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A supplement to the Alexandrov–Lester Theorem

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Abstract. Let V be the 4-dimensional Minkowski-space of special relativity over the reals with quadratic form Q. Consider a mapping $\psi\colon V \to V$ such that $Q(x-y)=0 \quad \Leftrightarrow \quad Q(x\psi - y\psi) = 0$ for all $x, y \in V$. Under the assumption that ψ is a bijection Alexandrov's theorem states that ψ is a linear bijection followed by a translation. Our results imply (as a special case) that the assumption of ψ being a bijection can be dropped.

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Introduction and result

Let V be a vector space over a field K endowed with a quadratic form Q and let $f\colon V \times V \to K$, $f(x,y) := Q(x+y) - Q(x) - Q(y)$ denote the associated symmetric bilinear form. Hence $f(x,x) = 2Q(x)$.

We assume that $V^\perp = \{x \in V \mid f(x,y) = 0 \quad \text{for all} \quad y \in V\} = \{0\}$ and V is isotropic, *i.e.* $Q(v) \neq 0$ for some $v \in V \setminus \{0\}$ and $\dim V \geq 3$.

Also, we assume for convenience from the beginning that V is finite-dimensional.

For any $a \in V$ let $C_a := \{a + v \mid v \in V, \quad Q(v) = 0\}$ denote the 'cone' with center a. Each cone has a unique center. The set of isotropic vectors of V is $C := C_0$.

For a mapping $\psi\colon V \to V$ we study the property

$$\text{(d)} \quad Q(x-y) = 0 \quad \Leftrightarrow \quad Q(x\psi - y\psi) = 0$$

for all $x, y \in V$.

Using cones the property can be rephrased: $y \in C_x$ if and only if $y\psi \in C_{x\psi}$, for all $x, y \in V$.

We quote a by now classical theorem.

1 Theorem (A. D. Alexandrov [1] and J. Lester [6]). *Let $\psi\colon V \to V$ be a bijection satisfying property* (d). *Then there is a semilinear bijection $\varphi\colon V \to V$ (with associated automorphism* $^-$ *of K) and a number $\lambda \in K \setminus \{0\}$ such that*

(1) $\psi = \varphi\tau$ where $\tau\colon V \to V$ is the translation $v \mapsto v + 0\psi$,

(2) $Q(x\varphi) = \lambda \cdot \overline{Q(x)}$ for all $x, y \in V$.

The assertion holds true even when V has infinite dimension. A brief proof to the theorem was published by E. M. Schröder [7].

We want to drop the assumption that ψ is a bijection and use only property (d). We shall see that injectivity of ψ can be proved without further assumptions. However, surjectivity seems to be a major problem and our proof of the following main result requires several arguments.

2 Theorem. *Suppose that K is a field such that each monomorphism of K is surjective.*

Let $\psi\colon V \to V$ satisfy

$$\text{(d)} \quad Q(x-y) = 0 \quad \Leftrightarrow \quad Q(x\psi - y\psi) = 0$$

for all $x, y \in V$.

Then one has a semilinear bijection $\varphi\colon V \to V$ (with associated automorphism $^-$ *of K) and a number $\lambda \in K \setminus \{0\}$ such that*

(1) $\psi = \varphi\tau$ where $\tau\colon V \to V$ is the translation $v \mapsto v + 0\psi$,

(2) $Q(x\varphi) = \lambda \cdot \overline{Q(x)}$ for all $x, y \in V$.

3 Remark. The assumption that each monomorphism of K is surjective is necessary. Indeed, if $e_1, \dots, e_n$ is a basis for V such that $f(e_i, e_j)$ is in the prime field and $^-$ is a monomorphism of K then the mapping $V \to V$, $\lambda_1 e_1 + \cdots + \lambda_n e_n \mapsto \overline{\lambda_1} e_1 + \cdots + \overline{\lambda_n} e_n$ satisfies property (d). Also the property $\dim V < \infty$ cannot simply be deleted.

In the sequel we will prove the above result.

1 Basic arguments and propositions

A subvectorspace is called isotropic if it is spanned by its isotropic vectors.

A subset $M \subseteq V$ is called a null set if $Q(x - y) = 0$ for all $x, y \in M$. Hence an affine subspace $a + U$ (where $a \in V$ and U is a subvectorspace of V) is a null set if and only if $Q(u) = 0$ for all $u \in U$.

A null subvectorspace U is also called a totally isotropic subvectorspace. This means that $Q(u) = 0$ for each $u \in U$.

Notation For $c \in V$ let $\tau_c \colon V \to V$, $v\tau_c = v + c$ denote the translation assigned to c.

For each mapping $\psi \colon V \to V$ and $a \in V$ let $\psi_a := \tau_a \psi \tau_{-a\psi}$.

Observe that $0\psi_a = 0$ for each $a \in V$ and $V\psi_a = V\psi - a\psi$. If ψ satisfies property (d) then ψ_a satisfies also (d).

Further, ψ is injective (surjective) if and only if ψ_a is injective (surjective).

If ψ fulfills $(x+y)\psi = x\psi + y\psi$ for all $x, y \in V$ then $\psi_a = \psi$ for each $a \in V$.

4 Lemma. *Let $\psi \colon V \to V$ be a mapping with the property*

$$\text{(c)} \quad C_{a\psi} \subseteq V\psi$$

*for each $a \in V$ (*i.e. *if the image of ψ contains the point $a\psi$ then it contains the cone whose center is $a\psi$). Then ψ is surjective.*

Proof. For each $b \in V$ the mapping ψ_b satisfies also property (c). Let $T := V\psi_0$. Then $0 \in T$ and $C_a \subseteq T$ for each $a \in T$. This implies $T = V$, hence ψ_0 and also ψ is surjective. $\boxed{QED}$

5 Proposition. *Let $\psi \colon V \to V$ satisfy property* (d) *(see introduction). Then ψ is injective.*

Proof. Let $x, y \in V$ such that $x\psi = y\psi$. Put $\varphi := \psi_y$. Then φ satisfies (d) and $0\varphi = 0 = a\varphi$ where $a := x - y$. As $0 = Q(0) = Q(a\varphi) = Q(a\varphi - 0\varphi)$ property (d) implies $a \in C$. Each $b \in C$ fulfills $Q(a\varphi - b\varphi) = Q(0 - b) = 0$, hence $Q(a - b) = 0$ and thus $b \in a^\perp$. As $\langle C \rangle = V$ we conclude that $V \subseteq a^\perp$, hence $a = 0$ and $x = y$. $\boxed{QED}$

6 Corollary. *If ψ fulfills properties (d) and (c) then ψ is a bijection.*

A physical interpretation The vectors of V are results of physical observations (measurements) obtained by an observer A; each result $w \in V$ corresponds to an 'event'. A second observer B obtains $w\psi$ when he applies his device to the same event.

Observer A watches a lightray joining two events if and only if the corresponding measurements fulfill $Q(x-y) = 0$. The analogue interpretation applies to observer B. Then (d) claims: A watches a lightray joining a pair of events if and only if B watches a lightray joining the events.

Property (d) implies (due to the previous proposition): if observer A distinguishes two events with his device then B will also obtain distinct results for the respective events.

Surjectivity of ψ means that both the devices of A and B have the same range.

Assumptions In the sequel let $\psi\colon V \to V$ satisfy (d).

Hence ψ is injective by the previous proposition 5.

We will **not assume surjectivity** of ψ.

The mapping $\varphi := \psi_0$ satisfies (d) and $0\varphi = 0$ and it is also injective. In the sequel we focus on φ.

7 Lemma. *(1) If $S \subseteq V$ is a null set then $S\varphi$ is also a null set.*

(2) A maximal null set of V is an affine subspace (maximal refers to $\subseteq$). A maximal null set of V containing 0 is a sub-vectorspace (of dimension $\operatorname{ind}(V, f)$, the Witt-index of (V, Q)).

(3) Let T be an affine null subspace. Then $T = \bigcap S$ where S is the set of all maximal null subsets of V containing T.

The proofs are straightforward.

8 Proposition. *Let U and W be null subvectorspaces of V such that $U \subset W$ (properly contained). Then $\langle U\varphi\rangle \subset \langle W\varphi\rangle$ (also properly contained).*

Proof. We may assume that $W = U \oplus \langle w\rangle$, $w \in W$. Clearly, $\operatorname{rad}(U^\perp) := U \cap U^\perp = U$ and $w \in U^\perp \setminus (\operatorname{rad}(U^\perp))$ is an isotropic vector. Hence $U^\perp$ is spanned by its isotropic vectors. Further, $U^\perp \not\subseteq w^\perp$. Hence there is some isotropic vector $v \in U^\perp \setminus w^\perp$. This implies $f(u, v) = 0 \neq f(w, v)$ for all $u \in U$, hence $Q(u-v) = 0 \neq Q(w - v)$ and by (d) $Q(u\varphi - v\varphi) = 0 \neq Q(w\varphi - v\varphi)$. Thus $f(u\varphi, v\varphi) = 0 \neq f(w\varphi, v\varphi)$ for all $u \in U$. We proved that $\langle U\varphi\rangle \subseteq v\varphi^\perp$ and $w\varphi \notin v\varphi^\perp$. So $w\varphi \notin \langle U\varphi\rangle$ and $W\varphi \not\subseteq \langle U\varphi\rangle$. $\boxed{QED}$

9 Corollary. *Let U be a null subvectorspace of V. Then $\dim\langle U\varphi\rangle = \dim U$. In particular, $\langle v\rangle\varphi \subseteq \langle v\varphi\rangle$ for each $v \in C$.*

Proof. Consider a proper chain $U_0 = O \subset U_1 \subset \cdots \subset U_k$ of maximal length consisting of null subvectorspaces of V where $U = U_j$ for some j. By the previous proposition $\langle U_0\varphi\rangle = O \subset \langle U_1\varphi\rangle \subset \cdots \subset \langle U_k\varphi\rangle$ is a proper chain of totally isotropic subspaces. Hence

$$k \leq \dim\langle U_k\varphi\rangle \leq \operatorname{ind}(V, f) = k$$

and therefore $\dim\langle U_i\varphi\rangle = i = \dim U_i$. - Our proof needs that $\operatorname{ind}(V, f)$ is finite. $\boxed{QED}$

10 Corollary. *Let $\Gamma = a + \langle v\rangle$ be an affine null line. Then $\Gamma\varphi \subseteq a\varphi + \langle v\varphi_a\rangle$.*

Proof. As φ_a satisfies the assumptions used for φ (namely (d) and $0\varphi = 0$) the above corollary holds also true for φ_a. We conclude $\Gamma\varphi = a\varphi + \langle v\rangle\tau_a\varphi\tau_{-a\varphi} = a\varphi + \langle v\rangle\varphi_a \subseteq a\varphi + \langle v\varphi_a\rangle$. $\boxed{QED}$

11 Lemma. *Let $v \in C$ (*i.e.* v is isotropic) and $w \in V$.*

(1) If $w \in C$ then

$$w \perp v \quad \Leftrightarrow \quad w\varphi \perp v\varphi$$

(2) If $w \notin C$ then

$$w \perp v \quad \Leftarrow \quad w\varphi \perp v\varphi$$

Proof. (1) We have $w \in v^\perp \Leftrightarrow Q(v-w) = 0 \Leftrightarrow Q(v\varphi - w\varphi) = 0 \Leftrightarrow w\varphi \in v\varphi^\perp$, where we take into account that $v\varphi$ and $w\varphi$ are isotropic vectors.

(2) Clearly, $w \notin v^\perp \Leftrightarrow Q(\lambda v - w) = 0$ for some $\lambda \in K \Leftrightarrow Q((\lambda v)\varphi - w\varphi) = 0$ for some $\lambda \in K$.
Now $(\lambda v)\varphi \in \langle v\rangle\varphi \subseteq \langle v\varphi\rangle$ by the previous corollary. Hence the last statement in the above chain of equivalences implies that $Q(\mu(v\varphi) - w\varphi) = 0$ for some $\mu \in K$. This holds true if and only if $w\varphi \notin v\varphi^\perp$.

$\boxed{QED}$

12 Corollary. *Let U be a totally isotropic subvectorspace of V. Then $\varphi|_U : U \to \langle U\varphi\rangle$ is an injective mapping of U into $\langle U\varphi\rangle$ such that $\dim U = \dim\langle U\varphi\rangle$ and $\varphi|_U$ preserves collinearity (given in the affine spaces assigned to U respectively to $\langle U\varphi\rangle$).*

This is an immediate consequence of the previous two corollaries.

We will use the following theorem (see e.g [4] page 104, A.3.1) that is based on a more general result by H. Schaeffer for 2-dimensional vector spaces.

Affine Collineation Theorem. Let U and W be n-dimensional vector spaces over the same field K (n an integer ≥ 2). Suppose that each monomorphism of K is surjective and $|K| \geq 3$.

Let $\alpha\colon U \to W$ be an injective mapping that preserves collinearity and such that $U\alpha$ is not contained in a hyperplane of W and $0\alpha = 0$.

Then α is is a semilinear bijection; *i.e.* there is an automorphism $\bar{\ }$ of K such that $(x+y)\alpha = x\alpha + y\alpha$ and $(\lambda x)\alpha = \bar{\lambda}(x\alpha)$ for all $x, y \in V$ and $\lambda \in K$.

If K is finite then the mapping φ under investigation is a bijection. Hence the condition $|K| \geq 3$ is irrelevant in our arguments.

The only monomorphism of $\mathbb{R}$ is the identity, hence the Affine Collineation Theorem subsumes $K = \mathbb{R}$.

The previous corollary and the Affine Collineation Theorem yield

13 Lemma. *Suppose that each monomorphism of K is surjective.*

Let U be a totally isotropic subvectorspace of V such that $\dim U \geq 2$. *Then $U\varphi$ is a totally isotropic subvectorspace of V such that* $\dim(U\varphi) = \dim U$ *and the restriction $\varphi|_U \colon U \to U\varphi$ is a semilinear bijection.*

14 Corollary. *Suppose that each monomorphism of K is surjective and* $\mathrm{ind}(V, f) \geq 2$. *Then $\langle v\rangle\varphi = \langle v\varphi\rangle$ for each isotropic vector v.*

Proof. Select a 2-dimensional totally isotropic sub-vectorspace of V containing v and apply the previous lemma. $\boxed{QED}$

15 Lemma. *Suppose that $\langle v\rangle\varphi = \langle v\varphi\rangle$ for each $v \in C$. Then all $v \in C$ and $w \in V$ satisfy*

$$(\perp) \quad v \perp w \quad \Leftrightarrow \quad v\varphi \perp w\varphi$$

In particular, $v^\perp\varphi = v\varphi^\perp \cap V\varphi$.

Proof. We must only consider the case $'\Rightarrow'$ when w is anisotropic; cf. 11. If the right-hand statement is not true then $Q(\mu(v\varphi) - w\varphi) = 0$ for some $\mu \in K$. The assumption $\langle v\rangle\varphi = \langle v\varphi\rangle$ yields that $\mu(v\varphi) = (\lambda v)\varphi$ for some $\lambda \in K$, hence $Q((\lambda v)\varphi - w\varphi) = 0$ and by (d) $Q(\lambda v - w) = 0$. This implies $v \not\perp w$. $\boxed{QED}$

16 Lemma. *Let U and W be subvectorspaces of V such that $U^\perp$ is isotropic and $U \subset W$ (properly contained). If $(\perp)$ (for $v \in C$ and $w \in V$) of the previous lemma holds true then $\langle U\varphi\rangle$ is properly contained in $\langle W\varphi\rangle$.*

Proof. We may assume that $W = U \oplus \langle w\rangle$. Then $U^\perp \not\subseteq w^\perp$. As $U^\perp$ is spanned by a set of isotropic vectors we find an isotropic vector $v \in U^\perp \setminus w^\perp$. Hence $u \perp v$ and $w \not\perp v$ for all $u \in U$. Now $(\perp)$ implies that $u\varphi \perp v\varphi$ and $w\varphi \not\perp v\varphi$ for all $u \in U$ as v is isotropic. Hence $\langle U\varphi\rangle \subseteq v\varphi^\perp$ but $w\varphi \in W\varphi \setminus v\varphi^\perp$. This implies $W\varphi \not\subseteq \langle U\varphi\rangle$. As $\langle U\varphi\rangle \subseteq \langle W\varphi\rangle$ the assertion follows. $\boxed{QED}$

17 Corollary. *Let U be a subvectorspace of V such that $U^\perp$ is isotropic. Suppose that* $(\perp)$ *holds true. Then* $\dim\langle U\varphi\rangle \leq \dim U$.

Proof. Select a basis $z_1, \ldots, z_m$ of isotropic vectors for $U^\perp$. Put $Z_i := \langle z_1, \ldots, z_i\rangle$ for $i \in \{0, \ldots, m\}$ and $W_i := Z_i^\perp$. We obtain a proper chain $U = W_m \subset \cdots \subset W_0 = V$ such that each $W_i^\perp = Z_i$ is isotropic. The previous lemma yields that $\langle U\varphi\rangle = \langle W_m\varphi\rangle \subset \cdots \subset \langle W_0\varphi\rangle = \langle V\varphi\rangle$. Hence $\dim\langle U\varphi\rangle \leq \dim U$. $\boxed{QED}$

18 Remark. We want to prove that (under the assumptions of our theorem) φ is a collineation.

Since $0\varphi = 0$ this implies that φ is a semilinear bijection of V.

Having accomplished this result standard arguments (which we will not repeat) provide a number $\lambda \in K \setminus 0$ such that $Q(x\varphi) = \lambda \cdot \overline{Q(x)}$ for all $x \in V$, and the proof of our theorem is finished.

2 The case of $\operatorname{ind}(V, f) \geq 2$

19 Lemma. *Suppose that K is a field such that each monomorphism of K is surjective and* $\operatorname{ind}(V, f) \geq 2$.

Then $\langle w \rangle \varphi \subseteq \langle w\varphi \rangle$ for each vector $w \in V$.

For each affine line $a + \langle w \rangle$ one has $(a + \langle w \rangle)\varphi \subseteq a\varphi + \langle w\varphi_a \rangle$. In particular, φ preserves collinearity in the affine space over V.

Proof. As $w^\perp$ is always isotropic the first assertion follows from the previous corollary. As to the second one we observe that φ_a fulfills the requirements imposed on φ and thus the assertion of the previous corollary. Hence $(a + \langle w \rangle)\varphi = a\varphi + \langle w \rangle \varphi_a \subseteq a\varphi + \langle w\varphi_a \rangle$. $\boxed{QED}$

Consider the proof of 17 in the case $U = 0$, hence $U^\perp = V$ and $m = \dim V$. Then

$$0 = \langle U\varphi \rangle = \langle W_m\varphi \rangle \subset \cdots \subset \langle W_0\varphi \rangle = \langle V\varphi \rangle$$

is a proper chain of sub-vector spaces of length m. Hence $\langle V\varphi \rangle = V$. This observation and the previous corollary provide the requirements of the Collineation-Theorem. Thus we proved

20 Proposition. *Suppose that K is a field such that each monomorphism of K is surjective and* $\operatorname{ind}(V, f) \geq 2$.

Then $\varphi\colon V \to V$ is a semilinear bijection.

We have proved our theorem when $\operatorname{ind}(V, f) \geq 2$.

3 The case of $\operatorname{ind}(V, f) = 1$

We assume that K is infinite; else injectivity of φ implies surjectivity and our main result follows.

Further, we assume that each monomorphism of K to K is an automorphism.

21 Definition. A set X is called admissible if $X \subseteq V \setminus C$ and $|X| \geq 2$ and X is contained in an affine null line that intersects C.

If $X \subseteq V$ is collinear and $|X| \geq 2$ let $[X]$ denote the unique affine line containing X.

If X is admissible then the affine line $[X]$ is a null line and it intersects C in a unique point $p(X) \in C$.

We call admissible sets X, Y parallel if the lines $[X], [Y]$ are parallel.

Let $\Gamma = a + \langle v \rangle$ be an affine null line.

Then $\Gamma \subseteq C$ or $\Gamma \cap C = \emptyset$ or $|\Gamma \cap C| = 1$. The property $|\Gamma \cap C| = 1$ is equivalent to $f(a, v) \neq 0$.

22 Remark. (1) Let $\Gamma = a+\langle v\rangle$ be an affine null line such that $|\Gamma\cap C| = 1$ (*i.e.* $f(a,v) \neq 0$). Put $X := \Gamma \setminus C$. Then X is admissible and $[X] = \Gamma$ and $p(X) = \Gamma \cap C$.

(2) If X is admissible then $X\varphi$ is admissible.

23 Definition. Let $\rho\colon V \setminus C \to V \setminus C$, $v \mapsto Q(v)^{-1}v$.

In a Euclidean plane, ρ is the reflection at the unit circle.

24 Lemma. *(1) Let $\Gamma = a + \langle v\rangle$ be an affine null line.*

If $\Gamma \cap C = \emptyset$ then $\Gamma\rho = Q(a)^{-1}a + \langle v\rangle$.

If $|\Gamma \cap C| = 1$ then we can assume that $a \in C$ and have

$$(\Gamma \setminus C)\rho = [(f(a,v))^{-1}v + \langle a\rangle] \setminus \{(f(a,v))^{-1}v\}$$

(2) Let X be admissible. Then $X\rho$ is admissible.

(3) Let X and Y be admissible. Then X is parallel to Y if and only if $p(X\rho)$, $p(Y\rho)$, 0 are collinear.

Proof. Statement 1 follows from the definition of ρ.

Statement 2 follows immediately from 1.

We prove 3.

Put $\Gamma := [X] = a + \langle v\rangle$ and $\Sigma := [Y] = b + \langle w\rangle$ where $a, b, v, w \in C$.

Statement 1 reveals that the following statements are equivalent.

X is parallel to Y; Γ is parallel to Σ ; $\langle v\rangle = \langle w\rangle$; $p(X\rho), p(Y\rho), 0$ are collinear.

Put $\omega := \rho\varphi\rho\colon V \setminus C \to V \setminus C$. Clearly ω is injective and if X is admissible then $X\omega$ is also admissible. $\boxed{QED}$

25 Lemma. *Let X and Y be admissible. If X is parallel to Y then $X\omega$ is parallel to $Y\omega$.*

Proof. Let X be parallel to Y. Then $p(X\rho), p(Y\rho), 0$ are collinear (previous lemma b)). There are (unique) affine null lines Γ , Σ containing $X\rho$ respectively $Y\rho$, and $\Gamma \cap C = \{p(X\rho)\}$, $\Sigma \cap C = \{p(Y\rho)\}$. Now $(X\rho \cup \{p(X\rho)\})\varphi \subseteq \Gamma\varphi$ and (as Γ is a null line) the image $\Gamma\varphi$ is contained in a (unique) null line that intersects C in a single point. Also the point $p(X\rho)\varphi$ is contained in this line and in C. Hence

(j) $p(X\rho\varphi) = (p(X\rho))\varphi$. Also $p(Y\rho\varphi) = (p(Y\rho))\varphi$.

As $p(X\rho), p(Y\rho), 0$ are collinear and the common line is a null line, the image points

(jj) $(p(X\rho))\varphi$, $(p(Y\rho))\varphi$, $0 = 0\varphi$ are collinear.

Clearly, (j) and (jj) yield that $p(X\rho\varphi)$, $p(Y\rho\varphi)$, 0 are collinear.

The previous lemma finally implies that $X\rho\varphi\rho$ is parallel to $Y\rho\varphi\rho$. $\boxed{QED}$

We proved that ω is injective and the ω-image of an admissible set is admissible and that ω preserves parallelity on the system of admissible sets.

If Γ is a null line then $(\Gamma \setminus C)\omega$ is contained in a null line.

If Γ is a null line with $|\Gamma \cap C| = 1$ then $(\Gamma \setminus C)\omega$ is contained in a null line with that property.

It is our objection to prove that ω preserves collinearity on $V \setminus C$.

26 Lemma. *Let $a, b, c \in V \setminus C$ such that $b - a, c - a \in C$. Then a, b, c are collinear if and only if $a\omega, b\omega, c\omega$ are collinear.*

Proof. We may assume that a, b, c are distinct.

Suppose that $a\omega, b\omega, c\omega$ are collinear. Hence the three points lie on a null line. Now 24, a), yields that $a\omega\rho = a\rho\varphi, b\rho\varphi, c\rho\varphi$ lie on a null line. Due to property (d) any two of the points $a\rho, b\rho, c\rho$ are joined by null lines. As $\operatorname{ind}(V, Q) = 1$ these points are collinear. Finally, 24, 1 implies that a, b, c are collinear. $\boxed{QED}$

27 Lemma. *Let H be a hyperbolic plane, $H = \langle u, v\rangle$ where $u, v \in C$, and $m \in H \setminus C$ such that $m + u, m + v \notin C$ (for example $m = u + v$ when $\operatorname{char}K \neq 2$). Let Y denote the affine plane containing $m\omega, (m + u)\omega, (m + v)\omega$. Then $(H \setminus C)\omega \subseteq Y$ and $(H \setminus C)\omega$ contains a triplet of non-collinear points.*

Proof. The points $m\omega, (m + u)\omega, (m + v)\omega$ are not collinear; cf. 26.

Let $x = m + \mu u + \nu v \in H \setminus C$. The set $\{m, m + v, m + \nu v\}$ is admissible (as it is contained in $V \setminus C$ and in the null line $a + \langle v\rangle$ that intersects C in a unique point). Hence the ω-image $\{m\omega, (m+v)\omega, (m+\nu v)\omega\}$ is admissible and in particular collinear. As two of the points lie in Y we obtain $(m + \nu v)\omega \in Y$.

The sets $\{m, m + u\}$ and $\{m + \nu v, x\}$ are admissible and parallel. Hence their images $\{m\omega, (m + u)\omega\}$ and $\{(m + \nu v)\omega, x\omega\}$ are admissible and parallel. Clearly, $\{m\omega, (m + u)\omega\} \subseteq Y$ and as we proved $(m + \nu v)\omega \in Y$ it follows that $x\omega \in Y$. $\boxed{QED}$

28 Lemma. *Let H be a hyperbolic plane, $H = \langle u, v\rangle$ where $u, v \in C$ and $A = a+H$ where $a, a+u, a+v \in V \setminus C$. Let Y denote the affine plane containing $a\omega, (a + u)\omega, (a + v)\omega$. Then $(A \setminus C)\omega \subseteq Y$.*

Proof. We may assume that $a \in H^\perp \setminus \{0\}$ (previous lemma and $Q(a) \neq 0$ as $\operatorname{ind}(V, f) = 1$). Let $x = a + \mu u + \nu v \in A \setminus C$. As the anisotropic vectors a, $a+\nu v$, $a+v$ lie on an null line their ω-images are collinear. Hence $(a+\nu v)\omega \in Y$. The admissible sets $\{a, a+u\}$ and $\{a+\nu v, x\}$ are parallel, hence their ω-images are also parallel. As $\{a\omega, (a + u)\omega\} \subseteq A$ and $(a + \nu v)\omega \in A$ this yields $x\omega \in A$. $\boxed{QED}$

29 Lemma. *Let $\Gamma = a + \langle v\rangle$ be any line. Then $(\Gamma \setminus C)\omega$ is contained in an affine line.*

Proof. We can assume that Γ is not a null line; cf. 24.

Select $a \in \Gamma \setminus C$ and hyperbolic planes H, H' such that $\langle v \rangle = H \cap H'$ (here we need $\dim V \geq 3$ and that (V, f) is isotropic). The previous lemma provides planes Y, Y' such that $((a+H) \setminus C)\omega \subseteq Y$ and $((a+H') \setminus C)\omega \subseteq Y'$. As $(\Gamma \setminus C)\omega \subseteq Y \cap Y'$ it remains to prove that $Y \neq Y'$. Assume the contrary. We write $H = \langle u, w \rangle$ and $H' = \langle u', w' \rangle$ where $u, w, u', w' \in C$ and $a+u, a+w, a+u' \notin C$. The three affine null lines joining $a\omega$ to $(a+u)\omega$ respectively to $(a+w)\omega$ respectively to $(a+u')\omega$ lie in the affine plane $Y = Y'$ and pass through a common point. As $\operatorname{ind}(V, f) = 1$ the three lines are not distinct. Suppose that *e.g.* that $(a+u)\omega, a\omega, (a+w)\omega$ lie on a common line (necessarily a null line). Then by 26 $a+u, a, a+w$ are collinear, a contradiction.

So $\omega\colon V \setminus C \to V \setminus C$ is an injective mapping with the property: if $a, b, c \in V \setminus C$ are collinear then $a\omega, b\omega, c\omega$ are also collinear. $\boxed{QED}$

The following theorem is based on results of Rolfdieter Frank.

Projective Collineation Theorem. Let Π be a projective space of finite dimension ≥ 2 over an infinite field K such that each monomorphism $K \to K$ is an automorphism.

Let $\triangle \subset \Pi$ (here Π denotes the point-set of the projective space) such that

(1) Each line Γ of Π satisfies $\Gamma \subseteq \triangle$ or $\Gamma \cap \triangle$ is finite.

Let $\omega\colon M := \Pi \setminus \triangle \to \Pi$ be an injective mapping such that

(2) If $a, b, c \in M$ are collinear then $a\omega, b\omega, c\omega$ are collinear.

(3) $M\omega$ contains a triplet of non-collinear points.

Then ω is the restriction of a collineation $\Pi \to \Pi$.

We will first apply the theorem in order to solve our problem and give a proof to the theorem in a subsequent section.

As the theorem requires a projective space we put $V' := V \times K$ and consider the projective space Π (projective extension of the affine space over V) over the K-vectorspace V'.

Put $\triangle := \{\langle (x,1) \rangle \mid x \in C\} \cup \{\langle (x,0) \rangle \mid x \in V\}$ (so $\triangle$ is the union of the affine isotropic points and the hyperplane at infinity).

Clearly, requirement 1 of the theorem holds true.

We have $M := \Pi \setminus \triangle = \{\langle (x,1) \rangle \mid x \in V, Q(x) \neq 0\}$, *i.e.* M consists of the anisotropic affine points.

Define $\omega\colon M \to M$, $\langle (x,1) \rangle \mapsto \langle (x\omega, 1) \rangle$ where the ω at the right side is the mapping defined under the first lemma of this section. So the new mapping ω corresponds (in the projective extension) to the original ω.

We proved previously that ω satisfies assumptions 2 and 3 of the theorem (29 and 26).

Hence, by the above theorem, ω admits an extension to a collineation of Π.

Let ω also denote such an extension.

Let $x \in V \setminus \{0\}$ such that $Q(x) = 0$. Select $a, b \in V \setminus C$ such that $f(a,x) \neq 0 \neq f(b,x)$ and $a - b \notin \langle x \rangle$. Then $(a + \langle x \rangle) \setminus C$ and $(b + \langle x \rangle) \setminus C$ are distinct admissible parallel sets (obtained from the two isotropic affine lines by deletion of a single point respectively) and (by 24) the ω-images are also admissible parallel sets. In projective terms, this means that the projective point $\langle (x,0) \rangle$ (the intersection of the lines $\langle (x,0) \rangle + \langle (a,1) \rangle$ and $\langle (x,0) \rangle + \langle (b,1) \rangle$) has an ω-image on the hyperplane at infinity. As this hyperplane is spanned by points of the form $\langle (x,0) \rangle$ where $x \in C$ and as ω is a collineation we proved:

(i) The collineation ω maps the hyperplane at infinity $H = \{\langle (x,0) \rangle \mid x \in V \setminus \{0\}\}$ onto the hyperplane at infinity.

Hence we may restrict ω to the affine points and we proved that the mapping $\omega \colon V \setminus C \to V \setminus C$ as defined under the first lemma of this section admits an extension to an automorphism (collineation) of the affine space over V.

Consider a projective line $\Gamma = \langle (x,0) \rangle + \langle (0,1) \rangle$ where $x \in C \setminus \{0\}$ (*i.e.* an affine null line passing through the origin together with its point at infinity). Then $\Gamma \subseteq \triangle$. Hence $\Gamma\omega^{-1} \subseteq \triangle$ (else $\Gamma\omega^{-1} \cap M \neq \emptyset$ and then $\Gamma \cap M \supseteq \Gamma \cap M\omega \neq \emptyset$, which is not true as $\Gamma \subseteq \triangle$). Further, due to (i), the line $\Gamma\omega^{-1}$ is not contained in H (hyperplane at infinity) since Γ is not contained in H ; cf. (i). This (together with the definition of $\triangle$ and $\operatorname{ind}(V,f) = 1$) implies that $\Gamma\omega^{-1} = \langle (y,0) \rangle + \langle (0,1) \rangle$ for some $y \in C \setminus \{0\}$. In particular, Γ and also $\Gamma\omega^{-1}$ pass through the point $\langle (0,1) \rangle$. As we find distinct lines of the above type we conclude

(ii) $\langle (0,1) \rangle\omega = \langle (0,1) \rangle$. In particular, if $a, b \in V \setminus C$ and the points $0, a, b$ of the affine space over V are collinear (which means that the projective points $\langle (0,1) \rangle, \langle (a,1) \rangle, \langle (b,1) \rangle$ are collinear) then $0, a\omega, b\omega$ are collinear (here ω denotes the mapping defined under 24).

Now consider distinct collinear points $0, a, b \in V$. If $a \in C$ then $0\varphi = 0$, $a\varphi$, $b\varphi$ are collinear; cf. 10. So assume that $a, b \notin C$. We have $a\varphi = a\rho\omega\rho$ and also $b\varphi = b\rho\omega\rho$. The points 0, $a\rho$, $b\rho$ are collinear (obvious from the definition of ρ). Hence (ii) yields that 0, $a\rho\omega$, $b\rho\omega$ are also collinear. This implies obviously that 0, $a\rho\omega\rho = a\varphi$, $b\rho\omega\rho = b\varphi$ are collinear. We proved

(iii) $\Gamma\varphi$ is contained in a line for each affine line Γ through $0 \in V$.

The argument used already in 10 yields that φ maps each affine line into an affine line, and the Affine Collineation Theorem implies that φ is a collineation of the affine space over V. As said at the end of section 2 this finishes the proof.

30 Remark. The 'Projective Collineation Theorem' and our subsequent arguments are only used in order to prove statement (ii) $\langle (0,1)\rangle\omega = \langle (0,1)\rangle$.

4 Proof of the Projective Collineation Theorem

In the sequel let Π be a Desarguesian projective space of projective dimension ≥ 2 (*i.e.* at least a plane) and infinite order. Let Π simultaneously denote the point set of Π.

We compile some technical tools used in [5].

Suppose that a non-trivial (*i.e.* non-discrete and more than 2 open sets exist) topology is given on each line (considered as a set of points) such that perspectivities between intersecting lines are continuous mappings. Such a system of topologies is called a linear topology on Π.

A subset $M \subseteq \Pi$ of projective points is called linearly open if for each line Γ of Π the intersection $\Gamma \cap M$ is open (in the topology given on Γ).

Let X be any set. We call $\mathbf{U} := \{U \subseteq X \mid \quad X \setminus U \text{ is finite or } U = \emptyset \quad \}$ the cofinite topology on X ($\mathbf{U}$ denotes the set of open subsets of X). If X is an infinite set then the cofinite topology is non-trivial.

Each injective mapping $X \to Y$ is continuous with respect to the cofinite topologies on X and Y.

31 Lemma. *Each line of* Π *endowed with its cofinite topology yields a linear topology on* Π*. Let* $\triangle \subset \Pi$ *be a set of points such that* $\Gamma \subseteq \triangle$ *or* $\Gamma \cap \triangle$ *is finite for each line* Γ *of* Π*. Put* $M := \Pi \setminus \triangle$*. Then* M *is linearly open.*

Indeed, the first statement follows from our previous observations. If $M \cap \Gamma$ is not open then $\Gamma \cap \triangle$ is infinite, hence $\Gamma \subseteq \triangle$ and $M \cap \Gamma = \emptyset$.

The lemma provides the basic assumption in Frank's study [5], namely that a linearly open set M is given in a projective space endowed with a linear topology.

In the sequel let the assumptions of the Projective Collineation Theorem hold true and $M \neq \emptyset$.

Let us take points $x, y, z \in M$ such that $x\omega, y\omega, z\omega$ are noncollinear (property (c)). Let Ω denote the projective plane (subspace of Π) containing x, y, z and Ω' the projective plane containing $x\omega, y\omega, z\omega$. Then Lemma 5 (e) of [5] asserts that

$$a, b, c \quad \text{collinear} \quad \Leftrightarrow \quad a\omega, b\omega, c\omega \quad \text{collinear}$$

for all $a, b, c \in M \cap \Omega$.

(In terms of [5] this means that $\omega|_{M\cap\Omega}$ is an embedding.)

Now Proposition 1 (a) of [5] supplies an extension of $\omega|_{M\cap\Omega}$ to a mapping $\omega' \colon \Omega \to \Omega'$ such that

$$a, b, c \quad \text{collinear} \quad \Leftrightarrow \quad a\omega', b\omega', c\omega' \quad \text{collinear}$$

holds true for all $a, b, c \in \Omega$. This yields that $\Omega\omega'$ is a subplane of Ω', and as K does not contain a subfield isomorphic to K we conclude that $\Omega\omega' = \Omega'$. In other words, ω' is a collineation of Ω onto Ω'.

Consider the line Γ joining x to y and let Γ' be the line joining $x\omega$ to $y\omega$. The set $M \cap \Gamma$ is open, hence $\Gamma \setminus (M \cap \Gamma)$ is finite (as $x \in \Gamma$ we have $M \cap \Gamma \neq \emptyset$). Then $\Gamma\omega' \setminus (M \cap \Gamma)\omega'$ is finite. As $\Gamma\omega' = \Gamma'$ and $(M \cap \Gamma)\omega' = (M \cap \Gamma)\omega$ we have that $\Gamma' \setminus (M \cap \Gamma)\omega$ is finite. Hence $(M \cap \Gamma)\omega$ is open (in Γ'). We proved:

There is a line Γ of Π such that $(M \cap \Gamma)\omega$ is open (in the line containing this set) and non-empty.

Now proposition 2 of [5] yields that ω is induced by a semilinear mapping $\eta\colon V \to V$.[1]

We claim that η is injective.

If kernel(η) is non-zero then kernel(η) contains a point (1-dimensional subspace of V') $z \in \Pi \setminus M$. Clearly there is a line (2-dimensional subspace of V') Γ of Π through z such that $\Gamma \not\subseteq \triangle$ (join z to some point in M). Hence $\Gamma \cap \triangle$ is finite and Γ contains two distinct (in fact infinitely many) points $r, s \in M$. As ω is injective $r\omega, s\omega$ are distinct points in $\Gamma\eta$, contradicting kernel(η) $\subseteq \Gamma$.

Thus we obtained that η is a semilinear bijection and the induced collineation extends ω. The proof of our Projective Collineation Theorem is finished.

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[1] We do not need Frank's proposition 2 in full generality. In our setting, ω maps the linearly open subset M of Π into the same projective space Π. R. Frank's proposition 2 requires only the little Desargues-axiom for Π and the assertion reads that ω is the restriction of a central projection (where the center is a projective subspace of Π) followed by an isomorphism of projective subspaces of Π. As we assume that Π is a Desarguesian projective space, a composition of such mappings is induced by a semilinear mapping of the underlying vector space.