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ON IMBEDDINGS OF WEIGHTED SOBOLEV SPACES

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Abstract. *We obtained the continuous (and also compact) imbedding*

$$W^{1,p}(\Omega, v_0, v_1) \mapsto W^{1,p}(\Omega, w)$$

under certain conditions on weights, where Ω *is a bounded domain in* $\mathbb{R}^N$*. Also, for the case* $v_0 = v_1$*, this imbedding is shown to exist under less restrictive conditions. Finally, the imbedding*

$$\mathbf{W}^{1,p}(\Omega, v) \mapsto L^q(\Omega, w),$$

already obtained by Gurka and Opic [3], is established under a different set of conditions.

1. INTRODUCTION

It is well known now that Sobolev spaces provide a natural framework for the modern theory of partial differential equations and the numerical solution of boundary value problems and more so if Sobolev spaces with weights are considered, for instance, see [2], [10], [12] etc. This theory has become richer and richer by the various possibilities of imbedding one Sobolev (or weighted Sobolev) space in a variety of other such spaces, the corresponding imbedding being continuous and/or compact. Different types of imbeddings have been obtained by various authors; for instance, Opic [11], Gurka and Opic [3,4,5], Kufner et. al. [8], Adams [1], Sobolev [12] and references therein.

In the present paper, we discuss conditions (necessary and sufficient) on the weight functions v_0, v_1 and w for the continuous imbedding

$$W^{1,p}(\Omega, v_0, v_1) \mapsto W^{1,p}(\Omega, w) \tag{1.1}$$

to hold. Also it is shown that in the case $v_0 = v_1 = v$ the imbedding

$$W^{1,p}(\Omega, v, v) \mapsto W^{1,p}(\Omega, w) \tag{1.2}$$

holds under less restrictive conditions. Not only this, the sufficient conditions for the imbedding (1.2) can further be weakened.

Further, Gurka and Opic [3] obtained necessary and sufficient conditions for the imbedding

$$W^{1,p}(\Omega, v, v) \mapsto L^q(\Omega, w). \tag{1.3}$$

We also show, in this paper, the existence of the imbedding (1.3) under a different set of conditions. The results of the similar nature have also been discussed in respect of compact imbeddings.

We give notations and terminology in Section 2, the lemmas which are required in the proofs of our main results are given in Section 3. In Section 4, we discuss the continuous imbeddings while, in Section 5, the special cases of the continuous imbedding (1.1) are discussed along with the existence of the continuous imbedding (1.3) under a different set of conditions and finally, in Section 6, the compact imbeddings are considered.

2. NOTATIONS AND TERMINOLOGY

Let Ω be a domain in $\mathbb{R}^N$. By a weight function we mean a function which is measurable and positive almost everywhere (a.e.) in Ω. We denote by $\mathcal{S}$, the set of weight functions on Ω.

For $w \in \mathcal{S}$, let us denote by $L^p(\Omega, w), 1 \leq p < \infty$, the set of all functions $u = u(x)$ on Ω such that

$$\|u\|_{p,w} = \left(\int_\Omega |u(x)|^p w(x) dx\right)^{1/p} < \infty.$$

Also, for $v_0, v_1 \in \mathcal{S}$, let us denote by $W^{1,p}(\Omega, v_0, v_1)$, the set of all $u \in L^p(\Omega, v_0)$ such that the distributional derivatives $\frac{\partial u}{\partial x_i} \in L^P\ (\Omega, v_1)$, $i = 1, \ldots, N$. The norm of the space $W^{1,p}(\Omega, v_0, v_1)$ is defined by

$$\|u\|_{1,p,v_0,v_1} = \left(\|u\|^p_{p,v_0} + \sum \left\|\frac{\partial u}{\partial x_i}\right\|^p_{p,v_1}\right)^{1/p}.$$

Given $x \in \mathbb{R}^N$ and $R > 0$, we put

$$B(x, R) = \{y \in \mathbb{R}^N\ ,\ |x - y| < R\}$$

and for $h > 0$, we write

$$hB(x, R) = B(x, hR).$$

We denote by $\mathcal{C}^{0,1}$, the class of all bounded domains in $\mathbb{R}^N$ with a Lipschitz boundary (in sense of Definition 5.5.6 in [9]).

Throughout this paper, we assume that

$$v_0, v_1 \in \mathcal{S} \cap L^1_{loc}(\Omega), v_0^{-1/p}, v_1^{-1/p} \in L^{p'}_{loc}(\Omega), \left(p' = \frac{p}{p-1}, p \neq 1\right)$$

and further that

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n, \tag{2.1}$$

where Ω_n are domains in $\mathbb{R}^N$ such that

$$\Omega_n \subset \Omega_{n+1} \subset \Omega, \qquad \Omega_{n+1} \neq \Omega$$

and we write $\Omega^n = \Omega \backslash \Omega_n, n \in \mathbb{N}$.

Finally, we shall be using the symbols $\mapsto$ and $\mapsto\mapsto$, respectively, for continuous and compact imbeddings.

3. LEMMAS

In this section, we collect certain results in the form of lemmas on which we rely heavily for the proofs of our results in Sections 4, 5 and 6.

Lemma 1. *Let $X(\Omega)$ and $Y(\Omega)$ be two Banach spaces of functions defined on Ω such that $Y(\Omega)$ has an absolutely continuous and monotone norm i.e.*

$$0 \leq f \leq g \text{ a.e. in } \Omega \Rightarrow \|f\|_{Y(\Omega)} \leq \|g\|_{Y(\Omega)},$$

where Ω is a domain satisfying (2.1) and (2.2). Let $X(\Omega_n) \mapsto Y(\Omega_n)$, $n \in \mathbb{N}$, where

(i) $X(\Omega_n)$ and $Y(\Omega_n)$ are the sets of restrictions to Ω_n of the functions, respectively, from $X(\Omega)$ and $Y(\Omega)$;

(ii) The space $X(\Omega_n)$ is equipped with the norm $\|\cdot\|_{X,\Omega_n}$ satisfying

$$\|f\|_{X,\Omega_n} = C_n \|f\|_{X,\Omega},$$

with an appropriate constant C_n independent of f; and

(iii) The space $Y(\Omega_n)$ is equipped with the norm

$$\|f\|_{Y,\Omega_n} = \|f \cdot \chi_{\Omega_n}\|_{Y,\Omega},$$

where χ_{Ω_n} is the characteristic function of the sets Ω_n; Then a necessary and sufficient condition for the imbedding $X(\Omega) \mapsto Y(\Omega)$ to hold is that

$$\lim_{n\to\infty} B_n = B < \infty, \tag{3.1}$$

where

$$B_n = \sup_{\|f\|_{X,\Omega}=1} \|f\|_{Y,\Omega^n}. \tag{3.2}$$

Proof. See [8]. ■

Lemma 2. *Let $X(\Omega)$, $Y(\Omega)$, $X(\Omega_n)$ and $Y(\Omega_n)$ be as in Lemma 1. If $X(\Omega_n) \mapsto\mapsto Y(\Omega_n)$, $n \in \mathbb{N}$, then a necessary and sufficient condition for the imbedding $X(\Omega) \mapsto\mapsto Y(\Omega)$ to hold is that*

$$\lim_{n\to\infty} B_n = 0,$$

where B_n is given by (3.2).

Proof. See [8]. ■

Lemma 3. (Besicovitch Covering Lemma). *Let $A \subset \mathbb{R}^N$ be a bounded set and ρ be a positive function defined on A. Then, there exists a sequence $\{x_k\} \subset A$ such that the sequence of balls $\{B_k\}$ with $B_k = B_k\,(x_k, \rho(x_k))$ satisfies:*

(i) $A \subset \bigcup_{k=1}^{\infty} B_k$.

(ii) $\exists$ *a number* Θ *depending only on the dimension* N *such that*

$$\sum_{k=1}^{\infty} \chi_{B_k}(z) \leq \Theta, \qquad \forall z \in \mathbb{R}^N,$$

χ_{B_k} *being the characteristic function of* B_k.

(iii) The sequence $\{B_k\}$ *can be divided into* ξ *families of disjoint balls (the number* ξ *depends only on* N*).*

Proof. See [6]. ■

Lemma 4. *Let* $1 \leq p < \infty, R > 0$ *and* $x \in \mathbb{R}^N$. *Then*

$$\int_{B(x,R)} |u(u)|^p dy \leq (KR)^p \left[R^{-p} \int_{B(x,R)} |u(y)|^p dy + \int_{B(x,R)} |\nabla u(y)|^p dy \right],$$

for all $u \in W^{1,p}(B(x,R))$, *where* $|\nabla u(y)|^p = \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i}(y) \right|^p$ *and* $K > 0$ *is a constant independent of* x, R *and* u.

Proof. See Lemma 3.5 in [3] with $p = q$. ■

From now onwards throughout the paper, unless specified otherwise, we shall be making the following assumptions:

(A) Ω is a bounded domain in $\mathbb{R}^N$.

(B) $\{\Omega_n\} \subset C^{0,1}$ is a sequence of domains such that

$$\left\{x \in \Omega, n^{-1} < d(x)\right\} \subset \Omega_n \subset \left\{x \in \Omega, (n+1)^{-1} < d(x)\right\}.$$

where $d(x) = \text{dist}\,(x, \partial\,\Omega)$, $\partial\,\Omega$ being the boundary of Ω and $x \in \Omega$.

(C) $\exists \# n_0 \in \mathbb{N}, n_0 \geq 3$, a positive measurable function r defined on Ω^{n_0} such that

$$r(x) \leq d(x)\,/\,3, \qquad x \in \Omega^{n_0}.$$

Note that (Ω_n) in (B) satisfies (2.1) and (2.2).

Lemma 5. *Suppose that* $n \geq 3m$, $m \in \mathbb{N}$ *and* $m \geq 3$. *If* $B(x, r(x)) \cap \Omega^n \neq \emptyset$, *then* $B(x, r(x)) \subset \Omega^m$.

Proof. See Lemma 3.6 in [3]. ■

4. CONTINUOUS IMBEDDINGS

Theorem 1. *Let the following conditions be satisfied:*

S1. $W^{1,p}(\Omega_n, v_0, v_1) \mapsto w^{1,p}(\Omega_n, w), \quad n \geq n_0.$

S2. $\exists$ *positive measurable functions* a_0, a_1 *defined on* Ω^{n_0} *such that*

$$w(y) \leq a_0(x), \text{ and} \tag{4.1}$$

$$\left(1 + r^{-p}(y)\right) a_1(x) \le v_1(y), \tag{4.2}$$

for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(x))$.

S3. *$\exists$ a constant $K_0 > 0$ such that*

$$v_1(x) r^{-p}(x) \le K_0 v_0(x), \qquad \text{for a.e. } x \in \Omega^{n_0}.$$

S4. *$\exists$ a constant c_r (depending upon r) such that*

$$x \in \Omega^{n_0}, y \in B(x, r(x)) \Rightarrow c_r^{-1} \le \frac{r(y)}{r(x)} \le c_r.$$

S5. $\lim_{n\to\infty} A_n = A < \infty$, *where*

$$A_n = \sup_{X \in \Omega^n} \frac{a_0(x)}{a_1(x)} r^p(x). \tag{4.3}$$

Then

$$W^{1,p}(\Omega, v_0, v_1) \mapsto W^{1,p}(\Omega, w) \tag{4.4}$$

holds.

Proof. If we take $X(\Omega) = W^{1,p}(\Omega, v_0, v_1)$, $Y(\Omega) = W^{1,p}(\Omega, w)$ and Ω_{3n} instead of Ω_n in Lemma 1, then the imbedding (4.4) is established if we verify (3.1).

Taking $A = \Omega^{n_0}$ and $\rho = r$ in Lemma 3, $\exists$ a sequence $\{x_K\} \subset \Omega^{n_0}$ such that

$$\Omega^{n_0} \subset \bigcup_{k=1}^{\infty} B_k, \quad B_k = B_k\left(x_k, r(x_k)\right), \tag{4.5}$$

$$\sum_{k=1}^{\infty} \chi_{B_k}(z) \le \Theta, \quad \forall x \in \mathbb{R}^N. \tag{4.6}$$

Let $n \ge n_0$ be fixed. Write

$$\mathbb{K}_n = \left\{k \in \mathbb{N}, B_k \cap \Omega^{3n} \ne \emptyset\right\}.$$

By (4.5), we have

$$\|u\|^p_{1,p,\Omega^{3n},w} = \int_{\Omega^{3n}} |u(y)|^p w(y) dy + \int_{\Omega^{3n}} |\nabla u(y)|^p w(y) dy$$

$$\le \sum_{k \in K_n} \left(\int_{B_k} |u(y)|^p w(y) dy + \int_{B_k} |\nabla u(y)|^p w(y) dy \right). \tag{4.7}$$

Using Lemma 4 and (4.1), we have

$$\int_{B_k} |u(y)|^p w(y) dy + \int_{B_k} |\nabla u(y)|^p w(y) dy$$

$$\leq a_0(x_k)\left(\int_{B_k}|u(y)|^p dy + \int_{B_k}|\nabla u(y)|^p dy\right)$$

$$\leq a_0(x_k)\Bigg((Kr(x_k))^p\Big\{r^{-p}(x_k)\int_{B_k}|u(y)|^p dy$$

$$+\int_{B_k}|\nabla u(y)|^p dy\Big\} + \int_{b_k}|\nabla u(y)|^p dy\Bigg)$$

$$= a_0(x_k)K^p r^p(x_k)\Bigg(r^{-p}(x_k)\int_{B_k}|u(y)|^p dy \tag{4.8}$$

$$+\int_{B_k}|\nabla u(y)|^p dy + K^{-p}r^{-p}(x_k)\int_{B_k}|\nabla u(y)|^p dy\Bigg).$$

$$\leq a_0(x_k)K_1 r^p(x_k)\Bigg(r^{-p}(x_k)\int_{B_k}|u(y)|^P dy$$

$$+(1+r^{-p}(x_k))\int_{B_k}|\nabla u(y)|^p dy\Bigg)$$

$$\leq a_0(x_k)K_1 r^p(x_k)\Bigg(c_r^p\int_{B_k}|u(y)|^p\frac{dy}{r^p(y)}$$

$$+c_r^p\int_{B_k}|\nabla u(y)|^p(1+\frac{1}{r^p(y)})dy\Bigg). \tag{4.9}$$

using **S4**, where $K_1 = \max(K^p, 1)$. Inequality (4.2) implies that

$$a_1(x) \leq v_1(y).$$

Using this together with **S3** and (4.3) in (4.9), we get

$$\int_{B_k}|u(y)|^p w(y)dy + \int_{B_k}|\nabla u(y)|^p w(y)dy$$

$$\leq \frac{a_0(x_k)}{a_1(x_k)}K_1 r^p(x_k)\Bigg(c_r^p\int_{B_k}|u(y)|^p\frac{v_1(y)}{r^p(y)}dy$$

$$+c_r^p\int_{B_k}|\nabla u(y)|^p v_1(y)dy\Bigg)$$

$$\leq \frac{a_0(x_k)}{a_1(x_k)}c_r^p K_1 r^p(x_k)\Bigg(K_0\int_{B_k}|u(y)|^p v_0(y)dy$$

$$+\int_{B_k}|\nabla u(y)|^p v_1(y)dy\Bigg)$$

$$\leq K_2 A_n\left(\int_{B_k}|u(y)|^p v_0(y)dy + \int_{B_k}|\nabla u(y)|^p v_1(y)dy\right).$$

where $K_\alpha = c_r^p K_1 \max(K_0, 1)$. Thus (4.7) reduces to

$$\|u\|^p_{1,p,\Omega^{3n},w} \le K_2 A_n \sum_{k \in K_n} \left(\int_{B_k} |u(y)|^p v_0(y) dy \right.$$

$$\left. + \int_{B_k} |\nabla u(y)|^p v_1(y) dy \right)$$

$$\le \Theta K_2 A_n \|u\|^p_{1,p,\Omega,v_0,v_1}, \tag{4.10}$$

in view of (4.6) and $\bigcup_{k=1}^\infty B_k \subset \Omega$. Hence condition **S5** verifies (3.1). ■

Towards the converse of Theorem 1, we prove the following

Theorem 2. *Let* **S4** *and the following conditions be satisfied*

N2. *∃ positive measurable functions $\hat{a}_0$, $\hat{a}_1$ defined on Ω^{n_0} such that*

$$w(y) \ge \hat{a}_0(x), \hat{a}_1(x) \ge v_1(y). \tag{4.11}$$

for all $x \in \Omega^{n_0}$ and for a.e. $\gamma \in B(x, r(x))$.

N3. *∃ a constant $K_0 > 0$ such that*

$$K_0 v_0(x) \le v_1(x) r^{-p}(x). \qquad \textit{for a. e. } x \in \Omega^{n_0}.$$

N5. $\lim_{n\to\infty} \hat{A}_n = \infty$, *where*

$$\hat{A}_n = \sup_{x \in \Omega^n} \frac{\hat{a}_0(x)}{\hat{a}_1(x)} r^p(x). \tag{4.12}$$

Then $W^{1,p}(\Omega, v_0, v_1)$ is not continuously imbedded in $W^{1,p}(\Omega, w)$.

Proof. By **N5**, ∃ an increasing sequence of natural numbers $\{n_k\}$ and a sequence $\{x_k\}$ with $x_k \in \Omega^{n_k}$, such that

$$\frac{\hat{a}_0(x_k)}{\hat{a}_1(x_k)} r^p(x_k) > k, \quad k \in \mathbb{N}, \tag{4.13}$$

Take

$$u_k R_{r(x_k)/8}(x_{3/4B_k}), \quad k = 1, 2, \dots \tag{4.14}$$

where R_ϵ is a mollifier with radius ϵ defined in sense of Gurka and Opic ([3], Theorem 2.4). We note that the functions $u_k, k = 1, 2, \dots$ satisfy the following:

$$(i) u_k \in C_0^\infty(B_k), \quad 0 \le u_k \le 1,$$

$$(ii) u_k \equiv 1 \quad on\ \frac{1}{2} B_k, \tag{4.15}$$

$$(iii) \exists \# c > 0 \textit{ such that}$$

$$\left| \frac{\partial u_k}{\partial x_i}(x) \right| \le \frac{c}{r(x_k)}, \quad x \in \Omega, i = 1, 2, \dots, N,$$

$$(iv) u_k \in W^{1,p}(\Omega, v_0, v_1).$$

Again, working on the lines of Gurka and Opic [ibid], it can be shown that

$$\int_{\Omega} |u_k(y)|^p v_o(y) dy + \int_{\Omega} |\nabla u_k(y)|^p v_1(y) dy$$

$$\leq L r^{N-p}(x_k) \hat{a}_1(x_k), \tag{4.16}$$

where $L = (K_0^{-1} c_r^p + N c^p)|B(0,1)|$ with c_r given by **S4**.

Further, as a consequence of (4.15), we have

$$\frac{\partial u_k}{\partial x_i}(y) = 0, \#y \in \frac{1}{2} B_k, \#i = 1, 2, \ldots, N, \#k = 1, 2, \ldots$$

which gives

$$|\nabla u_k(y)|^p = 0, \#y \in \frac{1}{2} B_k, \#k = 1, 2, \ldots \tag{4.17}$$

Now, from (4.11), (4.15) and (4.17), we get

$$\int_{\Omega} |u_k(y)|^p w(y) dy + \int_{\Omega} |\nabla u_k(y)|^p w(y) dy$$

$$\geq \int_{(1/2)B_k} w(y) dy$$

$$\geq 2^{-N} |B(0,1)| \hat{a}_0(x_k) r^N(x_k). \tag{4.18}$$

Now, if we suppose that the imbedding

$$W^{1,p}(\Omega, v_0, v_1) \mapsto W^{1,p}(\Omega, w), \tag{4.19}$$

holds, then from (4.16) and (4.18) it follows that

$$2^{-N} |B(0,1)| \hat{a}_0(x_k) r^N(x_k) \leq \tilde{K} L r^{N-p}(x_k) \hat{a}_1(x_k),$$

for $k \in \mathbb{N}$, where $\tilde{K}$ is the norm of the imbedding operator from (4.19). But this contradicts (4.13). Hence, the theorem is proved. ■

Remark 1. Theorem 1 gives sufficient conditions for the imbedding (4.4) to take place whereas Theorem 2 gives necessary conditions. None of the sets of conditions is both necessary and sufficient. It remains open to find intermediate conditions which are both necessary and sufficient.

Remark 2. Towards an example of the imbedding (4.4), we show in [7], however in some other context, that a sufficient condition for the imbedding

$$W^{1,p}(\Omega, d^{\eta-p}, d^{\eta}) \mapsto W^{1,p}(\Omega, d^{\epsilon})$$

to hold is $\epsilon - \eta \geq 0$ whilw a necessary condition is $\epsilon \geq \eta - p$, where d^{ϵ} etc. are the power type weights on the domain $C^{0,1}$.

5. SPECIAL CASES OF CONTINUOUS IMBEDDING

If we assume $v_0 = v_1 = v$, then the sufficient conditions for the imbedding (4.4) can be obtained under less restrictive conditions in the sense that we do not require condition **S4** and also the inequality (4.2) is replaced by a weaker one. More precisely, we prove the following

Theorem 3. *Let* **S1** *and* **S3** *with $v_0 = v_1 = v$ alongwith* **S5** *and the following condition be satisfied:*

$\overline{\mathbf{S2}}$. *$\exists$ positive measurable functions a_0, a_1 defined on Ω^{n_0} such that (4.1) and*

$$a_1(x) \le v(y) \quad (\textit{in place of } (\overline{4.2})) \tag{$\overline{4.2}$}$$

for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(x))$.
Then the imbedding

$$W^{1,p}(\Omega, v, v) \mapsto W^{1,p}(\Omega, w) \tag{5.1}$$

holds.

Proof. Using **S3** with $v_0 = v_1 = v$ and $(\overline{4.2})$ in (4.8), we get

$$\int_{B_k} |u(y)|^p w(y)dy + \int_{B_k} |\nabla u(y)|^p w(y)dy$$

$$\le \frac{a_0(x_k)}{a_1(x_k)} K_3 r^p(x_k) \left(\int_{B_k} |u(y)|^p v(y)dy + \int_{B_k} |\nabla u(y)|^p v(y)dy \right),$$

where $K_3 = K^p \max(K_0, (1 + K_0 K^{-p}))$. Now, the proof follows on the same lines as that of Theorem 1. ■

Towards the converse of Theorem 3, we have

Theorem 4. *Let* **N2** *with $v_0 = v_1 = v$ alongwith* **N5** *be satisfied. Then the space $W^{1,p}(\Omega, v, v)$ is not continuously imbedded in the space $W^{1,p}(\Omega, w)$.*

Proof. It can be obtained on the same lines as that of Theorem 2. ■

Remark 3. In this case also, as seen in Remark 1, there is a gap between the set of sufficient conditions and necessary conditions.

It is interesting to note that the sufficient conditions for the imbedding (5.1) can further be weakened but in that case the condition **S5** changes. More precisely, we prove the following:

Theorem 5. *Let* **S1** *with $v_0 = v_1 = v$ alongwith* $\overline{\mathbf{S2}}$ *and the following condition be satisfied:*

$\overline{\mathbf{S5}}$. $\lim_{n\to\infty} \overline{A}_n = \overline{A} < \infty$*, where*

$$\overline{A}_n = \sup_{x \in \Omega^n} \frac{a_0(x)}{a_1(x)}. \tag{5.2}$$

Then the imbedding (5.1) holds.

Proof. Since Ω is bounded and $r(x) \le d(x)/3$, $x \in \Omega^{n_0}$, it follows that

$$r^p(x_k) \le (\mathit{diam}\,\Omega / 6)^p, \quad k \in \mathbb{K}_n.$$

Using this in (4.8), we have

$$\int_{B_k} |u(y)|^p w(y)dy + \int_{B_k} |\nabla u(y)|^p w(y)dy$$

$$\leq a_0(x_k)K^p\Big(\int_{B_k} |u(y)|^p dy$$

$$+(K^{-p} + (diam\,\Omega\,/\,6)^p)\int_{B_k} |\nabla u(y)|^p dy\Big)$$

$$\leq a_0(x_k)K_3\left(\int_{B_k} |u(y)|^p dy + \int_{B_k} |\nabla u(y)|^p dy\right)$$

$$\leq \frac{a_0(x_k)}{a_1(x_k)}K_4\left(\int_{B_k} |u(y)|^p v(y)dy + \int_{B_k} |\nabla u(y)|^p v(y)dy\right)$$

using $(\overline{4.2})$, where $K_4 = \max(K^p, 1 + K^p(diam\,\Omega\,/\,6)^p)$. Now the proof follows on the lines of that of Theorem 1.

Gurka and Opic [3] have given necessary and sufficient conditions for the imbedding

$$W^{1,p}(\Omega, v, v) \mapsto L^q(\Omega, w) \tag{5.3}$$

to hold. In fact they proved the following result:

Theorem A. *Let* $1 \leq p \leq q < \infty$ *and* $\frac{1}{N} \geq \frac{1}{p} - \frac{1}{q}$. *Let* $\overline{\textbf{S2}}$ *and the following conditions be satisfied:*

S1'. $W^{1,p}(\Omega_n, v, v) \mapsto L^q(\Omega_n, w), \quad n \geq n_0.$

S5'. $\lim_{n\to\infty} B_n = B < \infty$, *where*

$$B_n = \sup_{x\in\Omega^n} \frac{a_0^{1/q}(x)}{a_1^{1/p}(x)} r^{\frac{N}{q}-\frac{N}{p}}(x).$$

Then the imbedding (5.3) is satisfied.

In what follows, we show the existance of the imbedding (5.3) under a different set of conditions than that given in Theorem A.

Theorem 6. *Let* p, q *be as in Theorem A. Let* **S3** *with* $v_0 = v_1$ *alonwith* **S1'**, $\overline{\textbf{S2}}$ *and the following condition be satisfied:*

$$\lim_{n\to\infty} B'_n = B' < \infty, \textit{ where}$$

$$B'_n = \sup_{x\in\Omega^n} \frac{a_0^{1/q}(x)}{a_1^{1/p}(x)} r^{\frac{N}{q}-\frac{N}{p}}(x) \tag{5.4}$$

Then the imbedding (5.3) holds.

Proof. Using **S3** with $v_0 = v_1$ and $\overline{\mathbf{S2}}$, we have

$$\int_{B_k} |u(y)|^q w(y)dy \leq a_0(x_k) \int_{B_k} |u(y)|^q dy$$

$$\leq \left[K a_0^{1/q}(x_k) r^{\frac{N}{q}-\frac{N}{p}+1}(x_k)\right]^q$$

$$+ \left[r^{-p}(x_k) \int_{B_k} |u(y)|^p dy + \int_{B_k} |\nabla u(y)|^p dy\right]^{q/p}$$

$$\leq \left[K_5 \frac{a_0^{1/q}(x_k)}{a_1^{1/p}(x_k)} r^{\frac{N}{q}-\frac{N}{p}+1}(x_k)\right]^q$$

$$+ \left[\int_{B_k} |u(y)|^p dy + \int_{B_k} |\nabla u(y)|^p dy\right]^{q/p},$$

where $K_5 = K(\max(K_0, 1))^{1/p}$. Now, the proof follows on the same lines as that of Theorem 1. ■

6. COMPACT IMBEDDINGS

We discuss below results concerning compact imbeddings. Since the proofs of these results are obtained following the lines as in Sections 4 and 5, we omit the details for conciseness.

Theorem 7. *Let* **S2**, **S3**, **S4** *and the following conditions be satisfied:*
S1* $W^{1,p}(\Omega_n, v_0, v_1) \mapsto\!\!\!\mapsto W^{1,p}(\Omega_n, w), \quad n \geq n_0.$
S5* $\lim_{n\to\infty} A_n = 0$, *where* A_n *is given by (4.3).*
Then, we have the imbedding

$$W^{1,p}(\Omega, v_0, v_1) \mapsto\!\!\!\mapsto W^{1,p}(\Omega, w). \tag{6.1}$$

Proof. It is obtained in view of Lemma 2 if we use **S5*** in the inequality (4.10). ■

Theorem 8. *Let* **S4**, **N2**, *and* **N3** *and the following condition be satisfied:*
N5* $\lim_{n\to\infty} \hat{A}_n > 0$, *where* $\hat{A}_n$ *is given by (4.12).*
Then the space $W^{1,p}(\Omega, v_0, v_1)$ *is not compactly imbedded in the space* $W^{1,p}(\Omega, w)$.

Proof. Setting $\hat{u}_k = u_k / \|u_k\|_{1,p,\Omega,v_0,v_1}$, $k \in \mathbb{N}$, where u_k are the functions given by (4.14), and working on the lines as in the proof of Theorem 2 with $\hat{u}_k$ instead of u_k, we get the assertion. ■

Theorem 9. *Let* **S1***, *and* **S3** *with* $v_0 = v_1 = v$ *alongwith* $\overline{\mathbf{S2}}$ *and* **S5*** *be satisfied. Then we have the imbedding*

$$W^{1,p}(\Omega, v, v) \mapsto\!\!\!\mapsto W^{1,p}(\Omega, w). \tag{6.2}$$

Theorem 10. *Let* **N2** *with* $v_0 = v_1 = v$ *alongwith* **N5*** *be satisfied then the space* $W^{1,p}(\Omega, v, v)$ *is not compactly imbedded in the space* $W^{1,p}(\Omega, w)$.

Theorem 11. *Let* **S1*** *with* $v_0 = v_1 = v$ *alongwith* $\overline{\textbf{S2}}$ *and the following condition be satisfied:*

$\overline{\textbf{S5*}}$ $\lim_{n\to\infty} \overline{A}_n = 0$, *where* A_n *is given by (5.2).*

Then the imbedding (6.2) holds.

Remark 4. As in the case of Theorems 1 and 2 (and similarly Theorems 3 and 4), here too, it is open to seal the gap between the sufficient conditions and the necessary conditions given in Theorems 7 and 8 (and also Theorems 9 and 10). Theorems 9 and 10 are the special cases of Theorems 7 and 8, respectively, which have been derived under weaker hypothesis analogous to Theorems 3 and 4. The hypothesis in Theorem 9 is further weakened to result in Theorem 11.

Analogous to Theorem 6, a result can be derived giving the sufficient conditions for the imbedding

$$W^{1,p}(\Omega, v, v) \mapsto\mapsto L^q(\Omega, w) \tag{6.3}$$

to hold. However, this imbedding has already been shown to exist by Gurka and Opic ([3], Theorem 6.3) but under a different set of conditions. We formally state the result and omit the details.

Theorem 12. *Let* p, q *be as in Theorem A. Let* **S3** *with* $v_0 = v_1 = v$ *along with* $\overline{\textbf{S2}}$ *and the following condition be satisfied:*

S1*'. $W^{1,p}(\Omega_n, v, v) \mapsto\mapsto L^q(\Omega_n, w), \quad n \geq n_0.$

S5*'. $\lim_{n\to\infty} B'_n = 0$, *where* B'_n *is given by (5.4).*

Then the imbedding (6.3) holds.

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