EXTENDING NORMS ON GROUPS

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Following Farkas, by a norm n on a group G I shall understand a function n on G to the set of non-negative real numbers, satisfying

1. $g \in G : n(g) = 0 \iff g = 1$,
2. $\forall g \in G : n(g^{-1}) = n(g)$, and
3. $\forall g, h \in G : n(gh) \leq n(g) + n(h)$.

I call n a seminorm, if only $n(1) = 0$ instead of the first condition.

Beginning with some general remarks on normed groups, I will discuss the possibility of extending norms from a normal subgroup to the whole group. A sufficient condition for the existence of such an extension is found, which demands that the inner automorphisms of the whole group do not move the elements of the subgroup too far and that there exists a factor set of the given extension of groups, which is bounded on one side.

So I get a counterpart to the necessary condition, which was found by Farkas in [3].

1. GENERAL PROPERTIES OF NORMED GROUPS

The first question one should ask is: Does a norm on a group define a unique topology on that group, and if so, do we get a topological group? The following theorem sums up the answer.

Theorem 1.1. *Let G be a group, equipped with a topology. Then the following statements are equivalent:*

1. G is a topological group, whose topology is Hausdorff and admits a countable basis for the system of neighborhoods of the identity.

2. G is a topological group, whose topology can be defined by a right-invariant metric.

3. There is a norm n on G, such that the topology on G is generated both by the right-invariant metric

$$d_r : G \times G \to [0, \infty[, (g,h) \mapsto n(gh^{-1})$$

any by the left-invariant metric

$$d_l : G \times G \to [0, \infty[, (g,h) \mapsto n(g^{-1}h).$$

Proof. The equivalence of the first and second statement is given by the Birkhoff-Kakutani theorem, see e.g. [1]. If we suppose these statements to be true, d shall denote the right-invariant metric, it is easy to see, that $n : G \to [0, \infty[, g \mapsto d(1, g)$ defines a norm on G. For this norm we get by right-invariance of our metric $d = d_r$. A simple calculation also proofs d_l to be a left-invariant metric, whenever n is a norm. So we have to show, that d_l gives rise to the same topology as d_r.

Let $\epsilon > 0$. Since we assume G to be a topological group by the topology of d, there is a $\delta > 0$, such that

$$\forall g, h \in G : d(g^{-1}, h^{-1}) < \delta \quad \Rightarrow \quad d(g,h) < \epsilon.$$

Formulating this in terms of the metrics d_r and d_l we get

$$\forall g,h \in G : d_l(g,h) < \delta \quad \Rightarrow \quad d_r(g,h) < \epsilon.$$

A similar conclusion gives the analogue result with d_r and d_l interchanged, which then proves the topologies defined by d_r and d_l to be equivalent.

The last thing to be proved is that the third statement implies the first two. As we have already seen for d_l and may easily be seen for d_r too, these functions are metrics on G, left-invariant and right-invariant respectively, whenever n is a norm. Since these metrics generate the same topology on G, we get

$$\forall \epsilon > 0, g \in G : \exists \delta_1(g) > 0 : \forall h \in G : n(h^{-1}g) < \delta_1(g) \Rightarrow n(gh^{-1}) < \epsilon.$$

and hereby $\forall \epsilon 0, g,h \in G : \exists \delta > 0 : \forall g', h', \in G$:

$$\max\{n(hh'^{-1}), n(g'^{-1}g)\} < \delta \Rightarrow n(gh(g'h')^{-1}) < \epsilon,$$

since for $\delta = \delta_1(g)/2$ we have $n((hh'^{-1}g'^{-1})g) < \delta_1(g)$ and so $n(g(hh'^{-1}g'^{-1})) = n(gh(g'h')^{-1}) < \epsilon$. This last relation implies the continuity of the product on G, since it is by assumption unimportant, which of the metrics we use.

Finally we get the continuity of the operation "taking inverses" by $d_r(g,h) = d_l(g^{-1}, h^{-1})$, which holds for all $g,h \in G$. q.e.d.

Due to the symmetry of the statements, even more is true. First, we can interchange "left" and "right" in all statements and, second, we may not claim the topologies in the third statement to be equal, since they will be equal, if they are only comparable.

The theorem also shows, that the theories of norms on groups and of right-invariant (or left-invariant) metrics on groups are equivalent, since there are canonical bijective mappings between these structures on a given group.

In abelian groups the definitions of the left-invariant and the right-invariant metric coincide of course, so any normed abelian group is a topological group.

Any group G can be endowed with the trivial norm, assigning 0 to the identity element and 1 to all other elements. This trivial norm is designated δ and gives an embedding of abstract groups into the class of normed groups.

The best known examples for norms on groups are the wordnorms, which are defined here together with "generalized wordnorms", a tool for the construction of norms.

Definition 1.1. *Let G be a group. For a generating set E of elements of G and a map $k : E \to [0,\infty[$, we call the map $|\cdot|_k : G \to [0,\infty[$, defined for $g \in G$ by*

$$|g|_k := \inf \left\{ \sum_{i=1}^{n} |a_i| k(x_i) \,|\, n \in \mathbb{N}, x_1, \ldots, x_n \in E, \alpha_1, \ldots, \alpha_n \in \mathbb{Z} \right.$$

$$\left. \textit{such that } \prod_{i=1}^{n} x_i^{\alpha_i} = g \right\},$$

the seminorm associated to the map k. If this seminorm is indeed a norm, we speak of the generalized wordnorm associated to the map k. If $k \equiv 1$, then $|\cdot|_k =: \ell_E$ is called the wordnorm (associated to the generator E).

It's easy to see that wordnorms are indeed norms and that seminorms associated to such a map are indeed seminorms. But in general, it's not simple to prove such a seminorm to be a norm or not.

Of course any norm can be constructed as a generalized wordnorm, if we take the generator E to be the whole group and the map k to be the norm.

There is a relation between norms, which allows to compare them and which gives rise to the well-known equivalence relation of quasiisometry.

Definition 1.2. *Let G be a group and $|\cdot|_1, |\cdot|_2$ seminorms on G. Then $|\cdot|_1$ is said to be coarser than $|\cdot|_2$ ($|\cdot|_2$ is then called finer than $|\cdot|_1$), iff there exists a real number $a > 0$ such that $|g|_1 \leq a|g|_2$ for all $g \in G$. In this case I write $|\cdot|_1 \preceq |\cdot|_2$.*

The seminorms are called quasiisometric, if the relation holds in both directions, which is written $|\cdot|_1 \approx |\cdot|_2$.

The relation "$\preceq$" is easy seen to be a quasiorder, so quasiisometry is an equivalence relation. Furthermore, any seminorm finer than a norm has to be a norm itself, so if one of two quasiisometric seminorms is a norm, the other is a norm too.

The following theorem gives a nice characterization of wordnorms on finitely generated groups.

Theorem 1.2. *Let G be a finitely generated group. Then all wordnorms related to finite generating sets of G are quasiisometric. These norms form a complete equivalence class of quasiisometric norms in the set of all wordnorms.*

Any norm on G is furthermore coarser than any element of this equivalence class.

Proof. This theorem is only another formulation of results from Farkas [3] and Gromov [4] q.e.d.

I finish this section with a simple lemma, giving a criteria for a seminorm associated to a map to be a norm.

Lemma 1.1. *Let G be a group, E a generating set of elements of G and $k : G \to [0, \infty[$ a map. If there is a subgroup H of G, such that $H \subseteq E$, $k|_H$ is a norm on H and $\inf_{e \in E \setminus H} k(e) > 0$, then the seminorm $|\cdot|_k$ is a norm.*

Proof. Let $g \in G$ be an element with $|g|_k = 0$. If the seminorm of g is approximated by sums over representations of g containing elements of $E \setminus H$, we would have $|g|_k \geq \inf_{e \in e \setminus H} k(e)$, contrary to our assumption. So we conclude $g \in H$ and $|g|_{k|_H} = 0$. But we already know $k|_H$ to be a norm, so by the triangle-inequality we get $k(g) = 0$ and by this $g = 1. q.e.d.$

This lemma will be used in the next section for extending norms. From the lemma we know the topology (defined by the right-invariant or left-invariant metric) of a normed subgroup to be extendable by the metric of a norm on the whole group, since one can take the defining map of a generalized wordnorm to be equal to the given norm on the subgroup and equal to 1 elsewhere. But this only leads to a norm on the whole group, which is less or equal than the given norm on the subgroup. And the topologies of left - and right - invariant metrics do in general not coincide on the whole group, even if they do so on the normal subgroup.

But continuing these thoughts leads to the result, that the extendability of a norm is a question on the class of norms quasiisometric to the given one. That is, if a norm is extendable, then all norms quasiisometric to it are also extendable.

2. A THEOREM ON EXTENDING NORMS ON NORMAL SUBGROUPS

Starting with some lemmas I'll now come to the result on the extendability of norms. The above sketched method for extending norm fails in general to give an extension because of "short cuts". These are small products (by the sum defining the generalized wordnorm) in the whole group which reach large elements of the subgroup (by the given norm on the subgroup). The idea of the proof is to reduce the number of those short cuts, which have to be investigated, in the case of a normal subgroup by showing, that other short cuts can be put together by those under investigation.

Lemma 2.1. *Let G be a group, $N \trianglelefteq G$ a normal subgroup and $|\cdot|$ a norm on N. Additionally, I suppose*

$$\forall g \in G: \quad \sup_{x\in N} \big||gxg^{-1}| - |x|\big| < \infty.$$

Then we have for all $n \in \mathbb{N}$, all $g_1, \dots, g_n \in G$, all $x_1, \dots, x_n \in N$ and all $s_1, \dots, s_n \in \mathbb{Z}$ such that

$$\prod_{i=1}^{n} g_i^{s_i} x_i \in N$$

the inequality

$$\left|\prod_{i=1}^{n} g_i^{s_i} x_i\right| \leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n-1} |s_i| \cdot \sup_{x\in N} \big||g_i x g_i^{-1}| - |x|\big| + \left|\prod_{i=1}^{n} g_i^{s_i}\right|.$$

Proof. First, we get from the triangle-inequality

$$\left|\prod_{i=1}^{n} g_i^{s_i} x_i\right| \leq \left|g_1^{s_1}\left(x_1 \cdot \prod_{i=2}^{n-1} g_i^{s_i} x_i \cdot \prod_{j=1}^{n-2} g_{n-j}^{-s_{n-j}}\right) g_1^{-s_1}\right| + \left|\prod_{i=1}^{n} g_i^{s_i}\right| + |x_n|.$$

The first term on the right side is by the triangle-inequality and by introducing a supremum

$$\begin{aligned}
&\leq \quad |s_1| \cdot \sup_{x\in N} \big||g_1 x g_1^{-1}| - |x|\big| + |x_1| + \left|g_2^{s_2}\left(x_2 \cdot \prod_{i=3}^{n-1} g_i^{s_i} x_i \cdot \prod_{j=1}^{n-3} g_{n-j}^{-s_{n-j}}\right) g_2^{-s_2}\right| \\
&\leq \quad \sum_{j=0}^{n-1} \left\{ |s_j| \cdot \sup_{x\in N} \Big|\big|g_j x g_j^{-1}\big| - |x|\Big| + |x_j| \right\}.
\end{aligned}$$

For the last inequality one has to notice that the last term in the first line has the same shape as the term we started with, so one can inductively repeat the first step to get the sum. Putting together these inequalities gives the assertion. q.e.d.

This lemma shows, as we shall see later, that for the extension of norms only those products have to be investigated, which only deal with elements of $G \backslash N$, if G is the whole group and N the normal subgroup. For this purpose it's useful to see G as a group extension of N by $H := G/N$. A fixed transversal of this extension is noted as $\bar{\cdot} : H \to G, x \mapsto \bar{x}$ and $\bar{H}$ notes the set of all elements $\bar{x}$, for $x \in H$. Similar conventions are used for the factor set, given both by a map $(.,.) : H \times H \to N$, $(x,y) \mapsto (x,y) = \bar{x}\bar{y}\overline{xy}^{-1}$ and by the set (H,H) of all elements of the form (x,y) for $x, y \in H$. For convenience I shall suppose $\bar{1} = 1$. Now the following lemma holds.

Lemma 2.2. *In the situation described above we have*

$$\langle \bar{H} \rangle \cap N = \langle (H,H) \rangle$$

and for $x \in \langle (H,H) \rangle$, having a representation

$$x = \prod_{i=1}^{n} \overline{u_i}^{\epsilon_i}$$

with $n \in \mathbb{N}$, $u_1, \ldots, u_n \in H$ and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$,

$$|x| \leq \delta_{1,-\epsilon_1} \cdot |(u_1^{-1}, u_1)| + \sum_{i=2}^{n} \sup_{v \in H} |(v, u_i)|.$$

Proof. By definition "$\subseteq$" is clear for the first assertion. The other inclusion will be proved together with the inequality by induction on the lenght n of the representation. For $n = 1$ we have $x \in N \cap (\bar{H} \cup \bar{H}^{-1}) = \{1\}$, so the claimed properties are trivial in this case.

Now suppose the assertion to be proved for $n-1, n > 1$. Let $x \in \langle \bar{H} \rangle \cap N$ with a representation of length n,

$$x = \prod_{i=1}^{n} \overline{u_i}^{\epsilon_i}.$$

Several cases may now occur.

1. Case $\epsilon_1 = \epsilon_2 = +1$.
Then

$$x = \underbrace{\overline{u_1}\,\overline{u_2}\,\overline{u_1 u_2}^{-1}}_{=(u_1,u_2)\in\langle (H,H)\rangle} \quad \underbrace{\overline{u_1 u_2}\,\overline{u_3}^{\epsilon_3} \ldots \overline{u_n}^{\epsilon_n}}_{\in\langle (H,H)\rangle \text{ by assumption}} \quad \in \langle (H,H) \rangle.$$

Since $|(u_1, u_2)| \leq \sup_{x \in H} |(x, u_2)|$ and for the remaining factors by assumption

$$|\overline{u_1 u_2}\,\overline{u_3}^{\epsilon_3} \ldots \overline{u_n}^{\epsilon_n}| \leq \sum_{i=3}^{n} \sup_{x \in H} |(x, \overline{u_i})|,$$

we get the asserted inequality.

2. Case $\epsilon_1 = +1$ and $\epsilon_2 = -1$.

Then

$$x = \underbrace{\overline{u_1u_2}^{-1}\overline{u_1u_2^{-1}}^{-1}}_{=(u_1u_2^{-1},u_2)^{-1}\in\langle(H,H)\rangle} \cdot \underbrace{\overline{u_1u_2^{-1}}\,\overline{u_3}^{\epsilon_3}\ldots\overline{u_n}^{\epsilon_n}}_{\in\langle(H,H)\rangle \text{by assumption}} \in \langle(H,H)\rangle.$$

Because of $|(u_1u_2^{-1},u_2)| \leq \sup_{x\in H}|(x,u_2)|$ we may procede like in the first case.

3. Case $\epsilon_1 = -1$.

Then

$$x = \underbrace{\overline{u_1}^{-1}\overline{u_1^{-1}}^{-1}}_{=(u_1^{-1},u_1)^{-1}\in\langle(H,H)\rangle} \cdot \underbrace{\overline{u_1^{-1}}\,\overline{u_2}^{\epsilon_2}\ldots\overline{u_n}^{\epsilon_n}}_{\in\langle(H,H)\rangle \text{ by the first or second case}} \in \langle(H,H)\rangle.$$

Because of $|(u_1^{-1},u_1)^{-1}| = |(u_1^{-1},u_1)|$ and the inequalities proved in the other two cases, we again get the asserted inequality. q.e.d.

The Kronecker-δ is only used for simplifying the proof of the lemma. Later I only use the inequality in its symmetric form

$$|x| \leq \sum_{i=1}^{n} \sup_{v\in H}|(v,u_i)|.$$

In particular this lemma proves $\ell_{\bar{H}}|_{\langle(H,H)\rangle} \preceq \ell_{(H,H)}$. Since the relation in the other direction is trivial by the definition of the factor set, we have indeed $\ell_{\bar{H}}|_{\langle(H,H)\rangle} \approx \ell_{(H,H)}$.

After these preparations I now come to the proof of a sufficient condition for the extendability of norms on normal subgroups.

Theorem 2.1. *Let G be a group, $N \trianglelefteq G$ a normal subgroup, $|\cdot|$ a norm on N and $H := G/N$. For the existence of an extension of $|\cdot|$ to a norm on G it is necessary, that for all $g \in G$*

$$\sup_{x\in N}\big||gxg^{-1}| - |x|\big| < \infty. \qquad \textit{(Farkas' condition)}$$

If there exists in addition to this condition a transversal $\bar{\cdot} : H \to G$, such that for the factor set $(.,.) : H \times H \to N$ defined by this transversal holds

$$\forall u \in H : \sup_{x\in H}|(x,u)| < \infty,$$

then there exists an extension of the norm.

Proof. Farkas' condition is by the triangle-inequality easy seen to be necessary. So we only have to show that all these conditions together are sufficient for the existence of an extension of the norm. Set $E := N \cup \bar{H}$ and

$$k : E \to \mathbb{R}_0^+, g \mapsto \begin{cases} |g| & \text{if } g \in N, \\ 1 + \sup_{x\in N}(|gxg^{-1}| - |x|) + \sup_{x\in H}|(x,g)| & \text{if } g \in E\backslash N \end{cases}.$$

By lemma 1.1 $|\cdot|_k$ is a norm on G and of course we have for $x \in N$ always $|x|_k \leq |x|$. The other inequality remains to be shown. Let $x \in N$ and

$$x = \prod_{i=1}^{n} g_i^{s_i} x_i$$

a representation of x with elements of E and the conventions of lemma 2.1, that is in particular $g_i \in E \backslash N$ for $i = 1, \dots, n$. This kind of representation can always be reached without enlarging the defining sum for the generalized wordnorm by multiplying together neighbouring elements in any other representation and filling in some identity elements.

Then one gets by lemma 2.1 and lemma 2.2

$$\begin{aligned} |x| &\leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n-1} |s_i| \cdot \sup_{x \in N} \left| |g_i x g_i^{-1}| - |x| \right| + \left| \prod_{i=1}^{n} g_i^{s_i} \right| < \\ &< \sum_{i=1}^{n} k(x_i) + \sum_{i=1}^{n} |s_i| \cdot k(g_i) + \left(\left| \prod_{i=1}^{n} g_i^{s_i} \right| - \sum_{i=1}^{n} |s_i| \cdot \sup_{x \in H} |(x, g_i)| \right) \leq \\ &\leq \sum_{i=1}^{n} k(x_i) + \sum_{i=1}^{n} |s_i| \cdot k(g_i). \end{aligned}$$

On the right side one recognizes the defining sum for the generalized wordnorm and so gets the assertion by transition to the infimum. q.e.d.

It's easy to see that the norm constructed above gives G the structure of a topological group, if N is a topological group and the automorphisms on N induced by G are continuos.

The following examples illustrate that the conditions in the theorem cannot be left out.

Example 2.1. *Consider the group $N = \mathbb{Z}^n$ as a normal subgroup of a group G. If a $g \in G$ induces by conjugation an automorphism of N, described ba a matrix $A \in Gl_n(\mathbb{Z})$, which does not permute the canonical basis and its inverses, then there is an element e_i of the canonical basis and an $a > 1$ such that $|Ae_i| = a|e_i|$, where $|\cdot|$ is the euclidean norm. So we get for any $k \in \mathbb{N}$*

$$|(ke_i)^g| = |A(ke_i)| = ka.$$

Subtracting $|ke_i|$ we see, that Farkas' condition is not satisfied, so no extension of the euclidean norm on N to a norm on G exists.

That's why Farkas condition cannot be left out. But there remains the question whether this condition is sufficient for the existence of extensions or not. The next example gives the answer.

Example 2.2. *Let*

$$H = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{pmatrix}$$

the discrete Heisenberg group. Consider the center of H,

$$C(H) = \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ & 1 & 0 \\ & & 1 \end{pmatrix} \cong \mathbb{Z}$$

and provide it with the usual absolut value as norm. Of course we now have for all $h \in H$

$$\sup_{z \in C(H)} \left| |z^h| - |z| \right| = 0,$$

so Farkas' condition is satisfied. Suppose there exists an extension of $|\cdot|$ to a norm on H and denote it also by $|\cdot|$. Farkas shows in [3, Theorem 7 etc.], that the map

$$\begin{aligned} N : H &\rightarrow \mathbb{R}_0^+ \\ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} &\mapsto \max\left\{ |x|, |z|, \sqrt{\left| y - \frac{1}{2}xz \right|} \right\} \end{aligned}$$

This is a contradiction to the supposed existence of an extension. So Farkas' condition alone is not sufficient for the extendability of a norm.

The final section of this paper is now dedicated to corollaries and conclusions from the last theorem and to results on products of normed groups.

3. COROLLARIES TO THE THEOREM AND SIMPLE CASES

I now take a closer look at those special cases, where the conditions of theorem 2.1 simplify significantly.

At first we start with splitting extensions of groups. In this case the factor set can be chosen to be identically 1, so our main theorem gives the equivalence of Farkas' condition and the extendability of a norm on the normal subgroup. That's why I now address a more far reaching question, that of extending simultaneously norms on N and H, both of which are subgroups of $G = N \times H$. One gets the following result.

Theorem 3.1. *Let H, N be groups, $\alpha \in \mathrm{Hom}(H, \mathrm{Aut}\,(N))$ and G the semidirect product of H and N defined by α. Furthermore let $|\cdot|_H$ and $|\cdot|_N$ be norms on H and N respectively. H and N are identified with their isomorphic images in G.*

For the existence of a norm $|\cdot|$ on G such that $|\cdot||_H = |\cdot|_H$ and $|\cdot||_N = |\cdot|_N$, it is necessary, that

$$\forall_x \in H : \sup_{n \in N} \left| |\alpha(x)(n)|_N - |n|_N \right| \leq 2 \cdot |x|_H.$$

On the other hand there is the sufficient condition

$$\forall x \in H : \sup_{n \in N} \left| |\alpha(x)(n)|_N - |n|_N \right| \leq |x|_H.$$

Proof. The necessity of the first condition is nothing else but the triangle inequality.

Now return to the proof theorem 2.1 to construct the extended norm by supposing the second condition. The transversal is in this case given by the identification of H with a subgroup of G, so the factor set is constantly 1.

So the generator of G defined there is given by $E = N \cup H$. I now give a new definition for the map k,

$$k : E \to R_0^+, x \mapsto \begin{cases} |x|_H & \text{if } x \in H, \\ |x|_N & \text{if } x \in N \backslash \{1\}. \end{cases}$$

By repeating the former arguments we get $|\cdot|_k$ to be an extension of $|\cdot|_N$, but not having a fixed distance to 0 on $H \backslash \{1\}$, we only know $|\cdot|_k$ to be a seminorm.

Let hx with $h \in H$ and $x \in N$ be any element of G. For any other representation of this element given by

$$hx = \prod_{i=1}^{n} h_i x_i \qquad \text{with } h_i \in H, x_i \in N \quad (1 \leq i \leq n)$$

we have

$$\prod_{i=1}^{n} h_i x_i = \left(\prod_{i=1}^{n} h_i\right) \left(\prod_{i=1}^{n} x_i^{h_{i+1} \dots h_n}\right),$$

so $h = \prod_{i=1}^{n} h_i$ and hereby

$$k(h) = |h|_H \leq \sum_{i=1}^{n} |h_i|_H = \sum_{i=1}^{n} k(h_i).$$

Setting $x = 1$ we get by this inequality $|h|_k = |h|_H$, so $|\cdot|_k$ also extends $|\cdot|_H$.

It remains to show $|\cdot|_k$ to be a norm. Let again x be any element of N and $|hx|_k = 0$. The inequalities above imply $k(h) = |h|_H = 0$, so $h = 1$. But then $|x|_k = |x|_N = 0$, so $x = 1$ too, which implies the assertion. q.e.d.

Another example of simple group extensions leads to interesting results, that are the cyclic extensions. This time we have a corollary to theorem 2.1.

Corollary 3.1. *Let G be a group and N a normal subgroup of G such that G / N is a cyclic group. Then any norm on N can be extended to a norm on G, if and only if it satisfies Farkas' condition.*

Proof. If $H = G / N$ is finite, then the supremum in the second condition of theorem 2.1 is that of a finite number of reals an so finite. So, applying this theorem, we get the extendability of the norm.

Now suppose $|G / N| = \infty$. Then, by chosing the group generated by an arbitrary element in the coset representing a generator of G / N as transversal, the extension is seen to split, and so the norm is extendable, again. q.e.d.

The last corollary invites one to try the tool of induction. But in general Farkas' condition is lost, when we ascend to a larger subgroup. However, in the case of abelian groups this method is successful.

Theorem 3.2. *Let G be an abelian group, H a subgroup of G and $|\cdot|$ a norm on H. Then $|\cdot|$ is extendable to a norm on G.*

Proof. If G/H is generated by the coset of $x \in G$ then by corollary 3.1 the norm is indeed extendable - Farkas' condition is satisfied for any abelian group. So we can always add another element of a generating set of G to the normed subgroup. Formally I apply the tool of transfinite induction.

Let E' be a subset of $G \backslash H$ such that $\langle H \cap E' \rangle = G$. Further let "$\leq$" be a well-ordering on E'. I suppose the existence of a compatible extension of the norm on $H_e := \langle H \cup \{e' \in E' | e' \leq e\} \rangle$ for all $e < x \in E'$. Compatible means that on any H_e an extension $|\cdot|_e$ is marked such that for all $f \leq e < x$ the norm $|\cdot|_e$ is an extension of $|\cdot|_f$ too.

Consider the group

$$\tilde{H} := \bigcup_{e<x} H_e.$$

Putting $n(h) := |h|_e$ for $h \in H_e \subseteq \tilde{H}$ and $e < x$, we get a norm n on $\tilde{H}$. So by our preparations we find an extension of n on

$$\left\langle \{x\} \cup \tilde{H} \right\rangle = H_x,$$

which is also a compatible extension of any $|\cdot|_e$ for $e < x$. q.e.d.

Finally I wish to address the extension problem for products of groups. The case of the semidirect product is already done, so I concentrate on those products with a possibly infinite number of factors.

Let $(g_i)_{i \in I}$ be a family of groups indexed by the set I and $|\cdot|_i$ a norm on G_i for any $i \in I$. Then there is the following table of conditions for the existence of a simultaneous extension of all $|\cdot|_i$ to the product of the groups.

Type of Product	Sufficient Condition	Necessary Condition
direct (restricted) product	always extendable	no condition
cartesian (unrestricted) product	either all groups are abelian or there is a finite set $I_0 \subseteq I$ such that $\sup_{i \in N_0} \sup_{x \in G_i} \lvert x \rvert_i < \infty$	no condition known, but there are examples of not extendable norms
amalgamated product along a subgroup $U \subseteq G_i$ for any $i \in I$	for all $i, j \in I$ we have $\lvert\cdot\rvert_i\lvert_U = \lvert\cdot\rvert_j\lvert_U$ and U is closed in G_i in the topology of the right-invariant metric or the left-invariant metric, both defined in theorem 1.1 for any $i \in I$	for all $i, j \in I$ we have $\lvert\cdot\rvert_i\lvert_U = \lvert\cdot\rvert_j\lvert_U$ and U is closed in G_i in the topology of the right-invariant metric or the left-invariant metric, both defined in theorem 1.1 for all but one $i \in I$
free product	always extendable	no condition

The - more or less - simple proof of these facts can be found in [2].

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