A NEW "WEIGHTED" FORM OF THE RIEMANN-VON MANGOLDT EXPLICIT FORMULA

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Abstract. *In this paper we show a new "weighted" form of the classical Riemann-von Mangoldt explicit formula obtained averaging with kernels that satisfy particular conditions; our results generalize the Kaczorowski-Perelli [K-[3](1)] new form of the formula.*

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this paper we extend the results given in [3] by Kaczorowski and Perelli obtaining a new "weighted" form of the Riemann-von Mangoldt explicit formula.

We indicate, as usual, with $\wedge(n)$ the von-Mangoldt function, namely

$$\wedge(n) = \begin{cases} \log p & \text{if } n = p^{\alpha}, \text{ for some } \alpha \in N \\ 0 & \text{otherwise} \end{cases}$$

and with $\psi_0(x,\chi)$ the sum

$$\psi_0(x,\chi) = \sum_{n \le x}{}' \wedge(n)\chi(n),$$

where the dash means that the term $\wedge(x)\chi(x)$, if present ($x \in N$), has to be halved (here χ is a Dirichlet character χ (mod q)).

The Riemann-von Mangoldt explicit formula is (see [2], ch. 19)

$$\psi_0(x,\chi) = \varepsilon(\chi)x - \sum_{|\gamma| \le T} \frac{x^{\rho}}{\rho} - \varepsilon_1(\chi)\log x - \varepsilon_2(\chi) + E(x,T,\chi),$$

where $x \ge 2, 2 \le T \le x, 1 \le q \le x, \chi = \chi$ (mod q) is a primitive Dirichlet characther, $\rho = \beta + i\gamma$ is a generic non-trivial zero of $L(s,\chi), \chi_0$ is the principal character (mod q),

$$\varepsilon(\chi) = \begin{cases} 1 & \text{for } \chi = \chi_0 \\ 0 & \text{otherwise,} \end{cases} \qquad \varepsilon_1(\chi) = \begin{cases} 1 & \text{for } \chi(-1) = 1, \chi \ne \chi_0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\varepsilon_2(\chi) = \lim_{s \to 0} \left(\frac{L'}{L}(s,\chi) - \frac{\varepsilon_1(\chi)}{s} \right)$$

and

$$E(x,T,\chi) \ll \frac{x \log^2 x}{T}.$$

In [3] Kaczorowski and Perelli have found a new form of the Riemann-von Mangoldt explicit formula (Theorem 1) using an approximate Perron's formula whose error term is estimated on average.

Here, with applications in mind, we obtain a generalization of this result (Lemma 1), which seems to be suitable for our purposes for its flexibility, due to the presence of a large class of weights at our disposal.

Setting $L = \log N$,

$$sgn(u) = \begin{cases} |u|/u & \text{for } u \neq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad w(u) = \begin{cases} 1 & \text{for } 0 \leq u \leq 1/2 \\ 2(1-u) & \text{for } 1/2 \leq u \leq 1, \end{cases}$$

$$G(x,T,n) = \frac{2}{T}\int_{T/2}^{T}\int_{\tau|\log\frac{x}{n}|}^{\infty}\frac{\sin u}{u}\,du\,d\tau,$$

the Corollary in [3] gives

$$\psi_0(x,\chi_0) = x - \sum_{|\gamma|\leq T} w\left(\frac{|\gamma|}{T}\right)\frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) + R(x,T,\chi_0),$$

$q = 1, 16 \leq N \leq x \leq 2N, \varepsilon > 0, N^\varepsilon \leq T \leq N^{1-\varepsilon}$ and for $1 \leq M \leq L^{-9}\min(N^{1/16}, T^{1/5})$

$$R(x,T,\chi_0) = \frac{1}{\pi}\sum_{x-\frac{MN}{T}<n\leq x+\frac{MN}{T}}\Lambda(n)sgn(x-n)G(x,T,n) + O_\varepsilon\left(\frac{N}{TM}\right).$$

We obtain (see the Corollary for the precise statement)

$$\psi_0(x,\chi_0) = x - \sum_{|\gamma|\leq T} w_Y\left(\frac{|\gamma|}{T}\right)\frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) + R_Y(x,T,\chi_0),$$

where now (here $q = 1$)

$$R_Y(x,T,\chi_0) = \frac{1}{\pi}\sum_{x-\frac{MN}{T}<n\leq x+\frac{MN}{T}}\Lambda(n)sgn(x-n)G_Y(x,T,n) + O_{Y,\varepsilon}\left(\frac{N}{TM^Y}\right),$$

with

$$w_Y(u) = \begin{cases} 1 & \text{for } 0 \leq u \leq 1/2 \\ \int_u^1 \phi_Y(Tv)dv \Big/ \int_{1/2}^1 \phi_Y(Tv)dv & \text{for } 1/2 \leq u \leq 1 \end{cases}$$

and

$$G_Y(x,T,n) = \frac{1}{\int_{T/2}^T \phi_Y(\tau)d\tau}\int_{T/2}^{T}\phi_Y(\tau)\int_{\tau|\log\frac{x}{n}|}^{\infty}\frac{\sin u}{u}\,du\,d\tau.$$

The new form of the explicit formula given by Kaczorowski and Perelli is obtained by setting $\phi_Y(\tau) = 1 (\forall\tau \in [T/2, T])$.

We point out that we obtain also an improvement for the error term in R_Y, which is due to the decay of the function ϕ_Y, see below.

As usual we let

$$N(\sigma, T, \chi) = \#\{\rho = \beta + i\gamma | L(\rho, \chi) = 0, \beta \geq \sigma, |\gamma| \leq T\}$$

and we assume the following "regularity conditions" on the test-functions ϕ_Y (true $\forall Y \in N$ fixed and for T "large"):

$$(i) \quad \phi_Y \in C^Y([T/2, T]),$$

$$(ii) \quad \left.\frac{d^r \phi_Y(\tau)}{d\tau^r}\right|_{\tau = \frac{T}{2}, T} = 0 \qquad \forall_r = 0, \ldots, Y-1,$$

$$(iii) \quad \frac{1}{\int_{T/2}^{T} \phi_Y(\tau) d\tau} \frac{d^r \phi_Y(\tau)}{d\tau^r} \ll_Y \frac{1}{T^{r+1}} \quad \forall \tau \in \left(\frac{T}{2}, T\right) \forall_r = 0, \ldots, Y.$$

Then our Theorem is as follows.

Theorem. *Let $16 \leq N \leq x \leq 2N$, $1 \leq q \leq N$, $\chi(\bmod q)$ be a primitive Dirichlet character and $4 \leq T \leq N/4$, $1 \leq M \leq T/4$. Then for any (fixed) positive integer Y*

$$\psi_0(x, \chi) = \varepsilon(\chi)x - \sum_{|\gamma| \leq T} w_Y\left(\frac{|\gamma|}{T}\right) \frac{x^\rho}{\rho} - \varepsilon_1(\chi) \log x - \varepsilon_2(\chi) + R_Y(x, T, \chi),$$

where

$$R_Y(x, T, \chi) = R_1(x, T, N, M, \chi) + R_2(x, T, N, M, \chi),$$

with R_1 defined as

$$R_1(x, T, N, M, \chi) = \frac{1}{\pi} \sum_{x - \frac{MN}{T} < n \leq x + \frac{MN}{T}} \Lambda(n)\chi(n) sgn(x - n) G_Y(x, T, n)$$

and, for $0 < \alpha \leq 1, \frac{1}{2} \leq \sigma < 1, R_2$ satisfying

$$R_2(x, T, N, M, \chi) \ll_Y \frac{NL}{TM^Y \log \frac{N}{T}} + \frac{MNL}{T^2 \log \frac{N}{T}} + \frac{N^{\frac{\sigma+3}{4}}}{T} L^4 + \frac{N}{T^{1+\alpha}} + \frac{NL^4}{T^{2-\alpha}} N(\sigma, T, \chi).$$

From the Theorem we get, setting $q = 1$ and $\chi = \chi_0$, the

Corollary. *Under the conditions of the Theorem with $q = 1$ we have, for each $4 \leq T \leq N/4$ and $Y \in N$ (fixed)*

$$R_Y(x, T, \chi_0) \ll_Y \frac{NL}{T \log \frac{N}{T}}.$$

Furthermore, let $\varepsilon > 0$, $N^\varepsilon \leq T \leq N^{1-\varepsilon}$ and for $Y \in N$ suppose that $1 \leq M \leq$ mir $(T^{1/(Y+1)}, (N^{1/16} L^{-4})^{1/Y}, (T^{1/5} L^{-9})^{1/Y})$.
Then

$$R_Y(x, T, \chi_0) = R_1(x, T, N, M, \chi_0) + O_{Y,\varepsilon}\left(\frac{N}{TM^Y}\right).$$

We think that a possible interest of these new explicit formulae arises from the application to the distribution of primes in short intervals, both for "everywhere" and "almost everywhere" results; for the latter see Theorem 2 in [4] and Theorem 2 in [1].

In particular, through the use of one of these Theorems, we get the same non-trivial connection between the density of primes in the interval $[x, x+H]$ (H is around N/T) and the symmetry of primes around x in the interval $[x - MN/T, x + MN/T]$ obtainable with the formula of [3] (see the Corollary); the class of formulae in our Corollary allows us to choose a smaller M and hence an interval closer to $[x, x+H]$; actually, we can choose $M = L^{\varepsilon}$ with $\varepsilon > 0$, instead of $M = L^c$ with $c > 1$ (of course every statement here holds "almost everywhere", i.e. regards "almost all $x \in [N, 2N]$"; for the exact hypotheses on the variables see [1] and [4]).

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The paper is organized as follows:

- in the following section we state and prove Lemma 1;
- we apply it in section 3 to get the Lemmas for the Theorem;
- finally this and the Corollary are proved in section 4.

2. A "WEIGHTED" FORM OF PERRON'S FORMULA

Lemma 1. *Let $\{a_n\}_{n\in N}$ be a sequence of complex numbers, $x \geq 4, \delta = \frac{1}{\log x}$, $A_0(x) = \sum_{n<x} a_n + \frac{a_x}{2}$ (here $a_x = 0$ for $x \notin N$) and $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be absolutely convergent in $\sigma = Re(s) > 1$; then for $4 \leq T \leq \frac{x}{2}$ and $\tau \in [T/2, T]$ we have, for each fixed natural number Y and for each ϕ_Y satisfying (i), (ii) and (iii)*

$$A_0(x) = \frac{1}{2\pi i} \int_{1+\delta-i\tau}^{1+\delta+i\tau} f(s)\frac{x^s}{s} ds + g(x,\tau), \tag{1}$$

where

$$\frac{1}{\int_{T/2}^{T} \phi_Y(\tau)d\tau} \int_{T/2}^{T} \phi_Y(\tau) g(x,\tau) d\tau \ll_Y \frac{x}{T^2} \sum_{n\leq \frac{x}{2}} \frac{|a_n|}{n^{1+\delta}} +$$

$$+ \sum_{x-\frac{x}{T} < n \leq x+\frac{x}{T}} |a_n| + \sum_{n\in I} |a_n| \left(\frac{1}{T^{Y+1} \left|\log \frac{x}{n}\right|^{Y+1}} + \frac{1}{T^2} \right) + \frac{x}{T^2} \sum_{n>2x} \frac{|a_n|}{n^{1+\delta}}, \tag{2}$$

with I the set of integers in the intervals $(x/2, x - \frac{x}{T}]$, $(x + \frac{x}{T}, 2x]$.

Proof. By Perron's inversion formula and the absolute convergence of $f(s)$ for $\sigma > 1$ we get $\forall \tau \in [T/2, T]$

$$A_0(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s} ds = \frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} f(s)\frac{x^s}{s} ds + \sum_{n=1}^{\infty} a_n I(n,x,\tau), \tag{3}$$

having defined $c = 1 + \delta$, $\ell = \log(x/n)$ and

$$I(n,x,\tau) = \frac{1}{2\pi i} \int_{\substack{\sigma=c \\ |t|>\tau}} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \frac{1}{\pi} \left(\frac{x}{n}\right)^c \int_{\tau}^{\infty} \frac{c\cos \ell t + t \sin \ell t}{c^2 + t^2} dt. \tag{4}$$

Clearly we have

$$n \notin \left(x - \frac{x}{T}, x + \frac{x}{T}\right) \Rightarrow \frac{1}{|\ell| T} \ll 1 \tag{5}$$

and

$$\left(\frac{x}{n}\right)^{c} \ll \begin{cases} 1 & \text{if } x/2 < n \leq 2x \\ x/n^{c} & \text{otherwise.} \end{cases} \tag{6}$$

If we let

$$\mathcal{J}(n, x, \tau) = \int_{\tau}^{\infty} \frac{c \cos \ell t + t \sin \ell t}{c^2 + t^2} dt,$$

we have, since $\tau \gg T$,

$$\mathcal{J}(n, x, \tau) = \int_{\tau}^{\infty} \frac{c \cos \ell t + t \sin \ell t}{t^2} dt + O\left(\frac{1}{T^2}\right) \quad \forall n \in N. \tag{7}$$

Thus, because of (4), (6) and (7), we will prove, say, that

$$\frac{1}{\int_{T/2}^{T} \phi_Y(\tau) d\tau} \int_{T/2}^{T} \phi_Y(\tau) \mathcal{J}(n, x, \tau) d\tau \ll_Y 1 \tag{8}$$

for $n \in (x - x/T, x + x/T]$ and, by (5), that

$$\frac{1}{\int_{T/2}^{T} \phi_Y(\tau) d\tau} \int_{T/2}^{T} \phi_Y(\tau) \mathcal{J}(n, x, \tau) d\tau \ll_Y \left(\frac{1}{T|\ell|}\right)^{Y+1} + \frac{1}{T^2} \tag{9}$$

for $n \notin (x - x/T, x + x/T]$.

If $n = x$ we have

$$\mathcal{J}(n, x, \tau) = \int_{\tau}^{\infty} \frac{c}{c^2 + t^2} dt \ll \frac{1}{T},$$

which proves (8), since, by (iii)

$$\frac{\phi_Y(\tau)}{\int_{T/2}^{T} \phi_Y(\tau) d\tau} \ll_Y \frac{1}{T} \qquad \forall \tau \in (T/2, T). \tag{10}$$

If $n \neq x$ and $n \in (x - \frac{x}{T}, x + \frac{x}{T}]$ then by (7)

$$\mathcal{J}(n, x, \tau) = sgn(x - n) \int_{|\ell|\tau}^{\infty} \frac{\sin t}{t} dt + O\left(\int_{\tau}^{\infty} \frac{c}{t^2} dt + \frac{1}{T^2}\right),$$

since in this interval $|\ell| \leq 2/T \leq 2/\tau$ and hence $|\ell|\tau < \pi$ we get

$$\int_{|\ell|\tau}^{\infty} \frac{\sin t}{t} dt \ll 1,$$

which, together with (10), gives (8). (Here a different choice of the kernel ϕ_Y is ininfluent for the estimate of the average, because of the linearity of the integral operator).

By (7), to prove (9) we have still to prove, say, that

$$\frac{1}{\int_{T/2}^{T} \phi_Y(\tau)d\tau} \int_{T/2}^{T} \phi_Y(\tau) \int_{\tau}^{\infty} \frac{\cos \ell t}{t^2} dt d\tau \ll_Y \frac{1}{T^{Y+2}|\ell|^{Y+1}} \tag{11}$$

and

$$\frac{1}{\int_{T/2}^{T} \phi_Y(\tau)d\tau} \int_{T/2}^{T} \phi_Y(\tau) \int_{\tau}^{\infty} \frac{\sin \ell t}{t} dt d\tau \ll_Y \frac{1}{T^{Y+1}|\ell|^{Y+1}}. \tag{12}$$

We will prove only (12) (the proof of (11) is similar).

Integrating $Y+1$ times by parts we get

$$\int_{\tau}^{\infty} \frac{\sin \ell t}{t} dt = \sum_{j=0}^{Y} (-1)^j j! \frac{\cos^{(j)}(\ell\tau)}{\ell^{j+1}\tau^{j+1}} + O_Y\left(\frac{1}{|\ell|^{Y+1}T^{Y+1}}\right)$$

(here $\cos^{(j)}(\ell\tau)$ stands for $\frac{d^j}{dv^j}\cos(v)(\ell\tau)$), whence, by (10)

$$\frac{1}{\int_{T/2}^{T} \phi_Y(\tau)d\tau} \int_{t/2}^{T} \phi_Y(\tau) \int_{\tau}^{\infty} \frac{\sin \ell t}{t} dt d\tau =$$

$$= \sum_{j=0}^{Y} \frac{(-1)^j j!}{\int_{T/2}^{T} \phi_Y(\tau)d\tau} \int_{T/2}^{T} \phi_Y(\tau) \frac{\cos^{(j)}(\ell\tau)}{\ell^{j+1}\tau^{j+1}} d\tau + O_Y\left(\frac{1}{|\ell|^{Y+1}T^{Y+1}}\right);$$

we are thus reduced to prove that

$$\frac{1}{\int_{T/2}^{T} \phi_Y(\tau)d\tau} \max_{0\le j\le Y} \int_{T/2}^{T} \phi_Y(\tau) \frac{\cos^{(j)}(\ell\tau)}{\ell^{j+1}\tau^{j+1}} d\tau \ll_Y \frac{1}{|\ell|^{Y+1}T^{Y+1}}. \tag{13}$$

Defining $\forall j = 0, \ldots, Y$

$$h_j(\tau) = \tau^{-j-1}\phi_Y(\tau)$$

and

$$k_j(\tau) = \cos^{(j)}(\ell\tau)$$

we obtain, by (ii), (iii) and the Leibniz rule ($K = 0, \ldots, Y$):

$$\frac{d^K}{d\tau^K} h_j(\tau) = \sum_{r=0}^{K} \binom{K}{r} \frac{d^r}{d\tau^r} \phi_Y(\tau) \frac{d^{(K-r)}}{d\tau^{(K-r)}} \tau^{-j-1},$$

the following two properties of h_j (in the sequel $j \in \{0, \ldots, Y\}$):

$$\frac{d^K}{d\tau^K} h_j(\tau)|_{\tau=T/2,T} = 0 \qquad \forall K = 0, \ldots, Y-1 \tag{14}$$

and

$$\frac{1}{\int_{T/2}^{T}\phi_Y(\tau)d\tau}\frac{d^Y}{d\tau^Y}h_j(\tau) \ll_Y \frac{1}{T^{Y+j+2}} \qquad \forall\tau \in (T/2, T). \tag{15}$$

Integrating Y times by parts in τ we get, by (14)

$$\frac{1}{\int_{T/2}^{T}\phi_Y(\tau)d\tau}\int_{T/2}^{T} h_j(\tau)k_j(\tau)d\tau = \frac{1}{\ell^Y\int_{T/2}^{T}\phi_Y(\tau)d\tau}\int_{T/2}^{T} h_j^{(Y)}(\tau)b_j(\tau)d\tau,$$

where $b_j(\tau)$ is a (bounded) function such that

$$\frac{d^Y}{d\tau^Y}b_j\tau) = \ell^Y k_j(\tau),$$

then, by (15) we have

$$\frac{1}{\int_{T/2}^{T}\phi_Y(\tau)d\tau}\int_{T/2}^{T}\phi_Y(\tau)\frac{\cos^{(j)}(\ell\tau)}{\ell^{j+1}\tau^{j+1}}d\tau \ll_y \frac{1}{|\ell|^{Y+j+1}T^{Y+j+1}},$$

which gives (13) (because of (5)) and hence the Lemma.

3. APPLICATION OF LEMMA 1 TO $\psi(x,\chi)$

Lemma 2. *Let N, q, χ, x, T, M be as in the Theorem, $\delta = 1/L$,*

$$U(s,\chi) = U(s,\chi,N,x,T,M) = \sum_{x-\frac{MN}{T} < n \le x+\frac{MN}{T}} \wedge(n)\chi(n)n^{-s}$$

and

$$f(s) = -\frac{L'}{L}(s,\chi) - U(s,\chi).$$

Then for $T/2 \le \tau \le T$ we have

$$\psi\left(x - \frac{MN}{T},\chi\right) = \frac{1}{2\pi i}\int_{1+\delta-i\tau}^{1+\delta+i\tau} f(s)\frac{x^s}{s}ds + g_1(x,\tau,\chi)$$

where $g_1(x,\tau,\chi)$ satisfies

$$\frac{1}{\int_{T/2}^{T}\phi_Y(\tau)d\tau}\int_{T/2}^{T}\phi_Y(\tau)g_1(x,\tau,\chi)d\tau \ll_Y \frac{NL}{TM^Y\log\frac{N}{T}} + \frac{N}{T^2}.$$

Proof. From Lemma 1 with

$$a_n = \begin{cases} \wedge(n)\chi(n) & \text{for } n \le x - MN/T \text{ and } n > x + MN/T \\ 0 & \text{otherwise} \end{cases}$$

we get

$$\frac{1}{\int_{T/2}^{T}\phi_Y(\tau)d\tau}\int_{T/2}^{T}\phi_Y(\tau)g_1(x,\tau,\chi) \ll_Y \frac{N}{T^2}\sum_{n\le\frac{x}{2}}\frac{\Lambda(n)}{n^{1+\delta}}+$$

$$+\left(\sum_{\frac{x}{2}<n\le x-\frac{MN}{T}}+\sum_{x+\frac{MN}{T}<n\le 2x}\right)\left(\frac{\Lambda(n)}{T^{Y+1}\left|\log\frac{x}{n}\right|^{Y+1}}+\frac{\Lambda(n)}{T^2}\right)+\frac{N}{T^2}\sum_{n>2x}\frac{\Lambda(n)}{n^{1+\delta}}.$$

Since

$$\sum_{n\le\frac{x}{2}}\frac{\Lambda(n)}{n^{1+\delta}}\ll 1 \qquad \text{and} \qquad \sum_{n>2x}\frac{\Lambda(n)}{n^{1+\delta}}\ll 1$$

the sums over $n \le x/2$ and $n>2x$ contribute $O_Y\left(\frac{N}{T^2}\right)$; the sum over $\frac{x}{2}<n\le x-\frac{MN}{T}$ gives, by the Brun-Titchmarsh inequality, at most

$$\frac{1}{T^{Y+1}}\sum_{M\le j\le T-1}\frac{T^{Y+1}}{j^{Y+1}}\sum_{x-\frac{(j+1)N}{T}<n\le x-\frac{jN}{T}}\Lambda(n)+O_Y\left(\frac{N}{T^2}\right)\ll_Y \frac{NL}{TM^Y\log\frac{N}{T}}+\frac{N}{T^2}.$$

Using the same argument for the sum over $x+\frac{MN}{T}<n\le 2x$ we get the same estimate, and hence the Lemma.

Lemma 3. *Let* $0<\alpha\le 1$ *and* $\frac{1}{2}\le\sigma<1$. *Under the conditions of Lemma 2, let*

$$h(x,\tau,\chi) = -\frac{1}{2\pi i}\int_{1+\delta-i\tau}^{1+\delta+i\tau}\frac{L'}{L}(s,\chi)\frac{x^s}{s}ds.$$

Then

$$h(x,\tau,\chi) = \varepsilon(\chi)x - \sum_{|\gamma|\le\tau}\frac{x^\rho}{\rho} - \varepsilon_1(\chi)\log x - \varepsilon_2(\chi) + g_2(x,\tau,\chi),$$

where

$$\frac{1}{\int_{T/2}^{T}\phi_Y(\tau)d\tau}\int_{T/2}^{T}\phi_Y(\tau)g_2(x,\tau,\chi)d\tau \ll_Y \frac{N^{(\sigma+3)/4}}{T}L^4+\frac{N}{T^{1+\alpha}}+\frac{NL^4}{T^{2-\alpha}}N(\sigma,T,\chi). \quad (16)$$

Proof. We remark that, by (iii), we have $\forall g\in L^1(T/2,T)$

$$\frac{1}{\int_{T/2}^{T}\phi_Y(\tau)d\tau}\int_{T/2}^{T}\phi_Y(\tau)g(\tau)d\tau \ll_Y \frac{1}{T}\int_{T/2}^{T}|g(\tau)|d\tau. \quad (17)$$

Hence each estimate over $|g(\tau)|$ ($\forall g\in L^1(T/2,T)$) can be used to obtain our weighted average from a direct average of $|g(\tau)|$ (and the estimates are the same).

The proof can therefore follow the proof of Lemma 3 in [3] except for the average estimate of $-2\Re e(\mathcal{F}_1(\tau))$, where

$$\mathcal{F}_1(\tau) = \frac{1}{2\pi i}\int_{\sigma+i\tau}^{1+\delta+i\tau} \sum_{n\le y^2} \frac{\wedge_y(n)\chi(n)}{n^s}\frac{x^s}{s}ds,$$

with (we will make the same choice of [3], $y = x^{1/4}$)

$$\wedge_y(n) = \begin{cases} \wedge(n) & \text{for } 1 \le n \le y \\ \dfrac{\wedge(n)\log\frac{y^2}{n}}{\log y} & \text{for } y \le n \le y^2. \end{cases}$$

Integrating by parts we have

$$-2\Re e(\mathcal{F}_1(\tau) = -2\Re e\left(\sum_{n\le y^2} \wedge_y(n)\chi(n)\frac{1}{2\pi i}\int_{\sigma+i\tau}^{1+\delta+i\tau}\left(\frac{x}{n}\right)^s\frac{ds}{s}\right) =$$

$$= -2\Re e\left(\sum_{n\le y^2}\frac{\wedge_y(n)\chi(n)}{2\pi i\log\frac{x}{n}}\frac{(x/n)^{1+\delta+i\tau}}{1+\delta+i\tau}\right) + O\left(\frac{N^\sigma}{TL}\sum_{n\le y^2}\frac{\wedge_y(n)}{n^\sigma} + \frac{N}{T^2L^2}\sum_{n\le y^2}\frac{\wedge_y(n)}{n^{1+\delta}}\right) =$$

$$= -2\Re e\left(\sum_{n\le y^2}\frac{\wedge_y(n)\chi(n)}{2\pi i\log\frac{x}{n}}\frac{(x/n)^{1+\delta+i\tau}}{1+\delta+i\tau}\right) + O\left(\frac{N^\sigma}{TL} + \frac{N}{T^2L^2}\right),$$

by (17) and (iii) the weighted average of $-2\Re e\mathcal{F}_1(\tau)$ is therefore

$$\frac{1}{\int_{T/2}^{T}\phi_Y(\tau)d\tau}\int_{T/2}^{T}\phi_Y(\tau)(-2\Re e\mathcal{F}_1(\tau))d\tau \ll_Y$$

$$\ll_Y \frac{N}{L}\sum_{n\le y^2}\frac{\wedge_y(n)n^{-1-\delta}}{\left|\int_{T/2}^{T}\phi_Y(\tau)d\tau\right|}\left|\int_{T/2}^{T}\phi_Y(\tau)\left(\frac{x}{n}\right)^{i\tau}\frac{d\tau}{1+\delta+i\tau}\right| + \frac{N^\sigma}{TL} + \frac{N}{T^2L^2}$$

$$\ll_y \frac{N}{T^2L^2}\sum_{n\le x^{1/2}}\frac{\wedge(n)}{n^{1+\delta}} + \frac{N^\sigma}{TL} + \frac{N}{T^2L^2} \ll_Y \frac{N^\sigma}{TL} + \frac{N}{T^2L^2},$$

this last term is less than first two terms of (16) (that are obtained as in Lemma 3 of [3]) and hence we have the Lemma.

Lemma 4. *Under the conditions of Lemma 2 let*

$$k(x,\tau,\chi) = \frac{1}{2\pi i}\int_{1+\delta-i\tau}^{1+\delta+i\tau} U(s,\chi)\frac{x^s}{s}ds.$$

Then

$$\frac{1}{\int_{T/2}^{T}\phi_Y(\tau)d\tau}\int_{T/2}^{T}\phi_Y(\tau)k(x,\tau,\chi)d\tau=\psi_0(x,\chi)-\psi\left(x-\frac{MN}{T},\chi\right) \tag{18}$$

$$-\frac{1}{\pi}\sum_{x-\frac{MN}{T}<n\le x+\frac{MN}{T}}\Lambda(n)\chi(n)sgn(x-n)G_Y(x,T,n)+O_Y\left(\frac{MNL}{T^2\log\frac{N}{T}}\right).$$

Proof. We have

$$k(x,\tau,\chi)=\sum_{x-\frac{MN}{T}<n\le x+\frac{MN}{T}}\Lambda(n)\chi(n)\frac{1}{2\pi i}\int_{\substack{\sigma=1+\delta\\|t|\le\tau}}\left(\frac{x}{n}\right)^s\frac{ds}{s}= \tag{19}$$

$$=\psi_0(x,\chi)-\psi\left(x-\frac{MN}{T},\chi\right)-\sum_{x-\frac{MN}{T}<n\le x+\frac{MN}{T}}\Lambda(n)\chi(n)I(n,x,\tau),$$

having defined $I(n,x,\tau)$ as in Lemma 1; by (6) and (7) of this Lemma we get $\forall n\in(x-MN/T,x+MN/T]$ ($\ell=\log(x/n)$ again)

$$I(n,x,\tau)=\frac{1}{\pi}\left(\frac{x}{n}\right)^{1+\delta}\int_{\tau}^{\infty}\frac{\sin\ell t}{t}dt+O\left(\frac{1}{T}\right)=$$

$$=\frac{sgn(x-n)}{\pi}\left(\frac{x}{n}\right)^{1+\delta}\int_{|\ell|\tau}^{\infty}\frac{\sin y}{y}dy+O\left(\frac{1}{T}\right),$$

that turns the last sum of (19) into

$$\frac{1}{\pi}\sum_{x-\frac{MN}{T}<n\le x+\frac{MN}{T}}\Lambda(n)\chi(n)sgn(x-n)\left(\frac{x}{n}\right)^{1+\delta}\int_{|\ell|\tau}^{\infty}\frac{\sin y}{y}dy+O\left(\frac{1}{T}\sum_{|n-x|\le MN/T}\Lambda(n)\right). \tag{20}$$

The same arguments used to prove (8) and (12) give clearly

$$G_Y(x,T,n)\ll_Y\min\left(1,\frac{1}{T^{Y+1}\left|\log\frac{x}{n}\right|^{Y+1}}\right), \tag{21}$$

since

$$\left(\frac{x}{n}\right)^{1+\delta}=1+O\left(\frac{M}{T}\right)\qquad\forall n\in\left[x-\frac{MN}{T},x+\frac{MN}{T}\right]$$

we get

$$\frac{1}{\pi}\sum_{x-\frac{MN}{T}<n\le x+\frac{MN}{T}}\Lambda(n)\chi(n)sgn(x-n)\left(\frac{x}{n}\right)^{1+\delta}G_Y(x,T,n)= \tag{22}$$

$$=\frac{1}{\pi}\sum_{x-\frac{MN}{T}<n\le x+\frac{MN}{T}}\Lambda(n)\chi(n)sgn(x-n)G_Y(x,T,n)+$$

$$+O_Y\left(\frac{M}{T}\sum_{|n-x|\le\frac{N}{T}}\Lambda(n)+\frac{M}{T}\sum_{\frac{N}{T}<|n-x|\le\frac{MN}{T}}\frac{\Lambda(n)}{\left|\log\frac{x}{n}\right|^{Y+1}T^{Y+1}}\right).$$

Then, splitting the last sum as in Lemma 2 and using the Brun-Titchmarsh inequality, the weighted average of (19) becomes by (20) and (22)

$$\frac{1}{\int_{T/2}^{T}\phi_Y(\tau)d\tau}\int_{T/2}^{T}\phi_Y(\tau)k(x,\tau,\chi)d\tau=\psi_0(x,\chi)-\psi\left(x-\frac{MN}{T},\chi\right)$$

$$-\frac{1}{\pi}\sum_{x-\frac{MN}{T}<n\le x+\frac{MN}{T}}\Lambda(n)\chi(n)sgn(x-n)G_Y(x,T,n)+O_Y\left(\frac{MNL}{T^2\log\frac{N}{T}}+\frac{MNL}{T^2\log\frac{MN}{T}}\right),$$

whence (18).

4. PROOF OF THE THEOREM AND THE COROLLARY

By Lemmas 2 and 3 we have

$$\begin{aligned}\psi\left(x-\frac{MN}{T},\chi\right) &= -\frac{1}{2\pi i}\int_{1+\delta-i\tau}^{1+\delta+i\tau}\frac{L'}{L}(s,\chi)\frac{x^s}{s}ds-\frac{1}{2\pi i}\int_{1+\delta-i\tau}^{1+\delta+i\tau}U(s,\chi)\frac{x^s}{s}ds+g_1= \\ &= \varepsilon(\chi)x-\sum_{|\gamma|\le\tau}\frac{x^\rho}{\rho}-\varepsilon_1(\chi)\log x-\varepsilon_2(\chi)+g_2(x,\tau,\chi) \\ &- k(x,\tau,\chi)+g_1(x,\tau,\chi),\end{aligned}$$

multiplying by $\phi_Y(\tau)$, integrating in $[T/2,T]$ and dividing by $\int_{T/2}^{T}\phi_Y(\tau)d\tau$ we then get the Theorem, by Lemma 4, since

$$\begin{aligned}\psi\left(x-\frac{MN}{T},\chi\right) &= \varepsilon(\chi)x-\sum_{|\gamma|\le T}w_Y\left(\frac{|\gamma|}{T}\right)\frac{x^\rho}{\rho}-\varepsilon_1(\chi)\log x-\varepsilon_2(\chi)-\psi_0(x,\chi) \\ &+\psi\left(x-\frac{MN}{T},\chi\right)+R_1+O_Y\left(\frac{N^{(\sigma+3)/4}}{T}L^4+\frac{N}{T^{1+\alpha}}+\frac{NL^4}{T^{2-\alpha}}N(\sigma,T,\chi)\right) \\ &+O_Y\left(\frac{NL}{TM^Y\log\frac{N}{T}}+\frac{MNL}{T^2\log\frac{N}{T}}\right).\end{aligned}$$

The proof of the Corollary is obtained by choosing $\sigma=\frac{3}{4}$, $\alpha=\frac{1}{5}$ in the Theorem and using Ingham's density estimate (see e.g. [5], Th. 9.19 (B))

$$N(\sigma,T,\chi_0)\ll T^{\frac{3(1-\sigma)}{2-\sigma}}\log^5 T,$$

then the first estimate follows from (21) and the Brun-Titchmarsh inequality, while the second is immediate.

As an example for a function $\phi_Y(\tau)$ satisfying (i), (ii) and (iii) we give

$$\phi_Y(\tau) = \sin^Y\left(\frac{2\pi\tau}{T}\right) \qquad \forall\tau \in [T/2, T],$$

in which the dependence on Y is explicit.

We hope to discuss other applications of the method to the mean-square

$$\int_N^{2N} |R_1(x, T, N, M, \chi)|^2 dx$$

in future papers.

REFERENCES

[1] G. COPPOLA and A. VITOLO, *On the distribution of primes in intervals of length* $\log^{\theta} N$, Acta Math. Hung. 70 (1-2) (1996).

[2] H. DAVENPORT, *Multiplicative Number Theory*, second edition - Springer Verlag 1980.

[3] J. KACZOROWSKI and A. PERELLI, *A new form of the Riemann-von Mangoldt explicit formula*, To appear in Boll. U.M.I.

[4] J. KACZOROWSKI and A. PERELLI, *On the distribution of primes in short intervals*, J. Math. Soc. Japan vol. 45, 1993, 447-458.

[5] E.C. TITCHMARSH, *The Theory of the Riemann Zeta Function*, second edition - Oxford U.P. 1986.

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