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A Personal Overview on the Reduction Methods for Partial Differential Equations

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Abstract. This paper is devoted to a personal overview about weak symmetries of partial differential equations. The relationship between weak symmetries and overdetermined systems of partial differential is considered. On the basis of this relationship it is shown how it is possible to use the concept of weak symmetries to unify several ad hoc reduction methods for partial differential equations.

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1 Introduction

Invariance under transformations is a key feature of the entire building of classical mechanics and mathematical-physics. For example, in the phenomenological theory of continuum mechanics our knowledge on constitutive equations is mainly based on the idea of material symmetry [20]. Moreover, invariance under transformations often enables the solution to the relevant equations of mathematical-physics to be obtained by symmetry arguments. This is the universal case of scaling symmetries; indeed by dimensional analysis is well known how to reduce the general field equations of the various theories of physics to ordinary differential equations with a simple and direct method [7], [3].

In fluid dynamics one of the first examples where invariance have been used to obtain special solutions has been given by H. Blasius in 1908, [45], in searching for exact solutions of the boundary layers equations. From this first attempt, many authors have used the so called *semi-inverse method* to reduce the Navier–Stokes equations in $(3+1)$-dimensions to a system having a minor number of independent variables. Generally, a semi-inverse method is one that reduces the basic equations to equations involving fewer independently or dependent variables, or both, for a limited set of solutions. A complete list of references about the use of the semi-inverse method in fluid dynamics can be found in the wonderful article by Berker in the Handbuch der Physik, [4], and in the survey papers by Wang, [49] and [48]. Because the equations of fluid dynamics are a very complex system of nonlinear differential equations, the exact solutions obtainable by the semi-inverse method are very valuable tools for the understanding the physics of the various problems and for benchmark purposes of the numerical codes used in solving more realistic flow situations where an exact solutions is not possible. A similar situation holds also in solid mechanics and in other fields of physics. For example, the well known Saint-Venant solutions of linear elasticity, [25], and Rivlin-Ericksen universal solutions for nonlinear elasticity are indeed obtained by a semi-inverse method [43].

It is important to point out, that the semi-inverse method is based on a *trial* and *error* procedure. Any application of the semi-inverse method is *ad hoc,* and no general theory seems to be possible at least at first sight. On the other hand, it is well known that Lie group theory provides a general, algorithmic and efficient method for obtaining exact solution of partial differential equations by a reduction method that shares many similarities with the semi-inverse method. For this reason many authors have tried to find a possible relationship between the Lie's classical method of reduction and the semi-inverse method. In the book on hydrodynamics by Birkhoff, [5], I was able to find a clear statement about the possibility of such a relationship and Ericksen in studying some special solutions for rods in his *Special Topics in Elastostatics* paper of 1978, [18] declares:

> Commonly, this (*a semi-inverse method*) involves exploiting some invariance of the equations. It seems probable that, by better developing the underlying group theory, one could make the search for such methods more routine.

In this framework, the main problem is that standard Lie method of symmetry reduction is not always applicable and it has to be generalized to recover all the solutions obtainable via ad hoc reduction methods. This has been clearly pointed out by Bluman and Cole in their 1969 celebrated paper on the general similarity solution of the heat equation, where a fundamental generalization of the classical Lie algorithm, [6], denoted as the *non-classical method*, has been proposed. Olver and Rosenau in [32] show that every solution of a given partial differential equation is indeed an invariant solution under the action of at least a Lie group of point transformation. This Lie group, generally speaking, it is *not* a symmetry group of the given equation, i.e. the action of this group on the set of solutions of the differential equations does not transform *all* the solutions of the given equation again in solutions. For these groups the invariance property is enjoyed only by a proper subset of all the solutions and for this reason Olver and Rosenau introduce the definition of *weak symmetry.*

The definition of weak symmetry is not a geometric one, but it is based on the analytic properties of the *overdetermined system* composed by the given partial differential equation and the characteristic system associated with the basis vector fields of the local Lie group. The local Lie group is a weak symmetry of the given differential equation if and only if this overdetermined system admits a non trivial solution. This is a very general definition denoted by Olver and Rosenau [31] as a *Pandora box*, to underline that if now it is clear that a group theoretic nature is indeed possible for every solution of a given partial differential equation it is still unknown how to obtain the relevant groups.

In the last years there has been a continuous flow of researches on the topic of non-classical and weak symmetries and surveys of these researches are reported in Olver and Vorobev, [33], and Clarkson, [15]. In many of these studies the authors claim the introduction of new generalizations of the basic ideas of non-classical and weak symmetries. It is clear that these ideas may be naturally generalized to higher order and non-local symmetries, but has already pointed out in Olver and Rosenau [31] the key question seems to recognize that

> *the reductions method can all be unified and significantly generalized by the concept of a differential equation with* ***side condition***

and therefore the unifying theme behind finding solutions to partial differential equations by reduction

> *is* ***not****, as is commonly supposed, group theory, but rather the more analytic subject of overdetermined systems of partial differential equations.*

In this paper we shall review some of the findings by Pucci and myself in the framework of the compatibility of systems of overdetermined equations. My aim is to show that among all possible side conditions that we may *append* to a given partial differential equation to determine new symmetries there is a *special* one. This is the characteristic equation associated with the vector field of the infinitesimal generators of a Lie group (the so called *invariant surface condition*). I agree with Olver and Rosenau that to unify all the various reduction methods scattered through the literature we need to study the analytical subject of overdetermined systems of partial differential equations. On the other hand, in my opinion, the key question of which side conditions are admissible providing genuine solutions to the given differential equation drives us back into group analysis.

The plane of the paper is the following. To basic equations we devote Section 2, whereas in Section 3 we review the compatibility problem for an overdetermined system of partial differential equations. In Section 4 we consider the relationship between direct methods and the compatibility of some special overdetermined systems. In Section 5 we consider the example of Navier–Stokes equations to show that indeed it is necessary to resort to the idea of weak symmetries to recover all the solutions found by the semi-inverse method. This shows that weak-symmetries are not only of academic interest, but of practical help in solving real flow problems. The reason, because we consider the side conditions associated to a classical Lie group special is discussed in Section 6. The last Section is devoted to concluding remarks.

2 Basic equations

The starting point of our discussion is the **standard method** of group analysis due to Sophus Lie ([34], [7], [30] among others). For the sake of simplicity we restrict our attention to point symmetries and we consider a scalar partial differential in $(1+1)$ independent variables

$$\Delta(x, t, u, u^{(k)}) = 0, \tag{1}$$

with $u^{(k)}$ denoting the derivatives of the unknown function u with respect to the variables x and t up to order k.

It is well known that the graph of a solution of the differential equation (1)

$$u = f(x, t), \tag{2}$$

defines a submanifold $\mathcal{M}_f$ in $\mathcal{E} \simeq \mathbb{R}^2 \times \mathbb{R}$. If in $\mathcal{E}$ we introduce the vector field

$$\mathbf{v} = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u} \tag{3}$$

we say that (2) is invariant under the one parameter group, $G_{\mathbf{v}}$, generated by $\mathbf{v}$ if and only if $\mathcal{M}_f$ is an invariant submanifold of this group.

In the *classical theory* of Sophus Lie we say (by definition) that $G_{\mathbf{v}}$ is a *symmetry group* of the differential equation (1) when all $\mathcal{M}_f \subset \mathcal{E}$ such that $\Delta(x, t, f, f^{(k)}) \equiv 0$ is an invariant submanifold of $G_{\mathbf{v}}$. Because this definition implies that **whenever** $u = f(x, t)$ is a solution to (1) the transformed function $G_{\mathbf{v}} \circ f$ is also a solution to (1), under mild regularity conditions, it is possible to show that all the symmetry groups of a given equation may be determined by requiring

$$\mathbf{v}^{(k)}(\Delta) = 0, \quad \text{whenever} \quad \Delta = 0. \tag{4}$$

The (4) is the **classical** infinitesimal symmetry condition and in this formula $\mathbf{v}^{(k)}$ is the $k-$th prolongation of the vector field. When we substitute in (4) the explicit formulas for the coefficients of the prolongation field $\mathbf{v}^{(k)}$ the relation (4) are the **determining equations** for the symmetry groups of (1) and they form a large, **overdetermined linear system** of partial differential equations for the coefficients ξ, τ and η.

Symmetry groups of differential equations have a huge list of important applications as for example: derive new solutions from old ones, classification of invariant equation, linearization theorems, determination of conservation laws and computation of invariant solutions. Here we restrict our attention mainly to the use of symmetries to obtain solutions of partial differential equations by a **reduction method**. A reduction method is an algorithm that allows to find solutions of a given partial differential equations in $(1+1)$ dimension by reducing it to **an** ordinary differential equation or to **a system** of ordinary differential equations. To the best of my knowledge all the exact solutions that have been recorded in the literature for nonlinear partial differential equations have been obtained by a reduction method.

We know that if (2) is invariant under $G_{\mathbf{v}}$ then it must be a solution of the quasi-linear first order differential equation denoted as the *invariant surface condition*

$$Q(x,t,u,u^{(1)}) \equiv \eta(x,t,u) - \xi(x,t,u)u_x - \tau(x,t,u)u_t = 0. \tag{5}$$

The (5) is the characteristic equation associated to the Lie group. The joint solutions of the overdetermined system $\mathcal{S}$, composed by (1) and (5) are the *invariant solutions* of $\Delta = 0$ under the action of the group $G_{\mathbf{v}}$. In the case of $(1+1)$ differential equations the invariant solutions may be always determined by reducing the given partial differential equation to an **ordinary** differential equation. This is a very important statement whose proof may be given using local rectifying coordinates. A basic theorem in the theory of vector fields, [30], ensures that, away from the singular points, there exists always in $\mathcal{E}$ a local system of coordinates $(\widetilde{x},\widetilde{t},\widetilde{u})$ such that $\mathbf{v} = \partial/\partial\widetilde{x}$. If $\Delta = 0$ is invariant under the group generated by $\mathbf{v}$ in the rectifying system of coordinates $(\widetilde{x},\widetilde{t},\widetilde{u})$ it must be of the form $\Delta(\widetilde{t},\widetilde{u},\widetilde{u}^{(k)}) = 0$. Indeed we know that for the translation group it is $\mathbf{v}^{(k)}(\cdot) \equiv \partial/\partial\widetilde{x}$ for any k and therefore to ensure (4) the equation (1) can not depend explicitly on $\widetilde{x}$. In this case invariant solutions may be determined by using the obvious reduction $\widetilde{u} = g(\widetilde{t})$, i.e. considering the solution of the overdetermined system

$$\Delta(\widetilde{t},\widetilde{u},\widetilde{u}^{(k)}) = 0, \quad \widetilde{u}_{\widetilde{x}} = 0. \tag{6}$$

Now, let us consider the following equation

$$\Delta_1 \equiv u_{tt} - u_{xx} + x u_x = 0. \tag{7}$$

In this case we have (7) is not invariant with respect the group generated by the vector field $\mathbf{v}_x = \partial/\partial x$ because

$$\mathbf{v}_x^{(2)}(\Delta_1) = u_x (\neq 0), \quad \text{whenever} \quad \Delta_1 = 0, \tag{8}$$

but the overdetermined system

$$\begin{cases} u_{tt} - u_{xx} + x u_x &= 0, \\ u_x &= 0, \end{cases} \tag{9}$$

admits the invariant solution (under the action of $\mathbf{v}_x = \partial/\partial x$)

$$u(t) = c_1 t + c_2. \tag{10}$$

Bluman and Cole in their study of the similarity solutions of the linear heat equation, [6], recognize that considering instead of the condition (4) the condition

$$\mathbf{v}^{(k)}(\Delta) = 0, \quad \text{whenever} \quad \Delta = 0,\ Q = 0, \tag{11}$$

new invariant solutions with respect the one corresponding obtained by classical Lie groups analysis may be found. The symmetries associated to vector fields determined via (11) are named by Bluman and Cole **non-classical**. The simple example (10) is indeed a nonclassical symmetry of (6). We point out that to find solutions of the given partial differential equation by the reduction method described in this Section non-classical symmetries are as good as the classical ones but for other purposes the non-classical symmetries may be useless. Here I means that the reductions we obtain using classical or non-classical symmetries are equivalent (in our case a single ordinary differential equation of the same order as the original partial differential equation), but non-classical symmetries are not intimately related to partial differential equation in the sense of Felix Klein's Erlanger Program.

An important remark is that there are equations for which is possible to find invariant solutions under Lie groups that are neither classical or non-classical symmetries for the given equation. An example is given by the equation

$$\Delta_2 \equiv u_{tt} - u_{xx} + x u_x + x^2 u_{ttt} = 0.$$

For this equation (10) is clearly still a solution but neither condition (4) nor (11) are satisfied. This is a simple example of *weak symmetry*. We point out that weak symmetries are not so good as classical and non-classical symmetries to find exact solutions of partial differential equations. Indeed, when we consider a weak symmetry, the reduction method here described reduces the given equation not to *a single* ordinary differential but to an *overdetermined system* of ordinary differential equations. Then, usually, we find a minor number of invariant solutions than in the case of a classical or non-classical symmetry, because different ordinary equations share only some of their solutions.

It is clear that it is possible to find solutions by reduction considering overdetermined system not directly associated with an invariant surface condition. For example we may consider the system

$$\begin{cases} u_{tt} - u_{xx} = 0, \\ u_{xt} = 0. \end{cases} \tag{12}$$

In this case it possible to read $u_{xt} = 0$, as the invariant surface condition associated with a **generalized** symmetry, but this is in some sense a formal definition, [7]. The complex geometrical theories that introduce a rigorous definition of generalized symmetries (as for example in [47]) to the best of my knowledge have been not able to produce explicit examples of interest that may suggest the advantage to abandon the formal point of view.

When we consider systems of partial differential equations the Lie's classical method of symmetry reduction is more complicated because a special maximal rank condition has to be satisfied. If this condition is not satisfied, only recently, it has been shown that under certain conditions it is possible to find again with a different method the invariant solutions. For the sake of simplicity in this note we consider only the scalar case.

3 Compatibility of overdetermined systems

The main point in the program by Olver and Rosenau [31, 32] is the study of the compatibility problem between the given differential equation and the invariant surface condition (5). In this Section we shall consider this point in details. We recall that the systematic study of system $\mathcal{S}$ has been reported in a direct way by Pucci and Saccomandi in [37] and by Vorob'ev (see the references in [33]) using the idea of modules of partial symmetries. We point out that Pucci and Saccomandi have worked out the details only for a single partial differential equation and for a generalization to any system in any number of variables, to the best of our knowledge we have to refer to the paper by Seiler [46].

The definition of compatibility here used is very simple and natural: we say, that for a given choice of ξ, τ and η the system $\mathcal{S}$ is compatible if there exists at least one solution shared between equations (1) and (5). Before considering the general theory let us consider an example. The overdetermined system

$$\left\{ \begin{array}{rcl} u_t - uu_x & = & 0, \\ \\ \eta(x,t,u) - \xi(x,t,u)u_x - \tau(x,t,u)u_t & = & 0, \end{array} \right. \tag{13}$$

is equivalent (when $\xi + \tau u \neq 0$) to

$$\left\{ \begin{array}{rcl} u_t & = & \frac{u\eta}{(\xi+\tau u)} \\ \\ u_x & = & \frac{\eta}{(\xi+\tau u)} \end{array} \right. . \tag{14}$$

To study if this system is compatible we use the Lagrange-Charpit method, [17]. Therefore we can easily deduce the compatibility (or integrability) condition as

$$D_x \left(\frac{u\eta}{(\xi+\tau u)} \right) + D_t \left(\frac{\eta}{(\xi+\tau u)} \right) = 0, \tag{15}$$

(here D_x and D_t are total derivatives respectively with respect x and t) i. e. when (14) holds

$$(\eta_x u - \eta_t)(\xi + \tau u) + \eta(\eta + \xi_x u - \xi_t + (\tau_x u - \tau_t) u) = 0. \tag{16}$$

We point out that

$$\mathbf{v}^{(1)}(u_t - uu_x) = 0, \quad \text{whenever} \quad u_t = uu_x, \tag{17}$$

is the relation

$$(\eta_x u - \eta_t) + u_x(\eta + \xi_x u - \xi_t + (\tau_x u - \tau_t) u) = 0, \tag{18}$$

and therefore

$$\mathbf{v}^{(1)}(u_t - uu_x) = 0, \quad \text{whenever} \quad u_t = uu_x, \ \eta = \xi u_x + \tau u_t, \tag{19}$$

is (16).

If (16) is satisfied for **any** choice of $u = u(x,t)$ solution of (13) then we have the system (13) is compatible and indeed the two equations composing the system are the same equation and therefore the system admits an infinite number of common solutions. In our example, this is the case of the choice $\eta \equiv 0$, which reduces the system (13) to

$$\left\{ \begin{array}{rcl} u_t & = & uu_x, \\ \\ (\xi + u\tau) u_x & = & 0, \end{array} \right. \tag{20}$$

and therefore (modulo trivial solutions)

$$\xi = -u\tau, \tag{21}$$

and

$$\eta - \xi u_x - \tau u_t = 0 \quad \rightarrow \quad \tau\,(uu_x - u_t) = 0. \tag{22}$$

On the other hand it is possible that (16) is **not** satisfied for any choice of $u = u(x,t)$ solution of (13) but only for **some** special choices. In this case we have to consider the compatibility of the **new** overdetermined system

$$\begin{cases} u_t + uu_x &= 0, \\ \eta(x,t,u) - \xi(x,t,u)u_x - \tau(x,t,u)u_t &= 0, \\ (\eta_x + \eta_t)\,(\xi - u) + \eta\,(\eta + \eta_u - u\eta_u - \xi_x - \xi_t) &= 0. \end{cases} \tag{23}$$

Now, only a discrete finite number of solutions are possible: the functions $u = u(x,t)$ implicitly defined by $(23)_3$. We have to check directly if these solutions are solutions of the equation $(23)_1$.

Lets go back to the general case and to study the compatibility problem in this setting. First of all we have to add to $\mathcal{S}$ the differential consequence of $Q = 0$ up to the order $k-1$, i.e. the set of all the total derivatives $D^{(k-1)}Q = 0$ and this because in the general case the given differential equation is not first order. In such a way we obtain a system equivalents to $\mathcal{S}$ with the following structure

$$\mathcal{S} \simeq \begin{cases} \Delta = 0, \\ \mathcal{S}_Q \begin{cases} Q = 0, \\ D^{(1)}Q = 0 \\ D^{(k-1)}Q = 0, \end{cases} \end{cases} \tag{24}$$

Here the sub-system $\mathcal{S}_Q$ is compatible by definition and to obtain the compatibility condition for $\mathcal{S}$ it is sufficient to pick a *kth*-order derivative from $\Delta = 0$ and then a *suitable kth*-order derivative derivative from $\mathcal{S}_Q$. The relation obtained by cross differentiation of these two derivatives is the searched integrability relation. It may be shown, [37] that this integrability relation is exactly

$$\mathbf{v}^{(k)}(\Delta) = 0. \tag{25}$$

If (25) is satisfied then the system $\mathcal{S}$ is complete in the sense that any differential consequence of this system is also an algebraic consequence and the system is integrable.

We point out that if we require that (25) is satisfied for any $u(x,t)$ solution of $\Delta = 0$ we come back to (4) and we obtain *classical symmetries*, if we require that is satisfied for any $u(x,t)$ solution of $\Delta = 0$ and $Q = 0, \ldots, D^{(k-1)}Q = 0$ we obtain the *non-classical symmetries* of Bluman and Cole. It is also clear that in this last step (25) will originates determining equations for the $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$ that are **nonlinear** and for this reason very hard to solve.

When the (25) is satisfied we have that the solutions shared by $\Delta = 0$ and $Q = 0$ may be obtained by reducing (1) to a *single ordinary differential equation.* This is for the same reason we have in the case of classical symmetries i.e the existence of a local system of rectifying coordinates.

It is also possible to require that (25) is not satisfied for any solution $u(x,t)$ and to consider the new system

$$\mathcal{S}^* \left\{ \begin{array}{l} \mathcal{S} \left\{ \begin{array}{l} \Delta = 0, \\ \mathcal{S}_Q \left\{ \begin{array}{l} Q = 0, \\ D^{(1)}Q = 0, \\ D^{(k-1)}Q = 0, \end{array} \right. \end{array} \right. \\ \mathbf{v}^{(k)}(\Delta) = 0. \end{array} \right. \tag{26}$$

The compatibility condition in this case is given by the relation

$$\mathbf{v}^{(k)} \left(\mathbf{v}^{(k)}(\Delta) \right) = 0. \tag{27}$$

Once again we may require that this relation is satisfied for any choice of $u(x,t)$ or we may iterate the above procedure defining a new system by appending to $\mathcal{S}^*$ the (27) and considering a new compatibility condition.

This procedure *must stop* at a certain step because the compatibility equation collapses to an equation of the kind

$$\mathcal{E}(x,t,u(x,t)) = 0, \tag{28}$$

in this case we have only two possibilities: the $u(x,t)$ defined implicitly by (28) **are solutions** of (1) and then the system is compatible or the $u(x,t)$ defined implicitly by (28) **are not solutions** of (1) and then the system is not compatible. After the first step when we find functions $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$ such that $\mathcal{S}$ is compatible we speak of *weak symmetries.*

At this point several remarks have to be done. First, if we recast the previous discussion in a geometrical framework it is clear that here we are using the classical Cartan-Kähler theorem. Then, it must be said that Olver and Rosenau [32] show that every solution of a given equation may be in principle obtained using the present method, but they do not clarify the different steps of the compatibility problem and they do not show the interesting property (27). Moreover, from our approach the weakness of this method is clear: there is no hope, in a general case, to solve the above determining equations for all steps. Indeed, I do not know a single non trivial example of an equation for which the method here illustrated has been completely solved.

This matter of fact opens another important problem, that in my opinion today is the central problem in the group analysis of partial differential equations. For which kind of groups and/or equations we know a priori that it is possible to completely solve some or all the above mentioned steps of the compatibility problem? It is well known that in the case of classical symmetries the determining equations are a linear system of partial differential equations where the unknowns are the components of the vector field associated with the symmetry. Because this system is linear and overdetermined, generally speaking, these determining equations can be easily solved and this also in an automatic way using computer algebra packages. When we consider non-classical or weak symmetries the general solution, for all the steps, seems to be impossible. On the other hand it is well known (see for example Olver and Vorob'ev [33] where may results are reported) that there are some special partial differential equations for which all non-classical symmetries may be determined and therefore the first compatibility step may be determined. If we analyze these *lucky* equations we discover that the corresponding determining equations for the non-classical symmetries are always in the special form

$$\sum_{i=1}^{m} \Lambda_i\left(\chi\right)\Gamma_i\left(\mu\right) = 0, \tag{29}$$

where the $\Lambda_i\left(\chi\right)$ and $\Gamma_i\left(\mu\right)$ are functions of a **different set of variables**. The determining equations in form (29), may be solved because geometrically they are orthogonality conditions. It is interesting to point out that Birkhoff, [5], discusses exactly equations (29) in his book on Hydrodynamics when he study the invariant solutions under the scaling group of the Navier–Stokes equations.

Therefore our claim is the following. The compatibility problem for the system $\mathcal{S}$ may be solved for all these steps where the determining equations may be written in the form (29). Obviously, this a only a sufficient condition, but all the determining equations for which a complete solution has been found that I have checked enjoys the condition (29). This is also the reason because methods based on the Riquier-Ritt theory of overdetermined system of partial differential equations and differential Gröbner bases are useful in solving some special systems $\mathcal{S}$, [40].

Another important fact to point out is that the method here presented have been rediscovered several time by different authors. For example in [26], Nucci has introduced the G-equations. These equations arise in the subcase of an invariant surface condition of the form

$$u_x = G(x, t, u). \tag{30}$$

In [27] Nucci discovered that if we consider the iteration of the non-classical method to (30) we may find special forms of G that are compatible with the given equation and this in an automatic way. This is a subcase of the general theory presented here, [19], but the idea of Nucci is very interesting to obtain explicit solutions of the determining equations also in very complicated situations. The interesting point about the Nucci analysis is the explicit connection between Bäcklund transformations and non-classical symmetries. Indeed non-classical symmetries of the form (30) may contain Bäcklund transformations and this is an interesting result that has been underestimated in the literature. In more recent years another paper where the results here presented are, once again, rediscovered are the **partial symmetries** introduced by Cicogna and Gaeta in [14]. The basic idea of Cicogna and Gaeta is, in some sense, to work by an inverse method. Let us fix a priori a vector field (i.e. a symmetry group) and consider its infinite prolongation (indeed, it is clear that only a finite terms are required and will appear in the computations). If we apply this prolongation to a given equation (or family of equations) and the vector field is not associated with a classical symmetry of this equation the group action will deliver a differential expression which is non zero on the manifold of the solutions. It is then possible to apply again the prolongation to the obtained differential expression and to iterate this action. If at a certain step we obtain that the prolongation applied to one of the differential expressions is zero we obtain an invariant solution. This compatibility result is obviously a byproduct of the property (27). Nevertheless the idea of Cicogna and Gaeta is smart, because it allows to classify all the equations in a given class admitting a given symmetry group (classical, non-classical or in a complete weak sense) in a direct and straightforward way.

It is unbelievable but still today in the literature we have papers that ignore the results contained in the paper by Olver and Rosenau, Pucci and Saccomandi, Vorob'ev, Seiler and so on. An example is the recent paper of Arrigo and Beckham [2] that proposes once again all the above results and ends up with the following remark [2, page 64]

> *Can the determining equations for the nonclassical symmetries of all partial differential equations be derived by imposing a condition of compatibility?*

This is a nice example of *devolution*: in a paper published in 2004 we find the position of a problem solved in 1992.

4 Direct methods and the system $\mathcal{S}$

The idea of non-classical symmetries was introduced in 1969 by Bluman and Cole, [6] but for many years was completely unnoticed and the only mention that I have recorded was in the book by W. Ames [1]. The reasons for this situation are indeed clear. The celebrated paper [6] was about the general similarity solution of the linear heat equation. In this paper, by application of the standard Lie groups tool kit, special solutions of the linear heat equation are obtained solving an overdetermined system of **linear equations** to obtain the admissible vector field. In some sense this is a strange situation, it seems that we are complicating our task: from a single linear equation we have to solve a system of linear equations. In the same paper it is shown that to find non-classical symmetries of the linear heat equation it was necessary to solve a **nonlinear system** of partial differential equations. This is the reason because the non-classical method was considered as a *monster*-method, a mathematical generalization of no practical value because we propose to solve linear equation via the solution of nonlinear systems.

The paper [31] has shown, by examples, that there was some simple situations where the method of non-classical symmetries was effective to find interesting solutions, but the true turning point was the Clarkson and Kruskal paper about the direct method, [16]. In a tour de force Clarkson and Kruskal were able to find all the similarity reduction of the Boussinesq equation from the ansatz

$$u(x,t) = U(x,t,w(z(x,t))). \tag{31}$$

This means that they were looking for a form of U and z such that by replacement of (31) into the given equation (1) one obtains a differential equation in $w(z)$. To understand how [16] ignited the interest in the paper [6] we refer to the citation of this paper. From 1970 to 1989 in the citation index we find only 44 citation for [6], whereas from 1990 to 1994 we find 58 citations. Indeed, in 1989 Levi and Winternitz, [22] shown that all the similarity reductions obtained by Clarkson and Kruskal are indeed non-classical symmetries in the sense of Bluman and Cole. The paper of Levi and Winternitz showed that there are equations for which the determining nonlinear equations for non-classical symmetries may be solved. In 1992 Pucci, [35] shows, using the method of the previous Section, that the direct method is contained in the non-classical method. The method of Bluman and Cole is indeed more general because it allows to handle the case where the similarity variable is implicitly defined. Clarkson and

Nucci, [27], have found an example of an equation for which the direct method fails to determine, for this last reason, all the similarity reductions that are possible considering the non-classical method. The result of Clarkson and Nucci is not truly important, indeed Hood [21] has shown easily how to modify the direct method to handle the case of similarity variables defined implicitly. In this framework, it is important to recall the contribution of Lou, [23], where it is noted that Clarkson and Kruskal missed to consider the case of similarity reduction that are in correspondence with vector fields of the kind defined in (30). Still today, people working with the direct method of Clarkson and Kruskal seem to ignore the Lou remarks and therefore they miss important classes of solutions. In [44] there is an example of this situation worked out for the steady-state boundary layer equations.

In any case the papers [16] and [22] have shown that the non-classical method, at least for some nonlinear equations, may be an effective method to find new class of solutions. From 1988 the Clarkson and Kruskal direct method and the Bluman and Cole non-classical method has been applied successfully to a huge number of partial differential equations. We refer to Clarkson [15] for a partial survey of these results. It is important to point out that for all these equations the ansatz (31) reduces to the special form

$$u(x,t) = \alpha(x,y) + \beta(x,t)w(z(x,t)). \tag{32}$$

This form is crucial. Indeed, when (32) is in force it easy to show that the determining equations for α, β and z are always in separated form. A similar situation is verified in the successful applications of the non-classical method: the determining equations allow to show that the vector fields must depend on a separated form with respect some of the variables and this allows to solve the problem..

The results of the previous Section are helpful to show that several direct methods are equivalent to the theory of non-classical and weak symmetries. I think interesting to report briefly on how it is possible to link a direct method to the theory of first order quasi-linear differential equations. This is the crucial step to recast a direct method as a compatibility problem for the system $\mathcal{S}$. In so doing the link with group theory is recovered considering that any quasi-linear first order differential equation may be read as a characteristic equation associated with a Lie point group of transformation.

From the theory of a single first order quasi-linear differential equation in (1+1) dimensions we know that (31) is the general solutions of the quasi-linear differential equation whose characteristic curves are the two-parameter family

$$z(x,t) = h, \quad H(x,t,u) = w, \tag{33}$$

wherein h and w are arbitrary parameters. Indeed, modulo a multiplicative factor, we have that to solve the first order partial differential equation

$$(z_t H_u)\, u_x - (z_x H_u)\, u_t + (z_x H_t - z_t H_x) = 0, \tag{34}$$

we consider the characteristic system

$$\frac{dx}{z_t H_u} = \frac{dt}{-z_x H_u} = \frac{du}{-\left(-z_x H_t + z_t H_x\right)}. \tag{35}$$

From

$$\frac{dx}{z_t H_u} = \frac{dt}{-z_x H_u}, \tag{36}$$

we obtain $z(x,t) = h$ the first integral in (33). Moreover, because

$$\frac{dt}{z_x H_u} = \frac{du}{\left(-z_x H_t + z_t H_x\right)}, \tag{37}$$

we have

$$-H_t dt + \frac{z_t}{z_x} H_x dt = H_u du, \tag{38}$$

i.e. $-H_t dt - H_x dx = H_u du$ and the second integral $H = w$ is recovered.

In such a way we have shown that any solution of the kind (31) is invariant under the action of the Lie group generated by

$$\xi = z_t H_u, \quad \tau = -z_x H_u, \quad \eta = \left(-z_x H_t + z_t H_x\right). \tag{39}$$

When

$$u(x,t) = \alpha(x,t) + \beta(x,t) w(z(x,t)). \tag{40}$$

we have that

$$z(x,t) = h, \quad \frac{u-\alpha}{\beta} = w, \tag{41}$$

and

$$\begin{aligned} &\xi = \tfrac{z_t}{\beta}, \quad \tau = -\tfrac{z_x}{\beta}, \\ &\eta = z_x \left(\tfrac{\alpha_t \beta + \beta_t (u-\alpha)}{\beta^2}\right) + z_t \left(\tfrac{\alpha_x \beta + \beta_x (u-\alpha)}{\beta^2}\right) \end{aligned} \tag{42}$$

Therefore the direct method is contained in the compatibility problem of the system $\mathcal{S}$ of the previous Section.

In [29] Olver shows that direct reductions of partial differential equations to systems of ordinary differential equations are in one-to-one correspondence with compatible differential constraints. To be more precise in [29] we find a proposition stating that there is a one-to-one correspondence between ansatz of the kind

$$u(x,t) = U(x,t,w_1(z),\dots,w_n(z)), \quad z = z(x,t), \tag{43}$$

and $n - th$ order differential constraints of the form

$$\mathbf{v}^n(u) = \Phi(x,t,u,\mathbf{v}(u),\dots,\mathbf{v}^{n-1}(u)), \tag{44}$$

where $\mathbf{v}(u)$ is a projectable vector field. This result seems to indicate generalized symmetries as the key tool to unify direct methods, but this is not truly the case. For example, in [42] it is shown that when a partial differential equation is written in conservative form the result by Olver, for $n = 2$, may be reconsidered in the framework of the potential symmetries of the given equation and therefore there is no need to consider generalized symmetries, but only to rewrite in a smart way the given partial differential equation.

We have to point out that both the method by Clarkson and Kruskal and the result by Olver stops the compatibility problem of the overdetermined system at the first step. Therefore, we are always considering non-classical point or generalized symmetries, but we never consider true weak symmetries. Direct methods related to weak symmetries are the method of quasisolutions by Rubel, [41] and the method proposed by Burde [9–11]. The equivalence of the method of quasisolutions with weak symmetries has been considered by Pucci and Saccomandi in [36]. About the interpretation from the point of view of group analysis of the method applied by Burde to the boundary layers equations there has been some confusion and many wrong ideas as explained in [44]. The methods of Rubel and Burde reduce the given partial differential equation not only to a single ordinary differential equation but also to a compatible system of overdetermined ordinary differential equations. Cianetti and Pucci have found some solutions for the Boussinesq equation invariant under weak symmetries in [13]. We have not noticed other papers where solutions invariant under weak symmetries are determined.

Another *strange* point about papers [16], [29] and other ones about direct methods is that they ignore the huge amount of literature devoted to the semi-inverse method. The only exception in this direction is the book by Ames [1]. As already pointed out in the Introduction many of the basic ideas in the effort toward the unification of the various direct methods of reduction was already contained in the book by Birkhoff [5] and the paper by Ericksen [18]. In these studies it is clear the reference to semi-inverse methods, but this cultural background has completely disappeared in recent papers.

5 Two-dimensional Navier–Stokes equations

In this Section we use the $(2+1)$-dimensional incompressible Navier–Stokes equations to show the importance of weak-symmetries in the quest for the unification of all the direct methods. Although we have introduced the theory of the compatibility of overdetermined systems for partial differential equations in $(1+1)$ variables there will be no problems in considering an equation in (2+1) variables. All our previous results are easily extended to this more general case.

It is well known that introducing the stream function $\Psi(x, y, t)$ the unsteady Navier–Stokes equations in the plane may be rewritten as a single fourth order partial differential equation

$$\begin{aligned}\Psi_{xxt} &+ \Psi_{yyt} - \Psi_x\left(\Psi_{xxy} + \Psi_{yyy}\right) + \Psi_y\left(\Psi_{xxx} + \Psi_{xyy}\right) \\ &- \nu\left(\Psi_{xxxx} + \Psi_{xxyy} + \Psi_{yyyy}\right) = 0.\end{aligned} \tag{45}$$

In the paper by Ludlow, Clarkson and Bassom [24] is possible to find a very nice survey about the history of the determination of the classical symmetry group for the Navier–Stokes equations. It is clear that the classical symmetry group for Navier–Stokes equations is not very rich, indeed the transformation that leave invariant these equations in the classical or strong sense are *trivial*. We have the usual Galilean transformations (shared by all the equations of classical physics) plus frame invariance (a property shared by many equations of continuum physics). All these groups may be guessed by physical intuition. Moreover not all the transformations associated with these groups are useful. This is clear in the case of the transformation corresponding to translational invariance of the velocity field, but also for other class of invariance. For example, using the invariance corresponding to frame indifference, a known solution is only rigidly translated and rotated, and therefore in the body the same state of stress correspond to the *old* and *new* solution. This fact has been unnoticed by several authors. For example Ames, Boisvert and Srinivasa, [8], claims that the infinite dimensional symmetry algebra of the incompressible Navier–Stokes equations may be used to transform any steady solution in a time dependent solution. The problem is that they transform the steady solution in a unsteady solution which differs from the steady one only by a rigid body motions and we know that continuum mechanics, when frame invariance is in force, is build up modulo rigid body motions (see for example [20]).

On the other hand in the literature, using the semi-inverse method, a huge list of exact solutions of the Navier–Stokes has been found. Therefore the contribution [24] where all the non-classical groups for the (45) are determined is highly valuable, but it is clear that the completion of the first step of the integrability of $\mathcal{S}$ is not sufficient to capture all the interesting solutions that have been proposed. Indeed the authors at page 234, formula (8.1), of their paper declare, with some disappoint, that some relatively simple exact steady solutions of the Navier–Stokes equations are not included in their results. These solutions are listed here

$$\begin{aligned}\Psi_1(x,y) &= ay+by^2+c\arctan\left(\tfrac{y}{x}\right),\\ \Psi_2(x,y) &= ay+by^2+c\ln\left(x^2+y^2\right),\\ \Psi_3(x,y) &= ax^2+by^2.\end{aligned} \tag{46}$$

It is a simple matter to realize that the solutions (46) are invariant under the point group of symmetries with infinitesimal generators

$$\mathbf{v}_1 = x\frac{\partial}{\partial x}+y\frac{\partial}{\partial x}+\left(ay+2by^2\right)\frac{\partial}{\partial \Psi},\quad \mathbf{v}_2 = -x\frac{\partial}{\partial x}+y\frac{\partial}{\partial x}+\left(ay+2by^2\right)\frac{\partial}{\partial \Psi} \tag{47}$$

for $(46)_1$ and $(46)_2$, whereas

$$\mathbf{v}_3 = \frac{\partial}{\partial x}+\left(\eta(u)\frac{df}{dx}\right)\frac{\partial}{\partial \Psi}, \tag{48}$$

where $\eta(u)$ and $f(x)$ are ad hoc functions, for $(46)_3$.

Indeed the general solution of

$$x\Psi_x + y\Psi_y = ay+2by^2$$

is given by

$$\Psi(x,y) = ay+by^2+F\left(\frac{y}{x}\right), \tag{49}$$

and the general solution of

$$y\Psi_y - x\Psi_x = ay+2by^2,$$

is given by

$$\Psi(x,y) = ay+by^2+G\left(x^2+y^2\right), \tag{50}$$

where F and G are arbitrary functions of their argument.

Introducing (49) in (45) we obtain

$$\nu^{-1}(2bzx^2+ax)\mathcal{E}_1+\mathcal{E}_2=0 \tag{51}$$

where

$$\mathcal{E}_1 = \left(z^2+1\right)zF''' + 2\left(3z^2+1\right)F''+6zF',$$

and

$$\begin{aligned}\mathcal{E}_2 &= \left(z^2+1\right)^2 F^{iv} + 12z\left(z^2+1\right)F''' \\ &+ 2\left(18z^2+\left(z^2+1\right)\nu^{-1}F'+6\right)F''+24zF'+4zF'^2.\end{aligned}$$

Here a prime denotes differentiation with respect to $z = y/x$.

On the other hand introducing (50) in (45), by means of some simple algebraic manipulations we obtain

$$\left[4b^2x^4+(a^2-4b^2z)x^2\right]\mathcal{E}_3^2+4\nu^2\mathcal{E}_4^2=0, \tag{52}$$

where

$$\mathcal{E}_3 = zG''' + 2G'',$$

and

$$\mathcal{E}_4 = z^2G^{iv} + 4G'''z + 2G''.$$

Here the prime denotes differentiation with respect to $z = x^2 + y^2$.

To solve for all x the equation (51) we have to solve the overdetermined system

$$\mathcal{E}_1 = 0, \quad \mathcal{E}_2 = 0.$$

The general solution of this system is clearly, up to an inessential constant,

$$F = c\arctan\left(\frac{y}{x}\right).$$

On the other hand the solution of (52) for all x is given by the overdetermined system

$$\mathcal{E}_3 = 0, \quad \mathcal{E}_4 = 0.$$

The general solution of this linear system is clearly

$$G = c\ln\left(x^2+y^2\right).$$

Therefore we have been able to recover by **weak invariance** the solutions $(46)_1$ and $(46)_2$.

The solution of

$$\Psi_x = \eta(\Psi)\frac{df}{dx} \tag{53}$$

is given by

$$\int\frac{d\Psi}{\eta(\Psi)} = f(x)+g(y),$$

and considering $\eta \equiv 1$, $f(x) \equiv ax^2$ and $g(y) \equiv by^2$ we recover $(46)_3$ as a **weak invariant** solution under

$$\mathbf{v} = \frac{\partial}{\partial x} + ax^2\frac{\partial}{\partial \Psi}.$$

Indeed the introduction

$$\Psi(x,y) = ax^2 + g(y),$$

in (45) gives

$$2axg''' + \nu g^{iv} = 0,$$

which again is not a reduction to a differential ordinary equation but to an overdetermined system of ordinary differential equations.

I think that these examples show clearly the need of weak symmetries to achieve the goal of unifying all the direct methods that have been proposed in the literature. Therefore, I think now clearly stated that stopping to the first step of compatibility of $\mathcal{S}$ is not sufficient.

6 A diffusion equation

In the Introduction I have stated that invariant surface condition associated with a Lie point group are in some sense special side conditions. In this Section using a diffusion equation I try to give an *empirical* explanation of this statement. Let us consider an overdetermined system composed by a given differential equation $\Delta = 0$ plus a side condition not of the first order. In this case, it may happens that to derive the compatibility relation for such a system it may be necessary to compute some differential consequences of the given equation. In so doing, we introduce some trivial solutions (solutions which satisfy the differential consequences of the equation but not the equation itself) and therefore it is possible to find the compatibility of the overdetermined system only in correspondence of these trivial solutions. For this reason we think that the use of first order side conditions is preferable. We remember that first order quasi-linear differential conditions are in correspondence with the invariant surface conditions of point Lie groups. This is the reason because we think that the overdetermined problem, if possible, must be recast as in the system $\mathcal{S}$.

Let us consider the second order evolution equation

$$u_t = \left(D\left(u\right)u_x^n\right)_x, \tag{54}$$

and the following class of solutions

$$u = F\left(\Phi(x)\Psi(t)\right) \tag{55}$$

a functional form that if we consider $F(\cdot) = \ln(\cdot)$ it is possible to rewrite as

$$u = \ln\Phi(x) + \ln\Psi(t) \equiv f(t) + g(x). \tag{56}$$

The solutions (56) may be characterized by the second order condition

$$u_{xt} = 0, \tag{57}$$

or the first order conditions

$$\begin{aligned} u_x &= g_x, \\ u_t &= f_t. \end{aligned} \tag{58}$$

If we transform (54) by the transformation

$$u = F(v), \tag{59}$$

we obtain

$$u_t = F' v_t, \quad u_x = F' v_x, \tag{60}$$

and therefore

$$F' v_t = \left[D\left(F\right) \left(F' v_x\right)^n \right]_x, \tag{61}$$

or

$$F' v_t = D_F \left(F' v_x\right)^{n+1} + nD \left(F' v_x\right)^{n-1} \left(F'' v_x^2 + F' v_{xx}\right), \tag{62}$$

We suppose $n \neq -1$ and we rewrite the equation (62) as

$$v_t = A(v) v_x^{n+1} + B(v) v_x^{n-1} v_{xx}, \tag{63}$$

where

$$\begin{aligned} A(v) &= D_F F'^n + nDF'^{n-2} F'' \\ B(v) &= nDF'^{n-1}. \end{aligned} \tag{64}$$

In the following we shall consider $B(v) \neq 0$. The equation (64) is remarkable because it is invariant under the action of the transformation

$$\widetilde{v} = \ln v. \tag{65}$$

Therefore the investigation of the compatibility problem of equation (64) and $v_{xt} = 0$, is equivalent to find the solutions of the form (55) for the equation (54). Now let us consider the two overdetermined systems

$$\mathcal{S}_1 : \left\{ \begin{aligned} v_t &= A(v) v_x^{n+1} + B(v) v_x^{n-1} v_{xx}, \\ v_{xt} &= 0, \end{aligned} \right. \tag{66}$$

and, for example,

$$\mathcal{S}_2 : \left\{ \begin{aligned} v_t &= A(v) v_x^{n+1} + B(v) v_x^{n-1} v_{xx}, \\ v_t &= f_t, \end{aligned} \right. \tag{67}$$

with the aim to point out the differences between the two approaches. It is clear that in the case of $\mathcal{S}_2$ we are considering invariant solutions under the action of the group associated with the vector field

$$\mathbf{v} = \frac{\partial}{\partial t} + f_t \frac{\partial}{\partial v}, \tag{68}$$

whereas $\mathcal{S}_1$ are solutions associated with a generalized non-classical (or weak) symmetry (in evolutionary form).

For $\mathcal{S}_1$ the integrability conditions is obtained considering the following differential consequences of (66)

$$\begin{cases} v_{xxx} &= D_x\left(\frac{v_t - A(v)v_x^{n+1}}{B(v)v_x^{n-1}}\right), \\ v_{xxt} &= D_t\left(\frac{v_t - A(v)v_x^{n+1}}{B(v)v_x^{n-1}}\right), \\ v_{xxt} &= 0, \\ v_{xtt} &= 0. \end{cases} \tag{69}$$

from which is obtained

$$D_{xt}\left(\frac{v_t - A(v)v_x^{n+1}}{B(v)v_x^{n-1}}\right) = 0. \tag{70}$$

From (70) considering the first set of the compatibility we have that

$$\begin{aligned} (n+1)\left(A'B - AB'\right) + B''B - B'^2 &= 0, \\ A''B - B'A' &= 0 \end{aligned} \tag{71}$$

a system that may be integrated once as

$$A' = \alpha B, \quad (n+1)A + B' = \beta B, \tag{72}$$

where α and β are integration constants. The non trivial compatibility of the system $\mathcal{S}_1$ at the first step (i.e. nonclassical generalized symmetries) is obtained in correspondence of the functions A and B that are solutions of (72). We point out that in (69) it was necessary to take the first order differential consequences of the equation (63).

For the system $\mathcal{S}_2$ we have to consider the following system

$$\begin{cases} v_{xx} &= \frac{v_t - A(v)v_x^{n+1}}{B(v)v_x^{n-1}}, \\ v_t &= f_t, \\ v_{xt} &= 0, \\ v_{tt} &= f_{tt}, \end{cases} \tag{73}$$

and therefore the integrability condition is obtained by cross differentiation as

$$D_t \left(\frac{v_t - A(v)v_x^{n+1}}{B(v)v_x^{n-1}} \right) = 0, \tag{74}$$

an equation which may be rewritten as

$$Bf_{tt} - B' f_t^2 - f_t v_x^{n+1} \left(A'B - AB' \right) = 0. \tag{75}$$

The equation (75) is in separated form and because we suppose that $n \neq -1$ we must have that

$$A'B - AB' = 0, \quad B' = k_1 B, \quad f_{tt} = k_1 f_t^2. \tag{76}$$

It is clear that if we consider (75) we have

$$D_x \left(\frac{B' f_t^2 + f_t v_x^{n+1} \left(A'B - AB' \right)}{B} \right) = 0, \tag{77}$$

a relation that when we consider

$$f_t = B(v)v_x^{n-1} v_{xx} + A(v)v_x^{n+1}, \tag{78}$$

is exactly the (70). This fact is not amazing, indeed the system (69) is a differential consequence of the system (73).

The solution of (76) is given by

$$A = k_2 \exp(k_1 v), \quad B = k_3 \exp(k_1 v), \tag{79}$$

and

$$f_t = \lambda \exp(k_1 f). \tag{80}$$

Therefore we have that *three* parameters family of equations

$$v_t = \exp(k_1 v) \left(k_3 v_x^{n-1} v_{xx} + k_2 v_x^{n+1} \right), \tag{81}$$

admits the reduction (56). Indeed by direct substitution of (56) into (81) we have

$$f_t = \exp(k_1(f+g))\left(k_3 g_x^{n-1} g_{xx} + k_2 g_x^{n+1}\right), \tag{82}$$

and considering (80) we have the reduction

$$\lambda = \exp(k_1 g)\left(k_3 g_x^{n-1} g_{xx} + k_2 g_x^{n+1}\right). \tag{83}$$

On the other hand the general solution of (72) gives a *five* parameter family of equations that allow the reduction (56). For example if $\beta = 0$ in (72) we have that

$$A' = \alpha B, \quad (n+1)\alpha B + B'' = 0, \tag{84}$$

and when $(n+1)\alpha > 0$ the family of equations we are considering is determined by

$$\begin{aligned} A &= \frac{\alpha k_4}{\sqrt{(n+1)\alpha}} \sin\left(\sqrt{(n+1)\alpha}v + k_5\right) + k_6, \\ B &= k_4 \cos\left(\sqrt{(n+1)\alpha}v + k_5\right). \end{aligned} \tag{85}$$

It is clear that the class of solutions (79) is recovered from (72) choosing

$$\alpha = \frac{k_2 k_1}{k_3}, \quad \beta = \frac{(n+1)k_2 + k_3 k_1}{k_3}. \tag{86}$$

Therefore the two methods *are not* equivalent and the result that are obtained from $\mathcal{S}_1$ seems to be more general. The equivalence is obtained only when we consider the second step of the compatibility of the system $\mathcal{S}_2$. In this case we have that the system to be considered is

$$\left\{ \begin{array}{lcl} v_t &=& A v_x^{n+1} + B v_x^{n-1} v_{xx}, \\ v_t &=& f_t, \\ B f_{tt} &=& B' f_t^2 + f_t v_x^{n+1}\left(A'B - AB'\right). \end{array} \right. \tag{87}$$

By introducing the notation $\Omega = A'B - AB'$ and considering the subsystem

$$\left\{ \begin{array}{lcl} f_t &=& A v_x^{n+1} + B v_x^{n-1} v_{xx}, \\ B f_{tt} &=& B' f_t^2 + f_t v_x^{n+1}\Omega. \end{array} \right. \tag{88}$$

we obtain, by a long but straightforward computation, the integrability relation

$$\begin{aligned} B\left[\frac{\Omega'}{\Omega}-(n+1)A-B'\right] &= k_6\left[(n+1)(\Omega-AB')\right. \\ &+ \left. BB'\frac{\Omega'}{\Omega}-BB''\right]. \end{aligned} \tag{89}$$

The solutions of (72) satisfy the relation (89) and the equivalence is now established.

The conclusion of our computations is that to recover an equivalence between the two methods that are in correspondence with the same class of solutions it is necessary to consider the class of point weak symmetries of equation (29). This fact is very important. Indeed this means that when we work with generalized (or conditional) symmetries we cannot deduce a priori when a generalized non-classical symmetry will reduce the given equation to a single ordinary differential equation or to a system of overdetermined ordinary differential equations.

The example of this Section has been suggested to me by a paper by Chu and Qu [12]. The theorem 7 in page 6280 of this paper *is false.* It is not true that under the hypothesis of this theorem the equation (63) (equation (32) in the paper by Chu and Qu [12]) is reduced by the ansatz $v = f(t) + g(x)$ to a set of two ordinary differential equations. Indeed in [12] the two ordinary differential equations are indicated as

$$\begin{aligned} f_t &= B\,(f+g)\,g_x^{n-1}g'' + A\,(f+g)\,g_x^{n+1}, \\ 0 &= g_x g_{xxx} + (n-1)g_{xx}^2 + \alpha g_x^4 + \beta g_x^2 g_{xx}, \end{aligned} \tag{90}$$

but the first one of these equations is indeed not a differential equation.

To understand clearly this point let us consider the equation

$$v_t = (v_x)^{n-1}\,v_{xx} + (v_x)^{n+1}\,, \tag{91}$$

which is scale equivalent to (81) when $k_1 = 0$. This equation is obtained also as particular case of (43a) in the list of theorem 9 in [12]. To simplify further our computations we set $n = 1$. Introducing in (91) $v = f(t) + g(x)$, because these solution are invariant under a *non-classical symmetry* we have

$$f_t = g_{xx} + g_x^2, \tag{92}$$

which splits in two ordinary differential equations

$$f_t = h_1, \quad g_{xx} + g_x^2 = h_1. \tag{93}$$

Therefore supposing the separation constant $h_1 > 0$, we obtain the following three parameter (h_1, h_2, h_3) family of additively separable solutions

$$v(x,t) = h_1 t + h_2 + \log\left(\exp\left(2\sqrt{h_1}\,(x+h_3)\right) - 1\right) - \sqrt{h_1}(x+2h_3). \qquad (94)$$

Now let us consider the equation

$$v_t = v\exp(v)\,(v_x)^{n-1}\,v_{xx} + \frac{(v-1)\exp(v)}{n+1}\,(v_x)^{n+1}, \qquad (95)$$

i.e. equation (43f) in the list of theorem 9 in [12]. This equation cannot be recovered from (81) then additive separable solutions are invariant under *weak symmetries*. Once again let us consider $n = 1$, so that by introducing into (95) the ansatz $v = f + g$ we obtain

$$\exp(-f)\frac{df}{dt} = f\exp(g)\left(g_{xx} + \frac{g_x^2}{2}\right) + \exp(g)\left[g\left(g_{xx} + \frac{g_x^2}{2}\right) - \frac{g_x^2}{2}\right]. \qquad (96)$$

It is clear that (96) cannot be split into a set of two differential equations, but into the following three equations

$$\exp(-f)\frac{df}{dt} = h_1 f + h_2. \qquad (97)$$

and

$$\exp(g)\left(g_{xx} + \frac{g_x^2}{2}\right) = h_1, \quad \exp(g)\left[g\left(g_{xx} + \frac{g_x^2}{2}\right) - \frac{g_x^2}{2}\right] = h_2. \qquad (98)$$

A manipulation of $(98)_1$ and $(98)_2$ allows to recover the equation

$$h_2 = h_1 g - \exp(g)\frac{g_x^2}{2}, \qquad (99)$$

which is a first integral of $(98)_1$ and therefore is still possible to recover non trivial solutions of (95) because the overdetermined system (98) is compatible.

The problem is to understand if the situation of this example is an happenstance or on the other hand the compatibility of the overdetermined reduced system is always guaranteed by some mathematical structure enforced by the generalized non-classical invariance.

To find an answer to this problem let us consider the equation (43i) in the list of theorem 9 in [12] always for $n = 1$, i.e.

$$v_t = \cos(v)\exp(\delta v)v_{xx} + (\sin(v) + \delta\cos(v))\exp(\delta v)v_x^2, \qquad (100)$$

where δ is a constant. The usual procedure gives

$$\begin{aligned}\exp(-\delta f)\tfrac{df}{dt} &= \sin f\left(\cos g\exp(\delta g)g_x^2 - \sin g\tfrac{d(\exp(\delta g)g_x)}{dx}\right)\\ &+ \cos f\left(\sin g\exp(\delta g)g_x^2 + \cos g\tfrac{d(\exp(\delta g)g_x)}{dx}\right).\end{aligned} \tag{101}$$

This means that

$$\exp(-\delta f)\frac{df}{dt} = h_1\sin f + h_2\cos f, \tag{102}$$

and

$$\begin{aligned}\cos g\exp(\delta g)g_x^2 - \sin g\tfrac{d(\exp(\delta g)g_x)}{dx} &= h_1,\\ \sin g\exp(\delta g)g_x^2 + \cos g\tfrac{d(\exp(\delta g)g_x)}{dx} &= h_2.\end{aligned} \tag{103}$$

The combination of $(103)_1$ and $(103)_2$ gives

$$\exp(\delta g)g_x^2 = h_1\cos g + h_2\sin g \tag{104}$$

whereas the differentiation of this last equation gives

$$\delta\exp(\delta g)g_x^2 + 2\exp(\delta g)g_{xx} = -h_1\sin g + h_2\cos g, \tag{105}$$

or

$$2\exp(\delta g)g_{xx} = -(h_1+h_2)\sin g + (h_2-h_1)\cos g, \tag{106}$$

If we consider $(103)_1$ and we introduce (105) and (106)

$$\begin{aligned}\left(\tfrac14 h_1 - \tfrac14 h_2 + \tfrac12\delta h_2\right)\cos 2g &+\\ \left(\tfrac14 h_2 + \tfrac14 h_1 - \tfrac12\delta h_1\right)\sin 2g &+\\ \left(\tfrac14 h_2 - \tfrac14 h_1\right) &= 0,\end{aligned} \tag{107}$$

therefore

$$h_2 = h_1 = 0. \tag{108}$$

In this case the overdetermined system enforce trivial solutions (i.e. time independent solutions).

The end of the history is that considering side conditions which are not invariant surface conditions associated with point symmetries may be dangerous. The compatibility problem of the corresponding overdetermined system may be tackle with the same tools as in the case of the system $\mathcal{S}$, but we loose the control on the reduction procedure. This is because only for first order quasi-linear equations we have a complete and satisfying theory that allows to understand the procedure of reduction.

7 Concluding remarks

In this paper we have reviewed some aspects of the compatibility problem for overdetermined system of differential equations and we have shown how this problem may be connected to the study of various kind of symmetries for a given differential equation.

The correct implementation of all the steps of the compatibility problem of a system composed by appending to a given differential equation a side equation which is an invariant surface condition allows to obtain, in principle, all the solutions of the given equation. For this reason many generalizations proposed in recent years of the symmetry methods are indeed already contained in this compatibility problem. For example, only to cite some recent papers, the extended rotation and scaling group for nonlinear evolution equations proposed by Qu and Estevez in [39] consists only in some ready to work formulae for the implementation of the compatibility problem for the side condition $u_x = xF(u)$, a side condition for which the determining equations for the compatibility problem end up in a separated form as shown by Pucci and Saccomandi in [38].

Usually the compatibility problem for the system $\mathcal{S}$ has been performed by stopping to the first order, here we have shown that weak symmetries are indeed useful to find some special solutions of equations for interest. It may be shown that all the solutions of the kind

$$w(x,y,t) = \sum_{k=1}^{n} f_k(x) g_k(y,t),$$

and

$$w(x,y,t) = \sum_{k=1}^{n} f_k(x,t) g_k(y),$$

may be found for a large class of equations by considering a compatibility problem with determining equations that are in separated form. Therefore considering all the steps of the compatibility problem it is possible to recover all the exact solutions of this kind proposed by Polyanin in [28] for the $(2+1)$ Navier–Stokes equations.

We have also shown, by examples, that the use of higher order differential constraints may be misleading and therefore the use of side conditions related to invariant surface conditions is in my opinion preferable. This is because, we have not a general method to solve higher order differential equations and therefore there is no a clear definition of invariant solutions under the action of generalized symmetries. It is well know that a formal definition of invariant solutions in the case of generalized symmetries is possible only in special cases and in these special case it is often possible to introduce equivalent formulations where point symmetries are used, [7].

This paper was only a personal tour in the complex world of reduction methods for partial differential equations, it was not a survey and therefore the reference list is not complete. I apologize for all those peoples that feel their contribution fundamental and cannot find their name in the list. For example, I have completely ignored the application of Lie group to ordinary differential equations, but I am sure that this is the more urgent field of application for group analysis.

My personal idea is that symmetry reduction methods are indeed sufficient to unify all reduction methods and that the key question is to characterize invariant surface conditions for which the compatibility problem admits determining equations in separated form as in (29). It is clear that other point of views are possible and may be necessary to complete the picture. I have to admit that when I use group analysis I am not truly interested in the important and beautiful problems associated with the geometrical picture of the solution set of a partial differential equation, but only with the more utilitarian idea to obtain some exact, and if possible useful, solutions of the equations I am studying. For this reason, I cover only a little of the theory, just enough to make clear issues that interest me.

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