 4.9 we get:

$$\lambda(V) \subset \lambda(\rho U + B) \subset 2\rho\lambda(U) + \lambda(B) \subset \rho W + \lambda(B).$$

This implies that $\lambda(E)$ is quasinormable.

5. BARRELLEDNESS CONDITIONS FOR VECTOR-VALUED SEQUENCE SPACES

Let $(\lambda, \|\ .\|_\lambda)$ be a normal Banach sequence space and Ict E be a locally convex Hausdorff space. In this chapter we will consider descriptions for bounded subsets of the space $\lambda(E)$ especially if E is metrizable or **a** DF space. **As** already noted the space $\lambda(E)$ is metrizable (or normed) whenever E has this property. In [17] the completeness of the space $\lambda(E)$ was characterized: $\lambda(E)$ is complete if and oniy if E is complete.

We are now interested in barrelledness conditions for Fréchet- and DF space-valued sequence spaces. Precisely, we waiit to investigate the barrelledness of $\lambda(E)$ for DF spaces E and the distinguishedness of $\lambda(E)$ for Frécliet spaces E.

If φ is dense in λ we noted **in** remark 2.17 iv) that λ is the regular subspace (in the sense of T. Kōmura & Kōmura, [23]) of the perfect space $\lambda^{(\delta)}$. In this case M. Florencio & P.J. Paúl [16], 3.18, proved that $\lambda(E)$ is **a** (barrelied) DF space if and only if E is a (barrelled) DF space. If, in addition, φ is dense in $(\lambda^\times, \beta(\lambda^\times, \lambda))$ we can again apply the result of M. Florencio & P.J. Paúl [16], 3.18, to ensure for every Fréchet space E that $\lambda(E)$ is distinguished if **anti** only if so is E.

For $\lambda = l$, S. Dierolf ([12], (5.13)) proved that $l_\infty(E)$ is a DF space whenever E is **a** DF space. K.D. Bierstedt & J. Bonet ([2], 1.5) could show that $l_\infty(E)$ is a quasibarrelled DF space if and only if E is a DF space satisfying the dual density condition. in **a** previous paper

K.D. Bierstedt & J. Boiiet [1] proved for metrizable spaces E that $l_1(E)$ is distinguished if aiid only if E satisfies **the** density condition.

First we consider barrelled metrizable spaces E **aiid** we prove that in this case $\lambda(E)$ is barrelled. To do this we need **a** lemma, which is based on an idea of **A.** Defant & W. Gavaerts [8], it appears in a wcaker form in [2] and it is contained in aii abstract Banach-Steinhaus theorein due to S. Díaz, **A.** Fernández, M. Florencio & P.J. Paúl (see [9], theoreiii 1). But for the sake ofcompleteness we will prove here version we need.

Lemma 5.1. *Let $(\lambda, \|\cdot\|_\lambda)$ be ci normal Banach sequence space and let E be ci locally convex Hausdorff space. If $\lambda(E)$ is quasibarrelled and E is barrelled, then $\lambda(E)$ is also barrelled.*

Proof. Let T be a barrel in $\lambda(E)$. We have to **show** that T absorbs the bouiided sets. Let $B \in \mathcal{B}(\lambda(E))$ be arbitrary. Withoiit loss of generality we assume that $B = \bigcap_{p\in cs(E)} \rho_p\lambda(p^{-1}([0,1]))$ with positive ρ_p, $p \in cs(E)$. To abbreviate we define for $n \in N$:

$$L_n := \prod_{k\le n} E \times \prod_{k>n} \{0\} \text{ and } M_n := \prod_{k\le n}\{0\} \times \prod_{k>n} E.$$

For $n \in N$ it is immediate that $\lambda(E)$ **is** topologically isomorphic to the direct sum $(\lambda(E)\cap L_n)\oplus(\lambda(E)\cap M_n)$ and $\lambda(E)\cap L_n$ is barrelled (it is isoiiiorphic to E^n). Hence, T absorbs the set $B\cap L_n$ for every $n \in N$. From $B = \bigcap_{p\in cs(E)} \rho_p\lambda(p^{-1}([0,1]))$ we obtain $B \subset B\cap L_n + B\cap M_n$. So, the proof will be coinplete if we can show that T absorbs $B \cap M_n$ for some $n \in \boldsymbol{N}$.

Assume this is iiot truc. Theii. for every $n \in \boldsymbol{N}$, there exists $x^{(n)} := (x_k^{(n)})_{k\in N} \in (B\cap M_n)\setminus 2^{2^n}T$. If $\tilde{E}$ deiiotes the completion of E, theii **the** space $\lambda(\tilde{E})$ is coinplete (see [17]) and we may define a map $f : l_1 \to \lambda(\tilde{E})$ by

$$f(\alpha) := \sum_{n=1}^{\infty} \alpha_n 2^{-n} x^{(n)}.$$

A standard argument (cf. [29], **3.2.13)** shows that $f(B_{l_1})$ is the closiire of the absolutely convex hull of $\{2^{-n}x^{(n)} : n \in N\}$. So, $f(B_{l_1}$ is ti compact (and absolutely coiivex) subset of $\lambda(\tilde{E})$, heiice **a** Baiiach disc. But we coinputc for every $\alpha \in l_1$ that

$$\sum_{n=1}^{\infty} \alpha_n 2^{-n} x^{(n)} = \left(\sum_{n=1}^{k} \alpha_n 2^{-n} x_k^{(n)}\right)_{k\in N}$$

Hence, $f(B_{l_1})$ is even contained in $\lambda(E)$. Since every barrel absorbs the Banach discs, $f(B_{l_1})$ is absorbed by T. In particular, T **absorbs** $\{2^{-n}x^{(n)} : n \in N\}$, **a** contradiction to $x_n \notin 2^{2^n}T, n \in N$.

Remark 5.2. *i) Let E ci locally convex Hausdorff space. Let* **T** *be an absolutely convex subset of $\lambda(E)$, which absorbs every Banach disc in $\lambda(E)$. If T absorbs, in addition, every bounded subset of $\prod_{k\le n} E \times \prod_{k>n}\{0\}$ for every $n \in N$, then the proof of lemma 5.1 shows that* T *absorbs every bounded subset of $\lambda(E)$.*

ii) According to i), we can replace in lemma 5.1 (quasi)barrelled by $\aleph_0$-(quasi)barrelled or by (ultra)bornological.

If E is a inetrizable locally convex space, then $\lambda(E)$ is quasibarrelled. Using this we get with the help of lemma 5.1:

Corollary 5.3. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space and let E be a metrizable locally convex space. If E is barrelled, then* $\lambda(E)$ *is barrelled.*

It is much harder to obtain a result similar as 5.3 if E is a (quasi)barrelled DF space. It will turn out that in this case the property E) for normal Banach sequence spaces plays an important role. We first consider the situation if A satisfies E). The following theorem is contained in a result of M. Florencio & P.J. Paul (cf. [16], 3.8 (i)).

Theorem 5.4. *Let* $(\mathrm{A}, \|\cdot\|_\lambda)$ *be a normal Banach sequence space satisfying property* E) *and let E be a quasibarrelled locally convex Hausdorff space such that* $\{\lambda^\times(B) : B \in \mathcal{B}((E', \beta(E',E)))\}$ *is a fundamental system of bounded sets in* $\lambda^\times((E',\beta(E',E)))$. *Then* $\lambda(E)$ *is quasibarrelled.*

Proof. Fix a bounded subset B of the strong dual of $\lambda(E)$. Using theoreiii 4. I4 iii) we conclude that B is bounded in $\lambda^\times((E',\beta(E',E)))$. Now, there exists some $C \in \mathcal{B}((E',\beta(E',E)))$ with $B \subset \lambda^\times(C)$. Using that E is quasibarrelled, we find a closed and absolutely convex zero neighbourhood U in E whose polar U° contains C. Applying theoreiii 4.14 ii) we get $B \subset \lambda(U)^{\circ\lambda(E)'}$ and therefore B is equicontinuous.

Corollary 5.5. *Let* $(\mathrm{A}, \|\cdot\|_\lambda)$ *be a normal Banach sequence space satisfying property* E) *and let E be a quasibarrelled DF space. Then* $\lambda(E)$ *is quasibarrelled.*

Proof. $(E',\beta(E',E))$ is inetrizable and corollary 4.4 implies that $\{\lambda^\times(B) : B \in \mathcal{B}((E', \beta(E',E)))\}$ is a fundamental systcin of boundcd sets in $\lambda^\times((E',\beta(E',E)))$. An application of 5.4 yields the desired statement.

Corollary 5.6. *Let* $(\mathrm{A}, \|\cdot\|_\lambda)$ *be a normal Banach sequence space satisfying property* ε) *and let E be a barrelled DF space. Then* $\lambda(E)$ *is barrelled.*

Proof. The above corollary implies that $\lambda(E)$ is quasibarrelled. Applying lemma 5.I completes the proof.

Let us consider the situation in the case that E is a DF space and A does not satisfy property ε).

For exainple this situation appears, if E is the strong dual of a iiietrizable space F and λ is the space l; then $l_\infty(E)$ is the strong dual of the space $l_1(F)$. Hence, $l_\infty(E)$ is (quasi)barrelled if and only if $l_1(F)$ is distinguished. K.D. Bierstedt & J. Bonet [1] proved that $l_1(F)$ is distinguished if aiid only if F satisfies Stefan Heinrich's *density condition* (which will be introduced later). They obtained that this density conditioii is equivalent to the fact that every bounded subset of E (the strong dual of F) is inetrizable. In a subseqiient paper [2] K.D. Bierstedt & J. Bonet introduced the so-called dual density condition for locally convex spaces E aiid they could show, whenever E is the strong dual of a inetrizable space F, that E satisfies the dual density condition if and only if F has the density condition. Moreover, whenever E is a DF space, they proved that the space $l_\infty(E)$ is quasibarrelled if and only if E satisfies the dual density conditioii. We will prove that tlie same condition is sufficient and necessary even in the general case. Using results from the previous chapter we will see that the dual density condition is sufficient. Tliat it is even necessary will follow by a reduction to the case which was proved by K.D. Bierstedt & J. Bonet.

Now, we introduce Heinrich's density condition and the dual deiisity condition. S. Heinrich introduced the density conditioii in the context of ultrapowers of locally convex spaces.

Definition 5.7. *(cf. [19], 1.4) A locally convex Hausdorff space E with zero neighbourhood filter $\mathcal{U}_0(E)$ satisfies the* deiisity condition *if, given any function* p $:\mathcal{U}_0(E) \rightarrow (0,\infty)$ *and an arbitrary $V \in \mathcal{U}_0(E)$, there always exist a finite subset $\mathcal{F}$ of $\mathcal{U}_0(E)$ and a bounded set* **B** *in E such that*

$$\bigcap_{U\in\mathcal{F}} \rho(U)U \subset B + V.$$

It is easy to see and already noted by S. Heinrich that every quasinormable locally convex Hausdorff space satisfies the density condition.

Remark 5.8. *(cf. [19], 1.4) In tlie definition of the density condition* rve *may replace $\mathcal{U}_0(E)$ by any basis $\mathcal{U}$ of the zero neighbourhood filter of E.*

Indeed, the outcoming condition is not strictly stronger than the density condition:

Given $\delta : \mathcal{U} \rightarrow (0,\infty)$, for every $U \in \mathcal{U}_0(E)$ there exists $V(U) \in \mathcal{U}$ with $V(U) \subset U$. Define p $: \mathcal{U}_0(E) \rightarrow (0,\infty)$ by $\rho(U) := \delta(V(U))$. Heiice, for every finite subset $\mathcal{F}$ of $\mathcal{U}_0(E)$ there exists a finite subset $\mathcal{G}$ of $\mathcal{U}$ with $\bigcap_{V\in\mathcal{G}} \delta(V)V \subset \bigcap_{U\in\mathcal{F}} \rho(U)U$.

For inetrizable spaces we get therefore:

Proposition 5.9. *Let E be a metrizable locally convex space and let $(U_n)_{n\in N}$ be a basis of the zero neighbourhood filter in E, Then E satisfies tlie density condition if and only if for every sequence $(\rho_n)_{n\in N} \in (0,\infty)^N$ and every zero neighbourhood V in E there exist a bounded set B iii E and $m \in N$ such that*

$$\bigcap_{n=1}^{m} \rho_n U_n \subset B + V.$$

Strongly connected with the density condition are the strong dual density condition and the dual density condition which were introduced by K.D. Bierstedt & J. Bonet (cf. [2], 1.1).

Definition 5.10. *Let E be a locally convex Hausdorff space. E satisfies the* (strong) dual density condition *if for every function $\alpha : \mathcal{B}(E) \rightarrow (0,\infty)$ and* every *bounded subset C of E, these exist always a finite subset $\mathcal{F}$ of $\mathcal{B}(E)$ and a zero neighbourhood U in E such that*

$$C \cap U \subset \overline{\Gamma(\bigcup_{B\in\mathcal{F}} \alpha(B)B)}^{\sim}$$

$$(C \cap U \subset \Gamma(\bigcup_{B\in\mathcal{F}} \alpha(B)B), \text{ respectively}).$$

The following remark is taken from [2], 1.2.

Remark 5.11. *i) Every locally convex Hausdorff space with the strict Mackey conditioii satisfies tlie strong dual density coiiclitioii.*

ii) Using tlie same technique as in remark 5.8 we obtain that in the definition of the dual density condition one can replace $\mathcal{B}(E)$ by any fundamental system of bounded sets $\mathcal{B}$ in E. Taking polars this yields:

iii) The strong dual of a quasibarrelled locally convex Hausdorff E satisfies the dual density coiidition if and only if it satisfies the strong dual density coiidition if and only if E satisfies the density condition. Especially this equivalence holds if E is a metrizable locally convex space.

We now prove a characterization useful for DF spaces (cf. [2], 1.4).

Proposition 5.12. *Let E be a locally convex Hausdorff space possessing an increasing fundamental sequence $(B_n)_{n\in N}$ of absolutely convex and closed bounded sets. Then E satisfies the (strong)dual density condition if and only if for every sequence $(\alpha_n)_{n\in N} \in (0,\infty)^N$ and every bounded set B in E there exist a zero neighbourhood U in E and in E N such that*

$$B \cap U \subset \overline{\sum_{n=1}^{m} \alpha_n B_n}^E$$

$$(B \cap U \subset \sum_{n=1}^{m} \alpha_n B_n, \text{ respectively}).$$

Proof. For every $(\alpha_n)_{n\in N} \in (0,\infty)^N$ and every $m \in N$ we get

$$\Gamma(\bigcap_{n=1}^{m} \alpha_n B_n) \subset \sum_{n=1}^{m} \alpha_n B_n \subset \Gamma(\bigcap_{n=1}^{m} 2^n \alpha_n B_n).$$

Using remark 5.11 ii) this implies the assertion.

The proof of the following theorem is essentialy contained in [2], 1.5.

Theorem 5.13. *Let E be a DF space and let $(B_n)_{n\in N}$ be an increasing fundamental sequence of closed, absolutely convex and bounded sets in E. A bounded and absolutely convex subset B of E is metrizable if and only if for every sequence $(\alpha_n)_{n\in N} \in (0,\infty)^N$ there exist a zero neighbourhoorl U in E and $m \in N$ such that*

$$B \cap U \subset \overline{\sum_{n=1}^{m} \alpha_n B_n}^E .$$

Proof. Let B be inetrizable.

Assume there is a sequence $(\alpha_n)_{n\in N} \in (0,\infty)^N$ such that for every $m \in N$ the set $\overline{\sum_{n=1}^m \alpha_n B_n}^E$ does not contain a zero neighbourhood in B. Since B is inetrizable, there exists a zero sequence $(x_n)_{n\in N}$ in B with $x_m \notin \overline{\sum_{n=1}^m \alpha_n B_n}^E$, $m \in N$. Hence, for every $m \in N$. there is an absolutely convex zero neighbourhood U_m in E with $x_m \notin U_m + \sum_{n=1}^m \alpha_n B_n$. Therefore the sequence $(x_n)_{n\in N}$ is not residually contained in tlie zero neighbourhood $\bigcap_{m\in N} U_m + \sum_{n=1}^m \alpha_n B_n$, a contradiction.

For the converse, we may assume that for every $\alpha = (\alpha_n)_{n\in N} \in (0,\infty)^N$ there is a natural number $m(\alpha)$ such that $\overline{\sum_{n=1}^{m(\alpha)} \alpha_n B_n}^E$ contains a zero neighbourhood in B. Hence, the sets $\overline{\sum_{n=1}^{m(\alpha)} \alpha_n B_n}^{E}$, α_n positive and rational for n E N, define a countable system of zero neighbourhoods in B and it is easy to see that this system is a zero basis in B.

Corollary 5.14. *(cf. [2], 1.5) Let E be a* ***DF*** *space. Then the following are equivalent:*

i) E satisfies the dual density condition.

ii) Every bounded subset of E is metrizable.

iii) There exists a continuous norm $\| \, . \|$ *on E such that on every bounded subset of E the topologies induced by E and by* $\| \, . \|$ *coincide.*

Proof. According to proposition 5.12 and theorem 5.13 it remains to show that ii) implies iii). Therefore let $(B_n)_{n\in N}$ be a fundamental sequence of bounded sets in E. Assuming ii) we get for every $n \in N$ a sequence $(U_{nm})_{m\in N}$ of closed and absolutely convex zero neighbourhoods in E defining a zero basis in B_n. E satisfies the countable neighbourhood property and therefore there exists $\rho_{nm} > 0$ such that $U := \bigcap_{n,m\in N} \rho_{nm} U_{nm}$ is a zero neighbourhood in E. The Minkowski functional $\| \, . \|$ of U has the required property.

Remark 5.15 i) *Let E be a* ***DF*** *space in which every bounded subset is metrizable or, equivalently, which satisfies the dual density condition. Then E is quasibarrelled. (This is due to* **A.** *Grothendieck, [18], p. 71, théorèm 5).*

ii) In every ***DFM*** *space E the bounded subsets are precompact and therefore metrizable (see e.g. [31]), so E satisfies the dual density condition and hence the strong dual densi9 condition.*

We are now able to characterize the barrelledness of $\lambda(E)$ for DF spaces E. We proved in 5.5 and 5.6:

If A satisfies property E), then $\lambda(E)$ is (quasi)barrelled if and only if E is (quasi)barrelled.

The remaining part is given in the next result which has been proved by K.D. Bierstedt & J. Bonet (see [2], 1.5) in the case $\lambda = l_\infty$:

Theorem 5.16. *Let* $(\lambda, \| \, . \|_\lambda)$ *be a normal Banach sequence space not satisfying property E) and let E be a* ***DF*** *space. Then the following are equivalent:*

i) E satisfies the dual density condition.

ii) Every bounded subset of E is metrizable.

iii) Every bounded subset of (E) *is metrizable.*

iv) $\lambda(E)$ *is quasibarrelled.*

Moreover, $\lambda(E)$ *is barrelled if and only if E is barrelled and one of the conditions i) to iv) is satisfied.*

Proof. The equivalence of i) and ii) was proved in 5.14. In 4.22 we could show that ii) implies iii). 4.10 yields that $\lambda(E)$ is a DF space and so iii) implies iv) by remark 5.15 i).

Let $\lambda(E)$ be quasibarrelled. λ does not satisfy property E) and by theorem 3.8 this implies that $l_\infty(E)$ is a complemented subspace of $\lambda(E)$. Therefore $l_\infty(E)$ is also quasibarrelled. (For the following part of the proof compare with [12], (4.5),(5.12) and with [2], 1.5" (3) implies (1)") Let $(B_n)_{n\in N}$ be an increasing fundamental sequence of bounded, closed and absolutely convex sets in E.

Let $(\alpha_n)_{n\in N}$ E $(0,\infty)^N$ be given.

We define

$$V := \bigcap_{n\in N} l_\infty(\overline{\sum_{k=1}^{n} \alpha_k B_k}).$$

The sequence $(l_\infty(\overline{\sum_{k=1}^{n} \alpha_k B_k}))_{n\in N}$ is increasing, consists only of closed, absolutely convex and bounded sets and every bounded subset of the DF space $l_\infty(E)$ is absorbed by some $l_\infty(\overline{\sum_{k=1}^{n} \alpha_k B_k})$ (cf. 4.11). So, we inay apply **A.** Grothendieck [18], p.72, lemma 4, to get $\bar{V}^{l_\infty(E)}$ C $2V$. Now, $\bar{V}^{l_\infty(E)}$ is ***a*** bornivorous barrel in $l_\infty(E)$ and the quasibarrelledness of $l_\infty(E)$ implies that V is a zero neighbourhood. So there is an absolutely convex zero neighbourhood in E with $l_\infty(U) \subset V$. We get that for every bounded subset B in E there exists n E ***N*** such that

$$B \cap U \subset \overline{\sum_{k=1}^{n} \alpha_k B_k}.$$

(Otherwise there is a bounded sequence $(x_n)_{n\in N}$, coiitained in U, with $x_n \notin \overline{\sum_{k=1}^{n} \alpha_k B_k}$ for each n E ***N*,** a contradiction to $l_\infty(U)$ C V.) Because $(\alpha_n)_{n\in N}$ is arbitrary we get with proposition 5.12 that E satisfies the dual density condition. So iv) implies i). The second assertion follows by the first part aiid lemma 5.1.

Remark 5.17. *i) Let E be ci DF space with citi increasing fundamental sequence $(B_n)_{n\in N}$ of bounded, closecl and absolutely convex sets. If E satisfies the dual density condition, then, according to the proof of theorem 5.16, it satisfies the (formally) stronger condition:*

For every $(\alpha_n)_{n\in N}$ E $(0,\infty)^N$ there is ci zero neighbourhood U such tliat for every bounded set B tliere exists n E *N with $B \cap U \subset \overline{\sum_{k=1}^{n} \alpha_k B_k}$.*

ii) It is possible to apply directly the result of K.D. Bierstedt & J. Bonet mentioned above to show that **tlie** *dual density condition is necessary for the quasibarrelledness of $l_\infty(E)$.*

We now want to consider distinguished (vector valued) metrizable spaces. Grothendieck called a locally convex Hausdorff space E *distinguished* if its strong dual $(E',\beta(E',E))$ is barrelled. If E is, in addition, metrizable, then E is distiiiguished if and only if $(E',\beta(E',E))$ is quasibarrelled (bornological, ultrabornological, an LB space) (see [18], p. 73, theorein 7).

First we note:

Remark 5.18. *Let E be ci metrizable locally convex space. If E satisfies* **tlie** *density condition, then E is distinguished.*

Indeed, E satisfies the density conditioii if and only if the DF space $(E',\beta(E',E))$ satisfies the dual density conditioii (see 5.11 iii)). But reinark 5.15 i) implies that $(E',\beta(E',E))$ then has to be quasibarrelled.

We will give a complete characterization of the distinguishedness of $\lambda(E)$ whenever λ is ***a*** normal Banach sequence space and E is ***a*** metrizable locally coiivex space. Analogously to the previous results on vector valued DF spaces (which we will use) condition ε) for normal Baiiach sequence spaces will play an iinportant role. Let us first consider the situation for norinal Banach sequence spaces λ satisfying property ε):

Theorem 5.19. *Ler $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space satisfying property ε) aiid let E be ci metrizable locally convex space.*

i) If $(\lambda^\times, \|\cdot\|_{\lambda^\times})$ *also satifies property* ε), *then* $\lambda(E)$ *is distinguished if and only if* E *is distinguished.*

ii) If $(\lambda^\times, \|\cdot\|_{\lambda^\times})$ *does not satisfy property* ε), *then* $\lambda(E)$ *is distinguished if and only if* E *satisfies the density condition.*

Proof. Becaiise λ satisfies property ε) we get witli corollary 4.20 **that** tlie strong dual of $\lambda(E)$ is the DF spacc $\lambda^\times((E', \beta(E', E)))$(topologically).

i) If $\lambda^\times$ satisfies the property ε). then $\lambda^\times((E', \beta(E', E)))$ is quasibarrelled if **and** only if $(E', \beta(E', E))$is quasibarrelled (see 5.5) or, equivalently, if **aiid** only if E is distinguished.

ii) If $\lambda^\times$ does not satisfy property ε), then $\lambda^\times((E', \beta(E', E)))$is quasibarrelled if **and** only if $(E', \beta(E', E))$satisfies the dual density condition (see 5.16) or. equivalently, ifaiicl only if E satisfies the density coiiclition *(see* 5.1 I iii)).

Remark 5.20. *i) Theorem 5.19 i) describes* **tlie** *situation whenever* $\lambda = l_p$, $p \in (1, \infty)$ ***or*** $\lambda = c_0$.

ii) Theorem 5.19 part ii) covers the case $\lambda = l_1$, *which was considered by K.D. Bierstedt & J. Ronet [1].*

In remark **5.**15 i) we noted that **a** DF space E in which every bouiided subset is inetrizable (i.e a DF space E which satisfies thc dual deiisity coiidition) ***has*** to be quasibarrelled. We used that this property is stable under forming the vector valued sequeiice spaces $\lambda(E)$ to get that $\lambda(E)$ is quasibarrelled for every DF space E with the dual deiisity coiiditioii (aiid every normal Banacli seqiieiice space λ). If λ does not satisfy property ε). then $\lambda(E)$ contains $l_\infty(E)$ as a complemented subspace **aiid** we couid show **that** tlie dual deiisity coiiditioii is also necessary for tlie quasibarrellediiessof $l_\infty(E)$ **aiid** therefore necessary for the quasibarrellediiessof $\lambda(E)$. We will use tlie same ideas iii tlie exaininatioii of tlie distinguishedness of $\lambda(E)$ whenever E is metrizable **aiid** λ does **iiot** satisfy property ε). Wc first prove:

Proposition 5.21. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space and let* E *be a metrizable locally convex space which satisfies the density condition. Then* **tlie** *(metrizable) space* $\lambda(E)$ *also satisfies* **tlie** *density condition.*

Proof. We use the characterization 5.9 of the density condition for metrizable locally conves spaces. Let therefore $(U_n)_{n\in N}$ be ***a*** decreasing sequence of closed **ancl** absolutely convex zero neighbourhoods in E which defines ***a*** zero basis.

Let $(\rho_n)_{n\in N} \in (0, \infty)^N$ aiid ***a*** zero neighbourhood W in $\lambda(E)$ be given. We have to show that there are $m \in N$ aiid $C \in \mathcal{B}(\lambda(E))$ with $\bigcap_{n=1}^m \rho_n\lambda(U_n) \subset C + W$.

First we get tlie existence ofan absolutely coiivex zero neighbourhood U in E with $\lambda(U) \subset W$. Since E satisfies **tlie** deiisiiy coiiditioii there are $m \in N$ **aiid** $B \in \mathcal{B}(E)$ **with**

$$\bigcap_{n=1}^{m} \rho_n 2^{n+1} U_n \subset B + U.$$

Lemma 4.3 thcii implies

$$\bigcap_{n=1}^{m} \rho_n\lambda(U_n) \subset \frac{1}{2}(\sum_{k=1}^{m} 2^{-k})\lambda(\bigcap_{n=1}^{m} \rho_n 2^{n+1} U_n) \subset \frac{1}{2}\lambda(B + U).$$

Now, we can apply lemma 4.9 to get $\bigcap_{n=1}^{m} \rho_n\lambda(U_n) \subset \lambda(B) + \lambda(U) \subset \lambda(B) + W$ and this yields the coiiclusion.

We recali a result of J. Bonet, S. Dierolf & C. Fernandez (sec [7], lemma i), which gives a necessary condition for the distinguishedness of Frechet spaces. Using this result they could give the first example of **a** distinguished Frechet space whose bidual is not distinguished. With essentially tlie same proof as given in [7] we prove the resuit for metrizable spaces:

Theorem 5.22. *Let E be a metrizable locally convex space and let F be a topological subspace of the bidual E" of E which contains E as a closed subspace (if we consider E as a subspace of E").*

Let $q : F \to F/E$ be the quotient map. **Assunie** *F is distinguished, then*

i) F / E is also distinguished and

ii) the strong dual of F / E is a topological subspace of the strong dual of F, i.e. every bounded set B in F / E is contained in the closure of $q(C)$ for some $C \in \mathcal{B}(F)$.

Proof. The transposed map $q' : ((F/E)', \beta((F/E)', F/E)) \to (F', \beta(F', F))$ of q is a continuous inclusion with range $E^{\circ F'}$, so we may identify $(F/E)'$ (algebraically) with $E^{\circ F'} =: E^{\circ}$. We equip E° with the relative topology induced by $\beta(F', F)$

We recali that a metrizable space is distinguished if aiid only if its strong dual is an LB space. We only have to show that E° is an LB space. (Since the opeii mapping theorem then implies i) and ii), using the fact that the associated bornological topology to $\beta((F/E)', F/E)$ is also an LB space topology).

Let $j : E \to F$ and $i : F \to E''$ be the canonical iiiclusions aiid let $j' : F' \to E'$ and $i' : (E'', \beta(E'', E'))' \to F'$ be their trasposed mappings (which are continuous with respect to the strong topologies). First we note that E° is the kernel of j'. Using that the strong dual of F is barrelled, we even get that

$$j' : (F', \beta(F', F)) \to (E', \beta(E', E''))$$

is continuous. Because E'' is barrelled (it is a Frechet space) we have the contiiiuity of

$$i' : (E''', \beta(E''', E'')) \to (F', \beta(F', F)).$$

Moreover, the mapping $j' \circ i'$ is the transpose of $i \circ j$ and therefore its restriction to E' is the identity on $(E', \beta(E', E''))$. So $(F', \beta(F', F))$ is the topological direct sum of a copy of $(E', \beta(E', E''))$ and $ker(j') = E^{\circ}$. Thus E° is a compleinented subspace of an LB space and therefore itself an LB space.

We remark that J. Bonet & S. Dierolf [6] could show that **a** quotient innp between Frechet spaces, which lifts bounded sets with closure, lifts already bounded sets.

We now turn back to the examination of the distinguishedness of our spaces $\lambda(E)$.

Theorem 5.23. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space and let E be a metrizable locally convex space.*

If both λ and $\lambda^\times$ satisfy property ε), then $\lambda(E)$ is distinguished if and only if E is distinguished.

If λ or $\lambda^\times$ does not satisfy property ε), then $\lambda(E)$ is distinguished if and only if E satisfies the density condition.

Proof. According to theoreni 5.I9 we only have to prove the assertion in the case where λ does not satisfy property ε). E satisfies the density coiidition, proposition 5.21 yields that $\lambda(E)$ satifies it. Remark 5.18 implies that $\lambda(E)$ is distinguished.

Let $\lambda(E)$ be distinguishcd. Theii theorem 3.8 iinplies that $l_\infty(E)$ is **a** complemented subspace of $\lambda(E)$ and therefore $l_\infty(E)$ is distinguished. Because E is nietrizable and distinguished we get with corollary 4.20 that $l_\infty(E'')$ is the bidual of $c_0(E)$. Moreover, $c_0(E)$ is closed in $l_\infty(E)$ and $l_\infty(E)$ is a topological subspace of $l_\infty(E'')$. So, we may apply 5.22 ii) and get that the quotient map q : /,(E) $\rightarrow l_\infty(E) / c_0(E)$ lifts bounded sets with closure. We choose a decreasing zero basis $(U_n)_{n\in N}$ in E consisting of closed aiid absolutely convex sets.

Let $(\rho_n)_{n\in N}$ E $(0,\infty)^N$ be arbitrary. Then

$$B := \bigcap_{n\in N} \rho_n q(l_\infty(U_n))$$

is a bounded subset of /,(E) $/ c_0(E)$. Hence, because q lifts bounded sets with closure, there is a bounded set C in $l_\infty(E)$ with B C $\overline{q(C)}$. By corollary 4.4 we may assume that $C = l_\infty(D)$ for some closed D E $\mathcal{B}(E)$ aiid we get B C $\overline{q(l_\infty(D))}$.

Assume there is ***a*** closed and absolutely convex zero neighbourhood U in E such that there exists $x_m \in (\bigcap_{n=1}^m \rho_n U_n)\backslash D + 2U$ for every m E ***N***. Then we get $(x_m)_{m\in N}$ E $l_\infty(E)$ and $(x_m)_{m\in N} + c_0(E) \in B$ (by the definition of B). Bui there cannot exist a zero sequeiice $(y_m)_{m\in N}$ in E with $(x_m + y_m)_{m\in N}$ E $D^N + U^N$. Heiice $(x_m)_{m\in N} + c_0(E) \notin q(l_\infty(D)) + q(l_\infty(U))$, a contradiction to $B \subset \overline{q(l_\infty(D))}$. So, the assumption is wrong and we have shown that for every zero neighbourhood U in E there exists m E ***N*** with $\bigcap_{n=1}^m \rho_n U_n \subset D + U$. Because $(\rho_n)_{n\in N}$ is arbitrary, this means that E satisfies the density condition.

Remark 5.24. *i) Let E be a metrizable locally convex space with a decreasing zero basis $(U_n)_{n\in N}$. If E satisfies the density condition,* ***tke*** *proof of* ***theorem 5.23*** *shows that it satisfies tlie (formally) stronger coiiclition:*

For every $(\rho_n)_{n\in N}$ E $(0,\infty)^N$ there exists a bounded subset D of E such that for every zero neighbourhood U one can find m E N with $\bigcap_{n=1}^m \rho_n U_n$ C $D + U$.

ii)I recently got ***tlie*** *information, that J.C. Díaz had independently proved* ***rhe*** *case $\lambda = l$, (unpublished).*

Grothendieck [18], Questions non Resolues 5, posed the question, whether the bidual of a distinguished Frechet space is again distinguished. As noted above the first counter exainple was given by J. Boiiet, S. Dierolf & C. Fernandez [7] using Frechet spaces of Moscatelli type. Using theoreni 5.23 we can give an example in the context of vector valued sequence spaces:

Example 5.25. *Let E be a reflexive Fréchet space which does <u>not</u> satisfy tlie density coiidition (for such a space see [12], (4.7) or (4.8)). Then we get that $c_0(E)$ is distinguished.* ***Birt*** *the bidual of $c_0(E)$ is l, (E) and therefore not distinguished.*

By **a** result of **A.** Pcris (see [30], 3.1.6) a Frechet space satisfies the density conditioii whenever its bidual satisfies it. so one may replace in the above example E by any distingiiished Frechet space without density condition.

Remark 5.26. *Assume that λ is a normal Banach sequence space and that E is a reflexive Fréchet space without tlie density condition such that $\lambda(E)$ is distinguished But its bidual does not. Then $c_0(E)$ is complemented in $\lambda(E)$.*

Indeed, under the assumptions above, λ and $\lambda^\times$ have to satisfy property ε) (see theorem 5.23). Moreover, $\lambda^{\times\times}$ must not satisfy property ε) (otherwise $\lambda(E)$ would be reflexive). We may apply reinark 3.9 to get that $\lambda(E)$ contains a complemented copy of $c_0(E)$.

6. BORNOLOGICAL VECTOR VALUED SEQUENCE SPACES

In this chapter we consider sufficient conditions to guarantee that for a given (ultra) bornological space E the space $\lambda(E)$ is also (ultra)bornological. First we consider the situation if E is metrizable. In this case $\lambda(E)$ is again metrizable, hence bornological. Using remark 5.2 ii) we get:

Proposition 6.1. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence spnce and let E be a metrizable locally convex spnce. Then $\lambda(E)$ is ultrabornological if and only if E is ultrabornological.*

The situation is more complicated if E is a bornological DF space or even a DFM space. We will prove for DFM spaces E and normal Banach sequence spaces λ which satisfy property γ) that $\lambda(E)$ is bornological. This given a partial answer to the question, posed by K.D. Bierstedt and J. Schmets (see [35]), if for every bornological space E also $c_0(E)$ is bornological. We first consider the situation when λ satisfies ε) and δ). The following proposition is covered by a result due to M. Florencio & P.J. Paul (see [16], 4.9), which is proved in the context of perfect spaces.

Proposition 6.2. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space satisfying properties ε) and δ). Let E* **be** *a DF spnce. Then $\lambda(E)$ is bornological if and only if E is bornological.*

Proof. Let f be an arbitrary locally bounded linear form on $\lambda(E)$. (We will show that f is continuous). Define locally bounded linear forms $f_n, n \in N$ on E by $f_n(x) := f(((0)_{k<n}, x, (0)_{k>n}))$, $x \in E$. Since E is bornological, these mappings f_n are continuous on E.

Let $x = (x_n)_{n\in N} \in \lambda(E)$ be arbitrary. Because λ satisfies property δ) we get $x \in \lambda(B)$ for some $B \in \mathcal{B}(E)$. Property ε) yields $\lim_{n\to\infty} \|((0)_{k\leq n}, (p_B(x_k))_{k>n})\|_\lambda = 0$. Using that f is locally bounded, this implies $\lim_{n\to\infty} (f(x) - \sum_{k=1}^n f_k(x_k)) = \lim_{n\to\infty} f(((0)_{k<n}, (x_k)_{k\geq n})) = 0$. So, we may identify f with $(f_n)_{n\in N}$ via $f(x) = \sum_{n=1}^\infty f_n(x_n)$, $x = (x_n)_{n\in N} \in \lambda(E)$.

Let C be a bounded subset of $\lambda(E)$. Without loss of generality we may assume that $C = \bigcap_{p\in cs(E)} \rho_p \lambda(p^{-1}([0,1]))$, $\rho_p > 0$. We get

$$\bigcup_{n\in N} \{\sum_{k=1}^n f_k(x_k) : (x_k)_{k\in N} \in C\} \subset f(C)$$

and therefore $((f_k)_{k\leq n}, (0)_{k>n})$ is a strongly bounded sequence in $\lambda(E)'$ which is by corollary 5.5 even equicontinuous. Hence $f = (f_n)_{n\in N}$ is continuous.

We proved that every locally bounded linear form on $\lambda(E)$ is already continuous and this implies together with corollary 5.5 that $\lambda(E)$ is bornological.

Remark 6.3. *Analysing the proof of proposition 6.2 we see that the result is even true for normal Banach sequence spaces λ satisfying property ε) and DF spaces E whenever*

$$\lambda(E) = \bigcup_{B\in\mathcal{B}(E)} \lambda(B).$$

Example 6.4. *(cf. [26]) Let E be a bornological* ***DF*** *space in which every zero sequence converges locally, i.e.* $c_0(E) = \bigcup_{B \in \mathcal{B}(E)} c_0(B)$. *(Tliismeans that E is a retractive LN space). Then* $c_0(E)$ *is a bornological* ***DF*** *space.*

For the iiext result we need a reformulation of leinina 4.9:

Lemma 6.5. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space and let E be a locally convex Hausdforff space. Let* $B_1, \ldots, B_m$ *be bounded and absolutely convex subsets of E. For every* $\varepsilon > 0$ *it follows*

$$\lambda(\sum_{n=1}^{m} B_n) \subset (1+\varepsilon) \sum_{n=1}^{m} \lambda(B_n).$$

Proof. Let $\varepsilon > 0$ be arbitrary. Withoiit loss of generality we may assume that $m = 2$. Let $(x_n)_{n\in N} \in \lambda(B_1 + B_2)$ be given. Using the boundedness of B_1 and B_2 we get

$$x_n \in (1+\varepsilon) p_{B_1+B_2}(x_n)(B_1 + B_2) = (1+\varepsilon) p_{B_1+B_2}(x_n) B_1 + (1+\varepsilon) p_{B_1+B_2}(x_n) B_2$$

for every $n \in N$.

This iiiiplies the existence of $y_n \in (1+\varepsilon) p_{B_1+B_2}(x_n) B_1$ aiid $z_n \in (1+\varepsilon) p_{B_1+B_2}(x_n) B_2$ with $x_n = y_n + z_n$ for every $n \in N$. We now compute $\|(p_{B_1}(y_n))_{n\in N}\|_\lambda, \|(p_{B_2}(x_n))_{n\in N}\|_\lambda \leq (1+\varepsilon)$ and using again *the* bouiidediiess of B_1 and B_2, **we** get $(y_n)_{n\in N} \in (1+\varepsilon)\lambda(B_1)$ and $(z_n)_{n\in N} \in (1+\varepsilon)\lambda(B_2)$. *Tliis* implies *the* assertion.

Proposition 6.6. *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach sequence space satisfying property* δ) *but not property* ε). *Let E be a* ***DF*** *space. Then* $\lambda(E)$ *is bornological if and only if E satisfies the strong dual density condition.*

Proof. *Let* $(B_n)_{n\in N}$ be an increasing fiindainental seqiience of bouiided, absolutely coiivex and closed sets in E satisfying $2B_n \subset B_{n+1}$, $n \in N$.

Suppose E satisfies the strong dual density condition and let $(\alpha_n)_{n\in N} \in (0,\infty)^N$ be arbitrary. If C is **a** bounded subset of $\lambda(E)$, we may (in view of property δ), see 4.11) assume that $C \subset \lambda(B_k)$ for some $k \in N$. Using that E satisfies the strong **dual** density condition, we get the existence of an absolutely convex and closed zero neighbourhood U in E and $m \in N$ such that

$$B_k \cap U \subset \frac{1}{4} \sum_{n=1}^{m} \alpha_n B_n.$$

Using the lemmata 4.3 aiid 6.5 this yields

$$C \cap \lambda(U) \subset \lambda(B_k) \cap \lambda(U) \subset 2\lambda(B_k \cap U) \subset \lambda(\frac{1}{2} \sum_{n=1}^{m} \alpha_n B_n) \subset \sum_{n=1}^{m} \alpha_n \lambda(B_n).$$

Hence, $\lambda(E)$ satisfies the stroiig dual density condition. Since $\lambda(E)$ is **a** DF space this yields that $\lambda(E)$ is bornological.

For the converse let $\lambda(E)$ be bornological. Then $l_\infty(E)$ **is** (as a coinpleiiieiited subspace of $\lambda(E)$, see **3.8)** also bornological.

Let $(\alpha_n)_{n\in N}$ E $(0, \infty)^N$ be arbitrary. Then

$$V := \bigcup_{n\in N} l_\infty(\sum_{k=1}^{n} \alpha_k B_k)$$

is a bornivorous and absolutely convex subset of $l_\infty(E)$, therefore **a** zero neighbourhood. Hence there is a zero neighbourhood U in E with $l_\infty(U) \subset$ **V.** Proceeding as in the proof of 5.16 this implies that for every bounded subset B of E there is n E N with

$$B \cap U \subset \sum_{k=1}^{n} \alpha_k B_k.$$

Therefore E satisfies the strong dual density condition.

We remark that K.D. Bierstedt & J. Bonet (see [2], 1.5) proved the above result in the case A $= \dot{l}$ with ***a*** somewhat different proof.

Remark 6.7. *i) Lei E be* ci *DF space with an increasing fundamental sequence* $(B_n)_{n\in N}$ *of bounded, closed and absolutely convex sets. If E satisfies the strong dual density condition, then, according to* ***the*** *proof of proposition 6.6, it satisfies the (formally) stronger condition:*

For every $(\alpha_n)_{n\in N}$ E $(0,\infty)^N$ *there is a zero neighbourhood U such that for every bounded set B there esists n* $\in$ ***N*** *with* $B \cap U \subset \sum_{k=1}^{n} \alpha_k B_k$.

ii) Let (A, $\|\ .\ \|_\lambda$) *be a normal Banach sequence space satisfying property 6) and let E be* ci *DF space. Combining remark 5.2 ii) with proposition 6.2 and proposition 6.6 we get necessary and sufficient conditions deciding whether* A(E) *is ultrabornological.*

iii)In tlie previous results of this chapter one may replace "property 6)" by "property δ_w*)", see theorem 2.9 iv).*

We must confess that we can give only partial results in the case when (A, $\|\ .\ \|_\lambda$) does not satisfy property δ_w).

Up to now we have only remark 6.3 to ensure that in this case $\lambda(E)$ is bornological. But we can give a positive answer whenever $(\lambda, \|\ .\ \|_\lambda)$ satisfies property γ) and E is a DFM space.

Let (A, $\|\cdot\|_\lambda$) be a normal Banach space satisfying property γ). We recall that (A'"', $\|\ .\ \|_{\lambda^{(\delta)}}$) is defined by

$$\lambda^{(\delta)} := \{\alpha \text{ E } \omega : \sup_{n\in N} \|P_n(\alpha)\|_\lambda < \infty\},$$

$$\|\alpha\|_{\lambda^{(\delta)}} := \sup_{n\in N} \|P_n(\alpha)\|_\lambda,$$

where P_n denote tlie canonical projection onto the first n coordinates. Then (A, $\|\ .\ \|_\lambda$) is *a* subspace of $(\lambda^{(\delta)}, \|\cdot\|_{\lambda^{(\delta)}})$ (see 2.9 i)) and $(\lambda^{(\delta)}, \|\cdot\|_{\lambda^{(\delta)}})$ satisfies property 6). i.e. its closed unit ball is even closed in ω *(see* 2.9 iii) and 2.10). We now prove a technical lemma coiicerning this spaces.

Lemma 6.8. *Let* (A, $\|\ .\ \|_\lambda$) *be* ci *normal Banach sequence space satisfying property y) and let an arbitrary* $(\alpha_n)_{n\in N}$ E A *be given. Moreover, let* $\gamma : [0, 1] \rightarrow [0,1]$ *be increasing such that* $\lim_{x\rightarrow 0} \gamma(x) = 0$.

Then for every $(\delta_n)_{n\in N} \in [1,\infty)^N$ *satisfying* $(\alpha_n\delta_n)_{n\in N} \in \lambda^{(\delta)}$ *it follows* $(\alpha_n\delta_n\gamma(\delta_n^{-1}))_{n\in N} \in$ A.

Proof. Let for every subset S of N be $P_S : \omega \to \omega$ the canonical projection onto the coordinates belonging to S.

Without loss of generality we may assume that $(\alpha_n\delta_n)_{n\in N}$ is contained in the closed unit ball $B_{\lambda^{(\delta)}}$ of $\lambda^{(\delta)}$.

Let $\varepsilon > 0$ be arbitrary and define $S := S(\varepsilon) := \{n \in N : \gamma(\delta_n^{-1}) \leq \varepsilon\}$. Using that γ is increasing and satisfies $\lim_{x\to 0}\gamma(x) = 0$ we get the existence of $\rho > 0$ such that $|\alpha_n\delta_n| \leq \rho|\alpha_n|$ for every $n \in N\backslash S$. Hence, $P_{N\backslash S}((\alpha_n\delta_n\gamma(\delta_n^{-1}))_{n\in N}) \in \lambda$. Moreover, we coinpute $\|P_S((\alpha_n\delta_n\gamma(\delta_n^{-1}))_{n\in N})\|_{\lambda^{(\delta)}} \leq \varepsilon\|(\alpha_n\delta_n)_{n\in N}\|_{\lambda^{(\delta)}} \leq \varepsilon$. Therefore we get $(\alpha_n\delta_n\gamma(\delta_n^{-1}))_{n\in N} = P_{N\backslash S}((\alpha_n\delta_n\gamma(\delta_n^{-1}))_{n\in N}) + P_S((\alpha_n\delta_n\gamma(\delta_n^{-1}))_{n\in N}) \in A + \varepsilon B_{\lambda^{(\delta)}}$. Because A is closed in $\lambda^{(\delta)}$ and ε is arbitrary this iinplies the assertion.

We need another lemma concerning zero neighbourhoods of compact subsets of normed spaces.

Lemma 6.9. *(S. Dierolf, per-soneil communication) Let $(E,\|\cdot\|)$ be a normed space with closed unit ball $B_{\|\cdot\|}$. Let K_1 and K_2 be absolutely convex and compact subsets of E such that $2(K_1 + K_2) \subset B_{\|\cdot\|}$. Then there is an increasing mapping $\varepsilon : [0,1] \to [0,1]$ satisfying $\lim_{x\to 0}\varepsilon(x) = 0$ such that for every $x \in [0,1]$ we have*

$$(K_1 + K_2) \cap xB_{\|\cdot\|} \subset (2K_1 \cap \varepsilon(x)B_{\|\cdot\|}) + (2K_2 \cap \varepsilon(x)B_{\|\cdot\|}).$$

Proof. We will prove:

(*) For every $\varepsilon > 0$ there is $\delta > 0$ such that
$(K_1 + K_2) \cap \delta B_{\|\cdot\|} \subset (2K_1 \cap \varepsilon B_{\|\cdot\|}) + (2K_2 \cap \varepsilon B_{\|\cdot\|})$.

This implies iinmediately the assertion.

Assume that (*) is wrong. Then there is $\varepsilon > 0$ and a zero sequence $(z_n)_{n\in N}$ in E, contained in $K_1 + K_2$, such that $z_n \notin (2K_1 \cap \varepsilon B_{\|\cdot\|}) + (2K_2 \cap \varepsilon B_{\|\cdot\|})$ for every $n \in N$. Choose sequences $(x_n)_{n\in N}$ in K_1 and $(y_n)_{n\in N}$ in K_2 with $z_n = x_n + y_n$, $n \in N$. The sets K_1 and K_2 are (sequentially) compact, so there exists a stricly increasing sequence $(n_k)_{k\in N}$ of positive nuinbers, $x \in K_1$ and $y \in K_2$ such that $x_{n_k} \to x$ and $y_{n_k} \to y (k \to \infty)$. Since $(z_n)_{n\in N}$ is a zero sequence we coinpute $x + y = 0$. Hence, there is $m \in N$ such that

$$z_{n_m} = (x_{n_m} - x) + (y_{n_m} - y) \in (2K_1 \cap \varepsilon B_{\|\cdot\|}) + (2K_2 \cap \varepsilon B_{\|\cdot\|}),$$

a contradiction.

Proposition 6.10. *Let $(\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space satisfying property γ) and let E be a DFM space. Moreover, let $K_1, \dots, K_m$ be absolutely convex and compact subsets of E. Then we get*

$$\lambda^{(\delta)}\Big(\sum_{n=1}^{m} K_n\Big) \cap \lambda(E) \subset \sum_{n=1}^{m} 2^n(\lambda^{(\delta)}(K_n) \cap \lambda(E)).$$

Proof. We will prove

$$\lambda^{(\delta)}(K_1 + K_2) \cap \lambda(E) \subset 2(\lambda^{(\delta)}(K_1) \cap \lambda(E)) + 2(\lambda^{(\delta)}(K_2) \cap \lambda(E)),$$

the assertion follows then by induction.

Because E satisfies the dual density condition (cf. 5.15 ii)), corollary 5.14 yields the existence of a continuous norm $\|\,.\,\|$ on E, such that the topologies induced on $K := K_1 + K_2$ by $\|\,.\,\|$ and E coincide. Without loss of generality $2K = 2(K_1 + K_2)$ is contained in the closed unit ball $B_{\|\cdot\|}$. Using lemma 6.9 we get an increasing mapping $\varepsilon : [0, 1] \to [0, 1]$ with $\lim_{x\to 0} \varepsilon(x) = 0$ such that for every $x \in [0, 1]$

$$K \cap xB_{\|\cdot\|} \subset (2K_1 \cap \varepsilon(x)B_{\|\cdot\|}) + (2K_2 \cap \varepsilon(x)B_{\|\cdot\|}).$$

Let $(z_n)_{n\in N} \in \lambda^{(\delta)}(K) \cap \lambda(E)$ be arbitrary. We define for the reason of shortness for $n \in N$:

$$\delta_n := \begin{cases} \dfrac{p_K(z_n)}{p_{B_{\|\cdot\|}}(z_n)} : & z_n \neq 0 \\ 1 : & z_n = 0. \end{cases}$$

Then it follows for $n \in N$:

$$\begin{aligned} z_n &\in p_K(z_n)K \cap p_{B_{\|\cdot\|}}(z_n)B_{\|\cdot\|} = p_K(z_n) \cdot (K \cap \delta_n^{-1}B_{\|\cdot\|}) \\ &\subset p_K(z_n)((2K_1 \cap \varepsilon(\delta_n^{-1})B_{\|\cdot\|}) + (2K_2 \cap \varepsilon(\delta_n^{-1})B_{\|\cdot\|})). \end{aligned}$$

Hence, for every $n \in N$ we have $z_n = u_n + v_n$, where

$$\begin{aligned} u_n &\in p_K(z_n)(2K_1 \cap \varepsilon(\delta_n^{-1})B_{\|\cdot\|}) \text{ and} \\ v_n &\in p_K(z_n)(2K_2 \cap \varepsilon(\delta_n^{-1})B_{\|\cdot\|})). \end{aligned}$$

From this we first obtain $p_{K_1}(u_n) \leq 2p_K(z_n), n \in N$ and therefore $(u_n)_{n\in N} \in 2\lambda^{(\delta)}(K_1)$, if we take into account that K_1 is bounded. Analogously we show that $(v_n)_{n\in N}$ is contained in $2\lambda^{(\delta)}(K_2)$.

It remains to show that $(u_n)_{n\in N}$ and $(v_n)_{n\in N}$ are contained in $\lambda(E)$. We will do this explicitly for $(u_n)_{n\in N}$, the other case can be proved analogously.

Let an absolutely convex and closed zero neighbourhood V in E be given. We may assume that $2K = 2(K_1 + K_2)$ is contained in V and that $B_{\|\cdot\|}$ absorbs V. Because $\|\cdot\|$ induces on K_1 the same topology as E there exists an increasing mapping $\varphi : [0, 1] \to [0, 1]$ with $\lim_{x\to 0} \varphi(x) = 0$ such that for every $x \in [0, 1]$

$$2K_1 \cap xB_{\|\cdot\|} \subset \varphi(x)V.$$

We define $\gamma := \varphi \circ \varepsilon$ and we get for every $n \in N$:

$$\begin{aligned} u_n &\in p_K(z_n)(2K_1 \cap \varepsilon(\delta_n^{-1})B_{\|\cdot\|}) \\ &\subset p_K(z_n)(2K_1 \cap \gamma(\delta_n^{-1})V) \\ &\subset p_K(z_n)\gamma(\delta_n^{-1})V. \end{aligned}$$

Hence, by definition of the δ_n, $p_V(u_n) \leq p_{B_{\|\cdot\|}}(z_n)\delta_n\gamma(\delta_n^{-1})$. Taking into account that $(p_{B_{\|\cdot\|}}(z_n))_{n\in N} \in \lambda$ and $(p_{B_{\|\cdot\|}}(z_n)\delta_n)_{n\in N} = (p_K(z_n))_{n\in N} \in \lambda^{(\delta)}$ we may apply lemma 6.8 to get $(p_V(u_n))_{n\in N} \in \lambda$. Because V is arbitrary we have $(u_n)_{n\in N} \in \lambda(E)$ which completes the proof.

Theorem 6.11. *Let* $(A, \|\ .\|_\lambda)$ *be a normal Banach sequence space satisfying property* γ) *and let E be a DFM space. Then* $\lambda(E)$ ***is** bornological (and so a complete* ***LB*** *space).*

Proof. Let $(K_n)_{n\in N}$ bc a funclaiiiental sequeiice of closed, absolutely convex and bo-uiided sets in ***E***. Lct $(\alpha_n)_{n\in N}$ E $(0,\infty)^N$ be arbitrary. We have to show that $V := \bigcup_{m\in N}\sum_{n=1}^{m}\alpha_n(\lambda^{(\delta)}(K_n)\cap\lambda(E))$ is a zero neighbourhood iii $\lambda(E)$. Siiice E is a DFM space, the sets K_n are compact, so theorem 3.3 iv) implies that $C_m := \lambda^{(\delta)}(\sum_{n=1}^{m}\frac{1}{2^n+1}\alpha_n K_n)\cap\lambda(E)$ is closed iii $\lambda(E)$ for each m E N. Moreover, $(C_m)_{m\in N}$ is increasing aiid every bounded set is absoi-bed by soiiie C_m, see 4.12. So, we inay apply **A.** Grotheiidieck [18], p. 72, lemme 4, aiid get that $U := \bigcup_{m\in N} C_m$ is a zero iieighbourhood iii the quasibarrelled DF space $\lambda(E)$. Proposition 6.10 implies U C V aiid **we** are doiie.

Remark 6.12. *We* ***note*** *that one may replace* ***in the*** *previous theorem "property* γ)" *by "property* γ_w)", *see* ***2.9 i),ii).***

c_0 satisfies property ε) and therefore **property** y) and we obtain:

Corollary 6.13. *(S. Dierolf, unpublished) Let E be a DFM space. Then* $c_0(E)$ *is bornological.*

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Received October 31, 1994

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