SMALL LARGE SUBGROUPS OF A TOPOLOGICAL GROUP

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Abstract. *This paper presents the following general structure theorem for topological groups. Let G be a topological group of density κ, for any cardinal κ. Then each neighbourhood of the identity in G contains a subgroup of index less than or equal to $\kappa^{\aleph_0}$.*

1. INTRODUCTION

Recall that the *density* of a topological space is the smallest cardinal of a dense set in that space. Also recall that the *index* of a subgroup H of a group G is the cardinality of the set $\{xH : x \in G\}$ of left cosets of H in G (or, equivalently, of the set of right cosets).

The main result of this note is the following general structure theorem for topological groups.

Theorem 1. *Let G be a topological group of density κ. Then each neighbourhood of the identity in G contains a subgroup of index less than or equal to $\kappa^{\aleph_0}$.*

In the theory of topological groups one informally says that a topological group has 'small subgroups' of a particular type if each neighbourhood of the identity contains a subgroup of that type. Similarly, we might say that a subgroup of a group is 'large', relative to some cardinal, if its index is bounded above by that cardinal. We may therefore loosely paraphrase our result, in the words of the title, by saying that topological groups contain small large subgroups.

The *weight* of a topological space is the smallest cardinal of an open basis for the space. Clearly the density of a space is dominated by the weight, ant it is perhaps worth noting that our theorem therefore holds with 'density' replaced by 'weight'. A further special case is worth noting: a topological group G of density (or weight) κ which has no small subgroups must have cardinality at most $\kappa^{\aleph_0}$. This bound holds in particular for any Banach space G; in fact, by Sections 8 and 10 of [4], the cardinality of G is precisely $\kappa^{\aleph_0}$.

2. PROOF OF THE THEOREM

We prepare the proof through the sequence of lemmas below, and we note that the argument proceeds by entirely elementary means. We begin by recalling some basics of the theory of topological groups. For our purposes, a *neighbourhood* of a point in a topological space is any set containing an open subset containing the point. With application to topological groups in mind, BOURBAKI shows in [1], §1, N^0 2, p.14ff., how to characterise a topology on a set X in terms of the neighbourhood filters of its points as follows.

Lemma 2. *Assume that $x \mapsto \mathfrak{U}(x)$ is a function which assigns to each point of a set X a set of subsets of X satisfying the following conditions.*

(V_I) *If $U \in \mathfrak{U}(x)$ and $U \subseteq V$, then $V \in \mathfrak{U}(x)$.*

(V_{II}) *A finite intersection of members of $\mathfrak{U}(x)$ is in $\mathfrak{U}(x)$.*

(V_{III}) *If* $V \in \mathfrak{U}(x)$ *then* $x \in V$.

(V_{IV}) *For each* $V \in \mathfrak{U}(x)$ *there is a* $W \in \mathfrak{U}(x)$ *such that for each* $w \in W$ *one has* $V \in \mathfrak{U}(w)$.

Then $\mathfrak{O} = \{U \subset X : (\forall u \in U) U \in \mathfrak{U}\}$ *is the unique topology on* X *such that* $\mathfrak{U}(x)$ *is the set of all neighbourhoods of* x *respect to* $\mathfrak{O}$, *for all* $x \in X$.

With the aim of characterising a group topology $\mathfrak{O}$ on a group G in terms of a base $\mathfrak{B}$ of the filter $\mathfrak{U}$ of 'potential neighbourhoods' of the identity of G, BOURBAKI [2], §1, N^0 2, p. 12ff,. establishes the following facts.

Lemma 3. *Let* G *be a group, and* $\mathfrak{B}$ *a filter base satisfying the following conditions.*

(GV_I) *For each* $U \in \mathfrak{B}$ *there is a* $V \in \mathfrak{B}$ *with* $VV \subset U$.

(GV_{II}) *For each* $U \in \mathfrak{B}$ *there is a* $V \in \mathfrak{B}$ *with* $V^{-1} \subset U$.

Then $\mathfrak{O} = \{U \subseteq G : (\forall u \in U)(\exists V \in \mathfrak{B}) uV \subseteq U\}$ *is the unique topology on* G *such that*

(i) *for* $g \in G$ *a set* U *is a neighbourhood of* g *if and only if there is a* $V \in \mathfrak{B}$ *such that* $gV \subseteq U$,

(ii) *all translations* $x \mapsto gx : G \to G$ *are homeomorphisms, and*

(iii) *the function* $x \mapsto x^{-1} : G \to G$ *is continuous at 1.*

The main part of the proof is a direct verification that the conditions (V_I) – (V_{IV}) above hold when $\mathfrak{U}(g)$ is defined to be $\{U : (\exists V \in \mathfrak{B}) gV \subseteq U\}$, for each $g \in G$; properties (i) and (ii) secure uniqueness, and property (iii) follows at once from condition (GV_{II}).

We remark that $\mathfrak{O}$ is a group topology if in addition to (GV_I) and (GV_{II}) the following is satisfied:

$$(GV_{III}) \qquad (\forall g \in G)(\forall U \in \mathfrak{B})(\exists V \in \mathfrak{B}) \quad V \subseteq gUg^{-1}.$$

This condition is trivially satisfied if G is abelian. In fact, it suffices that

$$(SIN) \qquad (\forall U \in \mathfrak{B})(\forall g \in G) \quad U = gUg^{-1}.$$

In this case $(G, \mathfrak{O})$ is a *SIN-group*, or is *locally invariant* [5]; that is, a topological group with arbitrarily small identity neighbourhoods which are invariant under all inner automorphisms. (Alternatively, the group has equivalent left and right uniformities).

If H is a subgroup of a group G, we denote by G/H the set of all cosets $\xi = xH$, and we define the quotient map $q : G \to G/H$ by $q(x) = xH$. The natural transitive action $(g, x) \mapsto gx$ of G on G by left translation permutes the cosets xH and thus defines a natural transitive action $(g, \xi) \mapsto g \cdot \xi : G \times G/H \to G/H$ unambiguously via $g \cdot \xi = gxH$ for $\xi = xH$. Since $q(gx) = gxH = g \cdot (xH) = g \cdot q(x)$, the quotient map $q : G \to G/H$ is equivariant for the transitive actions of G on G and G/H, respectively.

Lemma 4. *Suppose that* G *and* $\mathfrak{B}$ *are as in Lemma 3. Then we have the following.*

(i) *The set* $H = \bigcap \mathfrak{B}$ *is a closed subgroup of* G *with respect to the topology* $\mathfrak{O}$.

(ii) *Every* $\mathfrak{O}$*-open set* U *satisfies* $UH = U$.

(iii) *In particular,* $q(\mathfrak{O}) = \{q(U) : U \in \mathfrak{O}\}$ *is the quotient topology on* G/H, *and* q *is a continuous open map.*

(iv) *The quotient topology on the coset space* $G/H = \{gH : g \in G\}$ *is Hausdorff.*

(v) *The natural transitive action of* G *on* G/H *gives homeomorphisms* $\xi \mapsto g \cdot \xi : G/H \to G/H$.

Proof. It is easy to see that H is a subgroup, proving part of (i). Let $U \in \mathfrak{O}$ and $u \in U$. Then there is a $V \in \mathfrak{B}$ such that $uV \subseteq U$. Hence $uH \subseteq U$, and so $UH = U$, proving (ii). From the fact that all $U \in \mathfrak{D}$ are stable under multiplication by H on the right, it is clear that the quotient topology on G/H is $q(\mathfrak{D}) = \{q(U) : U \in \mathfrak{D}\}$. Obviously, q is continuous and open under these circumstances, and we have (iii). Also, for $U \in \mathfrak{D}$ we have $g \cdot q(U) = q(gU) \in \mathfrak{D}$ and $g^{-1} \cdot q(U) = q(g^{-1}U)$. Thus the map $\xi \mapsto g \cdot \xi : G/H \to G/H$ is a $q(\mathfrak{D})$-homeomorphism for all g, which proves (v).

It remains to show that G/H is Hausdorff and that H is closed. But if G/H is Hausdorff, then in particular the singleton set $\{H\}$ is closed in G/H, whence $H = q^{-1}(\{H\})$ is closed in G, completing the proof of (i).

Finally, assume that $g_1H \neq g_2H$ in G/H. This is equivalent to $g_2^{-1}g_1 \notin H$, and so there is a $V \in \mathfrak{B}$ such that $g_2^{-1}g_1 \notin V$. Using (GV_I) and (GV_{II}), we find an $A \in \mathfrak{B}$ such that $AA^{-1} \subseteq V$. Then $g_2^{-1}g_1 \notin AA^{-1}$, and this implies that $g_1A \cap g_2A = \emptyset$. By Lemma 3, the set A is an $\mathfrak{D}$-neighbourhood of 1, and thus there is an $\mathfrak{D}$-open neighbourhood W of 1 such that $W \subseteq A$. Therefore $g_1W \cap g_2W = \emptyset$. But $q(g_1W)$ and $q(g_2W)$ are disjoint open neighbourhoods of g_1H and g_2H in G/H, respectively, and so G/H is Hausdorff. Thus (iv) holds, completing the proof.

Lemma 5. *If a topological space G of density κ satisfies the first axiom of countability and is Hausdorff, then card $G \leq \kappa^{\aleph_0}$.*

Proof. Let D be a dense subset of cardinality κ. Now for any $x \in G$, there is a sequence (d_n) of elements of D converging to x, since the neighbourhood filter of x has a countable basis. Let $C \subseteq D^{\mathbb{N}}$ denote the set of all sequences of D having a limit in G. Since G is Hausdorff, $(x_n)_{n\in\mathbb{N}} \mapsto \lim_{n\in\mathbb{N}} x_n : C \to G$ is a well-defined surjection. Hence card $G \leq$ card $C \leq$ card $D^{\mathbb{N}} = \kappa^{\aleph_0}$.

Note that Hausdorffness is essential here: if a set X with at least 2 points is equipped with the indiscrete topology then the density κ of X is 1, and so $\kappa^{\aleph_0} = 1 < 2 \leq$ card X.

We are now ready for the proof of Theorem 1. Assume that $(G, \mathfrak{D}_0)$ is a topological group and W an identity neighbourhood. Inductively define $\mathfrak{D}_0$-idendity neighbourhoods $U_0 = W, U_1, U_2, \ldots$ satisfying $U_n = U_n^{-1}$ and $U_nU_n \subseteq U_{n-1}, n = 1, 2 \ldots.$ Then $\mathfrak{B} = \{U_n : n = 0, 1, \ldots\}$ satisfies (GV_I) and (GV_{II}), and by Lemma 3 we have a unique topology $\mathfrak{D}$ on G such that $\mathfrak{B}$ is a basis for the neighbourhoods of the identity and all translations $x \mapsto gx : G \to G$ are homeomorphism. Since each U_n is a neighbourhood of 1 with respect to $\mathfrak{D}_0$, we have $\mathfrak{D} \subseteq \mathfrak{D}_0$. (Notice that we do not assert that the sets U_n are open in $\mathfrak{D}$, even if they are taken to be open in $\mathfrak{D}_0$; Lemma 4 shows that U_n cannot be open in $\mathfrak{D}$ unless $U_nH = U_n$). By Lemma 4, the intersection $H = \bigcap_{n=0}^{\infty} U_n$ is an $\mathfrak{D}$-closed subgroup, and because $\mathfrak{D} \subseteq \mathfrak{D}_0$, H is also $\mathfrak{D}_0$-closed. Obviously, H is contained in $U_0 = W$, and we shall show that H satisfies the requirements.

Let $q : G \to G/H$ denote the quotient map. Then by Lemma 4, the image $q(\mathfrak{D})$ is the quotient topology and $q : (G, \mathfrak{D}) \to (G/H, q(\mathfrak{D}))$ is a continuous open map onto a homogeneous Hausdorff space. Since $\mathfrak{B}$ is a basis for the neighbourhood filter of 1 in $(G, \mathfrak{D})$, the set $q(\mathfrak{B})$ is a basis of the neighbourhood filter of H in G/H. Hence $(G/H, \mathfrak{D})$ satisfies the first axiom of countability, by homogeneity. If D is dense in $(G, \mathfrak{D}_0)$ then $DH/H = q(D)$ is dense in $(G/H, q(\mathfrak{D}_0) \to (G/H, q(\mathfrak{D}))$ is continuous. Hence the density of G/H is less than or equal to κ, the density of G. Therefore, by Lemma 5, card $G/H \leq \kappa^{\aleph_0}$, as required.

3. REMARKS

A slight modification of the argument above shows that, in a SIN-group G of density κ, every identity neighbourhood contains a closed normal subgroup whose index is dominated by $\kappa^{\aleph_0}$. To see this, note that we can choose, in the proof of Lemma 5, identity neighbourhoods U_n, for $n = 1, 2, \ldots$, in such a fashion that, in addition to the conditions stated there, each U_n is invariant under all inner automorphisms. Accordingly, the group $H = \bigcap_{n=0}^{\infty} U_n$ is normal. (Also, as noted following Lemma 3, $\mathfrak{D}$ is a group topology on G with $\{U_n : n \in \mathbb{N}\}$ as a basis for the filter of identity neighbourhoods, and so G / H is a Hausdorff topological group).

We now present an example, for any infinite cardinal κ, of a Hausdorff topological group of density κ and cardinality 2^κ which has a neighbourhood of the identity containing no non-trivial normal subgroups.

The group $S(\kappa)$ of all permutations with finite support on the set κ acts on the product $\mathbb{R}^\kappa$ through permutations of the index set. (Explicitly, for $\sigma \in S(\kappa)$ and $x \in \mathbb{R}^\kappa$, we define $\sigma \cdot x$ by setting $(\sigma \cdot x)(\alpha) = x(\sigma(\alpha))$; for each $\alpha < \kappa$). With the product topology on $\mathbb{R}^\kappa$ and the discrete topology on $S(\kappa)$, this action is clearly continuous. Now let G be the semidirect product $\mathbb{R}^\kappa \times S(\kappa)$ with the product topology. Clearly G is a topological group, and it is easy to see that G has density (and indeed weight) κ and cardinality 2^κ.

Denote by V the open neighbourhood $V = \{x \in^\kappa : |x(0)| < 1\}$ of 0 in $\mathbb{R}^\kappa$. Then the set $W = V \times \{e\}$ is an open neighbourhood of the identity $(0, e)$ in G, and we claim that if H is a normal subgroup of W, then $H = \{0\}$. In fact, if H is normal, it must be contained in the maximal invariant subset $\bigcap\{\sigma(W) : \sigma \in S(\kappa)\}$ of W, which coincides in turn with

$$\{x \in \mathbb{R}^\kappa : |x(\alpha)| < \text{ for all } \alpha < \kappa\} \times \{e\}.$$

But the latter set, being the unit ball in the Banach space $l_\infty(\kappa)$ (which is embedded naturally as a linear subspace of $\mathbb{R}^\kappa$), has no small subgroups. Therefore, H is trivial.

We feel that a self-contained and entirely elementary proof of Theorem 1, such as we have presented, is of interest. We note, however, that a somewhat shorter argument is possible, at the cost of using more sophisticated machinery. Briefly, given an arbitrary neighbourhood U of 1 in G, there exists, by Theorem 8.2 of [3] for example, a left-invariant pseudometric ρ on G such that the unit sphere centred at 1 with respect to ρ is contained in U. The set of elements of distance 0 from 1 forms a closed subgroup H of G, and the homogeneous factor-space G / H supports a metric d, defined by $d(xH, yH) = \rho(x, y)$. (The well-definedness of d follows from the left-invariance of ρ). The argument may now be concluded in much the same way as that of §2.

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