INFINITE GROUPS SATISFYING A NORMALIZER CONDITION

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Summary. *In this article infinite groups G are studied with the property that if H is a non-normal subgroup of G then every normal subgroup of H is normal in the normalizer $N_G(H)$.*

1. INTRODUCTION

A subgroup H of a group G is said to satisfy the *lower N / C-extremal condition* if every normal subgroup of H is also normal in the normalizer $N_G(H)$ of H. It is clear that a group G is a $\bar{T}$-group (i.e. a group in which normality is a transitive relation) if and only if all its normal subgroups satisfy the lower N / C-extremal condition. In particular, G is a $\bar{T}$-group (i.e. a group in which all subgroups are T-groups) if and only if every subgroup of G satisfies the lower N / C-extremal condition.

Let X be the class of groups in which every non-normal subgroup is N / C-low. The investigation of the structure of X-groups was started in [3] and [4]; the results obtained there mostly concern the case of finite groups. In particular, it was proved that every finite X-group is soluble with derived length at most 3. On the other hand, the consideration of Tarski groups shows that arbitrary X-group need not be soluble. Here we shall consider infinite soluble X-groups, and in particular we shall prove that soluble X-groups have derived lenght at most 4. A well-known result of Robinson [6] states that a finitely generated soluble T-group either is finite or abelian. The situation is completely different in the case of soluble X-groups: the direct product $Z \times S_3$ is an infinite finitely generated soluble X-group. On the other hand, we shall prove that the elements of finite order of a soluble X-group form a subgroup, and that the torsion-free soluble X-groups are abelian.

For our considerations it will be useful to observe that X is contained in the class B_2 of groups in which every subnormal subgroup has defect at most 2. The structure of B_2-groups (and more generally of groups in which subnormal subgroups have bounded defect) has been investigated by many authors. In particular, Casolo [1], [2] has proved that finite (respectively: periodic) soluble B_2-groups have derived lenght at most 5 (respectively: at most 10), while Mahdavianary [5] showed that nilpotent B_2-groups have class at most 3 (and so they are metabelian).

Most of our notation is standard and can for instance be found in [7].

2. STATEMENTS AND PROOFS

It is clear that subgroups and homomorphic images of X-groups are likewise X-groups. Our first lemma deals with centralizers of elements of infinite order of an X-group.

Lemma 2.1. *Let G be an X-group, and let x be an element of infinite order of G. Then $N_G(\langle x\rangle) = C_G(x)$*

Proof. Assume that G contains an element a such that $\langle x\rangle^a = \langle x\rangle$ but $xa \neq ax$. Then a acts as the inversion on x, and so $\langle a,x\rangle$ has a quotient isomorphic to the infinite dihedral group D_∞, a contradiction since D_∞ is not an X-group.

Lemma 2.2. *Let G be a torsion-free nilpotent X-group. Then G is abelian.*

Proof. Assume that G is not abelian, and let x be an element of G. Then the normalizer $N_G(\langle x\rangle)$ is subnormal in G, and so even normal, since G is an X-group. Then $C_G(x)$ is a normal subgroup of G by Lemma 2.1, and hence the identity $[y,x,x] = 1$ holds in G. It follows that G has class at most 2 (see [7], 7.14). Without loss of generality it can be assumed that $G = \langle a,b\rangle$, where $[a,b] \neq 1$. Let m,n be coprime integers >1. Since $[a^m,b^n] \neq 1$, it follows from Lemma 2.1 that b^n does not normalize $\langle a^m\rangle$. On the other hand,

$$\langle a^m\rangle \lhd \langle a^m,[a^m,b^n]\rangle \lhd \langle a^m,b^n\rangle,$$

and hence $\langle a^m,[a^m,b^n]\rangle$ is a normal subgroup of the X-group G. Similary $\langle b^n,[a^m,b^n]\rangle$ is normal in G, and so also $\langle a^m,b^n\rangle$ is a normal subgroup of G. Clearly the factor group $G/\langle a^m,b^n\rangle$ is abelian, so that $[a,b] = a^{mh}b^{nk}[a,b]^{mnl}$, where h,k,l, are integers. It follows that $a^{mh}b^{nk}$ belongs to the centre of G, and hence in particular $1 = [a^m,b^{nk}] = [a,b]^{mnk}$. Then $k = 0$, and similary $h = 0$, so that $[a,b] = [a,b]^{mnl}$. Therefore $[a,b] = 1$, and this contradiction proves the lemma.

Corollary 2.3. *Let G be a locally nilpotent X-group. Then the commutator subgroup G' is a periodic abelian group.*

Proof. Clearly it can be assumed that G is finitely generated, and so nilpotent. Let T be the subgroup of all elements of finite order of G. Then G/T is abelian by Lemma 2.2, so that $G' \leq T$, and G' is periodic. Moreover G' is abelian by the quoted result of Mahdavianary [5].

Lemma 2.4. *Let G be an X-group containing an abelian normal subgroup A such that G/A is finite cyclic. Then the commutator subgroup G' of G is periodic.*

Proof. Without loss of generality it can be assumed that G is finitely generated and has no periodic non-trivial normal subgroups. Then A is a free abelian group of finite rank. Let G be a counterexample with G/A of minimal order, so that in particular A is a maximal abelian normal subgroup of G. Let x be an element of G such that $G = \langle x,A\rangle$, and let p be a prime dividing the order of G/A. Then $\langle x^p,A\rangle$ is a proper subgroup of G, and so $\langle x^p,A\rangle'$ is periodic. Since $\langle x^p,A\rangle'$ is normal in G, it follows that $\langle x^p,A\rangle' = 1$. Then $\langle x^p,A\rangle$ is abelian, and hence $\langle x^p,A\rangle = A$. Therefore $x^p \in A$ and G/A has order p. For each positive integer n, the finite p-group G/A^{p^n} belongs to X, and so it is a nilpotent B_2-group. Then G/A^{p^n} has class at most 3 (see [5]), and so $\gamma_4(G) \leq \bigcap_{n\in N} A^{p^n} = 1$. Then G is a torsion-free nilpotent X-group, and Lemma 2.2 yields that G is abelian, a contradiction.

It is now possible to prove that the elements of finite order of a locally soluble X-group form a subgroup.

Proposition 2.5. *Let G be a locally soluble X-group. Then the set of all elements of finite order of G is a subgroup.*

Proof. Let x and y be elements of finite order of G. Without loss of generality it can be assumed that $G = \langle x,y\rangle$, so that in particular G is soluble. Let N be the smallest non-trivial term of the derived series of G. By induction on the derived length of G we obtain that G / N is finite, so that also N is finitely generated. Let a be an element of N, and consider the subgroups $H = \langle a\rangle^G\langle x\rangle$ and $K = \langle a\rangle^G\langle y\rangle$. Then H' and K' are periodic by Lemma 2.4, and there exists a positive integer m such that $[a,x]^m = [a,y]^m = 1$. It follows that $[a^m,x] = [a^m,y] = 1$, so that $a^m \in Z(G)$. Therefore $G / Z(G)$ is periodic, and hence finite, so that also G' is finite (see [7], 4.12). It follows that G is finite.

Lemma 2.6. *Let p be a prime, and let $\langle x\rangle$ be a cyclic p-group. If y is an automorphism of order p^n of $\langle x\rangle$ such that the semidirect product $G = \langle y\rangle \propto \langle x\rangle$ is an X-group, then $n \leq 1$.*

Proof. Let p^m be the order of x, and assume that $n \geq 2$. Then $x^y = x^{1+sp^t}$, where p does not divide s and $t \leq m-2$. Put $k = m-1-t$, and consider the non-normal subgroup $H = \langle x^{p^{m-1}}, y\rangle$ of G. Clearly $H = \langle x^{p^{m-1}}\rangle \times \langle y\rangle$ and x^{p^k} normalizes H, a contradiction, since G is an X-group and $[x^{p^k}, y] \neq 1$.

Lemma 2.7. *Let A be a reduced torsion-free abelian group, and let σ be a non-trivial automorphism of A. Then the semidirect product $G = \langle\sigma\rangle \propto A$ is not an X-group.*

Proof. Assume that G is an X-group, and let a be an element of A such that $a^\sigma \neq a$. Since $H = \langle\sigma\rangle\langle a\rangle^G$ is also an X-group, and $\langle a\rangle^H = \langle a\rangle^G$, it can be assumed without loss of generality that $A = \langle a\rangle^G$. The automorphism σ has infinite order by Lemma 2.4. For every interger i put $a_i = a^{\sigma^i}$, so that $A = \langle a_i | i \in kZ\rangle$. Let k be a positive integer, and assume that $A_k = \langle a_i | i \in kZ\rangle$ is properly contained in A. As $a = a_0 \in A_k$, the subgroup A_k is not normal in G, and so $\langle a\rangle$ is normal in $N_G(A_k)$. Clearly σ^k fixes a by Lemma 2.1. Then $\sigma^k = 1$, a contradiction. Therefore $A = A_k$ for every $k \geq 1$. In particular, the set $\{a_i | i \in Z\}$ is dependent, and there exist integers r and s, with $r < s$ such that $\{a_r, \dots, a_s\}$ is independent and $\{a_r, \dots a_{s+1}\}$ is dependent. Thus $a_{s+1}^m = a_r^{m_r} \dots a_s^{m_s}$, where $m, m_r, \dots, m_s$ are integer and $m \neq 0$. Let D be the divisible hull of A, and let D_0, be the smallest divisible subgroup of D containing $\langle a_r, \dots, a_s\rangle$. Then σ can be extended to an automorphism τ of D. Since $a_{s+1} \in D_0$, we obtain $\langle a_r, \dots, a_s\rangle^\tau \leq D_0$. Moreover, D_o has the same rank of $\langle a_r, \dots, a_s\rangle$, so that $D_0 / \langle a_r, \dots, a_s\rangle$ is periodic and $D_0^\tau \leq D_0$, since D / D_0 is torsion-free. It follows that $D_0^\tau = D_0$, so that $A_0 = A \cap D_0$ is a subgroup of finite rank of A containing $\langle a_r, \dots, a_s\rangle$, and $A_0^\sigma = A_0$. Clearly $a = a_r^{\sigma^{-r}} \in A_0$, so that $A = A_0$ and A has finite rank. Thus the counterexample G can be chosen in such a way that A has minimal rank. As A is reduced, there exists a prime p such that $A^p \neq A$. Let k be the order of the automorphism induced by σ on the finite group A / A^p. If $i \in kZ$, we obtain that $a_iA^p = aA^p$. Since $A = A_k = \langle a_i | i \in kZ\rangle$, it follows that A / A^p is cyclic. Then A / A^{p^n} is cyclic of order p^n for every $n \geq 0$. Application of Lemma 2.6 yields that the automorphism induced by σ on A / A^{p^n} has order dividing $p(p-1)$. Therefore $\sigma^{p(p-1)}$ acts trivially on A / A^{p^n} for each $n \geq 0$. Put $B = \bigcap_{n\geq 0} A^{p^n}$, so that $[A, \sigma^{p(p-1)}] \leq B$. Cleraly $[A, \sigma^{p(p-1)}] \neq 1$, and hence it follows from Lemma 2.2 that $\sigma^{p(p-1)}$ does not act trivially on B. Moreover, A / B does not have finite exponent, so that $\langle a\rangle \cap B = 1$, and B has rank less than A. By the minimal choise of A, we obtain that the subgroup $\langle\sigma^{p(p-1)}, B\rangle$ does not belong to X. This contradiction proves the lemma.

We can now prove our main result.

Theorem 2.8. *Let G be a torsion-free locally soluble X-group. Then G is abelian.*

Proof. Clearly it can be assumed that G is finitely generated, and hence soluble. Thus by induction the derived lenght of G we may also suppose that the commutator subgroup G' is abelian. As a finitely generated metabelian group, it is well-know that G is residually finite (see [8], 9.51) and so in particular reduced. Assume that G is not abelian, so that G is not nilpotent by Lemma 2.2, and $C = C_G(G')$ is properly contained in G. Let x be an element of G' and y an element of $G\backslash C$ such that $[x,y] \neq 1$, and consider the subgroup $H = \langle x,y\rangle$. Clearly $\langle x\rangle^H$ is contained in G', and so is abelian. If $\langle x\rangle^H \cap \langle y\rangle \neq 1$, then $H/\langle x\rangle^H$ is a finite cyclic group, and H is abelian by Lemma 2.4. This contradiction shows that $\langle x\rangle^H \cap \langle y\rangle = 1$, and hence Lemma 2.7 can be applied to prove that the factor group $H/C_{\langle y\rangle}(\langle x\rangle^H)$ does not belong to X. This last contradiction completes the proof.

The above theorem has the following consequence.

Corollary 2.9. *Let G be a locally soluble X-group. Then the commutator subgroup G' of G is periodic, and G is soluble with derived length at most 4.*

Proof. The set T of all elements of finite order of G is a subgroup by Proposition 2.5, and it follows from Theorem 2.8 that the factor group G/T is abelian. Thus G' is periodic, and hence locally finite. Application of Theorem 3.4 of [3] yields now that $G^{(4)} = 1$.

We leave as an open question whether there exist soluble X-groups with derived lenght 4.

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