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Chern–Simons field theory on Non-Commutative Plane

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Abstract. New developments about the symmetries properties and their actions on special solutions allowed by certain field theoretical models on the non commutative plane are reported. In particular we are looking for Galilei invariant models. The analysis indicates that this requirement strongly restricts the admissible interactions. Moreover, looking for the transformed vortex-like solutions by the Galilei boosts, a geometrical phase emerges.

Keywords: Chern–Simons theory, non-commutative plane

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The study of the noncommutative solitons (finite action solutions of the classical equations of motion of noncommutative field theories) has attracted a great interest in the last few years, mainly in connection with strings and brane dynamics [1]. However, at very low energy (i.e. in condensed matter physics), the analysis of the Fractional Quantum Hall Effect (FQHE) [2], has suggested that the phenomenology can be expressed in terms of quasi-particles, related to states of strongly correlated electrons, in the Lowest Landau Level. These quasi-particles are imbedded into an effective gauge connections, the nature of which is completely quantum mechanical (related to the Berry phase argument [3]. As a consequence, the quasi-particles (anyons) satisfy a fractional statistics. In a field theoretical approach, the topological origin of this effect is encoded into a non dynamical vector potential $\vec{A}$, which reproduces an Abelian Chern–Simons term in the action, minimally interacting with the massive field ψ of a Landau-Ginzburg theory. So the original quasi-particles are represented as vortex like solutions, carrying fractional electric charge and unit magnetic flux. On the other hand, the classical Lagrangian for a system of interacting and charged particles (electrons) in the plane, in the limit of mass $m_e \to 0$, reduces to first order in time derivatives, providing Hamiltonian equations of motion for non commuting variables [4]. Analogous Lagrangian and equations, also Hamiltonian [6], can be obtained for the mean values of the position and momentum for wave packets in magnetic Bloch bands [5]. Also in this case a Mead-Berry connection [7] appears, depending on the quasi-momentum, again encoding a geometric phase on the semiclassical description of the microscopic system. Similar, simplified models of particles in the plane were introduced on a purely axiomatic mechanical setting in [8], resorting to an acceleration-dependent Lagrangian, and in [9], where a generalized Lagrangian formalism *á la Souriau* was used. Independently from these aspects, the models were proved equivalent and, more interestingly, they provide equations of motion, which reduce to the Hall equations for a charged particle into an electric and a magnetic field, by setting a fine tuning of the magnetic field with a new parameter (k) specifying the particle. Three phenomena appear: i) the dimensional reduction of the the phase space, ii) the non commutativity of the reduced "configurational" variables, iii) the parameter k results to be a second central extension of the Galilei group in the plane, which can be seen by computing the Poisson brackets of the boost generators

$$\{G_1, G_2\} = k. \tag{1}$$

It is long time when it was proved [10] that the Galilei group in 2+1 dimensions admits a 2-dimensional central extension, in contrast with the usual one associated to the particle mass. However, its physical meaning remained obscure for a long time and the result was considered a mere mathematical curiosity. Thus, at this stage it is natural to propose a Chern–Simons theory on a non commutative plane (NC-plane) as a better description of the FQHE , in order to reproduce detailed properties of the Laughlin quasi-particles. This idea was stressed in [11]. The NC-plane is represented as the C^*-algebra of the bounded operators generated by the Heisenberg algebra

$$[\hat{x}_i, \hat{x}_j] = \imath\varepsilon_{ij}\,\theta, \qquad (i, j = 1, 2) \tag{2}$$

where θ is a characteristic scalar parameter, playing the same role as $\hbar$ in the phase space, and $\hat{\varepsilon} = (\varepsilon_{ij})$ represents the antisymmetric tensor in two dimensions.

Between the space $\mathcal{S}$ of the Schwarzian functions ψ on $\mathbf{R}^2$ and the C^*-algebra there is a one-to-one mapping, defined by the Weyl quantization formula $\hat{\psi} = \int \psi(\vec{x})\hat{\Delta}(\vec{x})d^2(\vec{x})$, where $\hat{\Delta}(\vec{x}) = \frac{1}{(2\pi)^2}\int e^{(\imath\vec{k}\cdot(\hat{\vec{x}}-\vec{x}))}\,d^2\vec{k}$ is the point-like quantizer operator. The inverse is given by the Wigner de-quantization formula $\psi(\vec{x}) = \mathrm{Tr}(\hat{\psi}\hat{\Delta}(\vec{x}))$, where the translation invariant trace map $\mathrm{Tr}(\hat{\psi}) = \int \psi(\vec{x})d^2\vec{x}$ can be introduced. It satisfies the relation $\mathrm{Tr}(\hat{\Delta}(\vec{x})\hat{\Delta}(\vec{y})) = \delta(\vec{x} - \vec{y})$. Thus, one is lead to a new associative non abelian algebra in the space S in terms of the Moyal ($\star$) product

$$\psi \star \varphi\,(\vec{x}) = \mathrm{Tr}\left(\hat{\psi}\hat{\phi}\hat{\Delta}\,(\vec{x})\right). \tag{3}$$

This result allows us to rephrase any field theory, defined by an action

$$S\left[\hat{\psi}_\alpha\right] = \int dt\;Tr\left[\mathrm{L}\left(\hat{\psi}_\alpha, \left[\hat{\partial}_i, \hat{\psi}_\alpha\right], \ldots,\right)\right] = \int dt d^2\vec{x}\mathcal{L}\left(\psi_\alpha, \partial_i\psi_\alpha, \ldots,\right) \tag{4}$$

for the operators in C^*, in terms of a nonlocal Lagrangian density, involving the classical fields ψ_α, their derivatives and their $\star$-products.

The noncommutative version (see [11–13]) of the non relativistic scalar field theory ($m = 1$, $e = 1$) coupled to the Chern–Simons gauge field is given by the Lagrangian density

$$\mathcal{L} = i\bar{\psi} \star D_t\psi - \tfrac{1}{2}\overline{\vec{D}\psi} \star \vec{D}\psi + \kappa\left(\tfrac{1}{2}\epsilon_{ij}\partial_t A_i \star A_j + A_0 \star F_{12}\right) - V\left(\psi, \bar{\psi}\right). \quad (5)$$

In (5) one has introduced the $\star$-covariant derivative and the $\star$ field-strength tensor

$$D_\mu\psi = \partial_\mu\psi - iA_\mu \star \psi, \quad \overline{D_\mu\psi} = \partial_\mu\bar{\psi} + i\bar{\psi} \star (A_\mu) \quad (6)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\big(A_\mu \star A_\nu - A_\nu \star A_\mu\big), \quad (7)$$

respectively. According to (6) the matter field ψ is in the fundamental representation of the gauge group $U(1)_*$, i.e. the locally gauged fields are given by $\tilde{\psi} = e^{\imath\lambda(\vec{x})} \star \psi$, $\tilde{A_\mu} = e^{\imath\lambda(\vec{x})} \star (A_\mu + \imath\partial_\mu) \star e^{-\imath\lambda(\vec{x})}$ and $\tilde{F_{\mu\nu}} = e^{\imath\lambda(\vec{x})} \star F_{\mu\nu} \star e^{-\imath\lambda(\vec{x})}$. A remarkable feature of the $U(1)_*$ gauge theory (5) is the quantization of the coupling constant [14] $\kappa = \frac{n}{2\pi}$, $n \in \mathbf{Z}$, corresponding to the quantized filling factor in the FQHE [11, 14]. Vortex-like solutions of such a model were discussed in [12, 13].

As shown in (4), the Lagrangian can be expressed as a trace over a Hilbert space, namely

$$\begin{aligned} \mathrm{L} = \imath\pi\kappa\mathrm{Tr}\left[K^\dagger D_t K - K D_t K^\dagger\right] - 2\pi\kappa\mathrm{Tr}\left[A_0\right] + \\ 2\pi\theta\mathrm{Tr}\left[\imath\hat{\psi}^\dagger D_t\hat{\psi} - \frac{1}{2\theta}\left(D\hat{\psi}\left(D\hat{\psi}\right)^\dagger + \bar{D}\hat{\psi}\left(\bar{D}\hat{\psi}\right)^\dagger\right) + V\left(\hat{\psi}, \hat{\psi}^\dagger\right)\right], \end{aligned} \quad (8)$$

where we have redefined the gauge field operators as

$$K = \frac{1}{\sqrt{2\theta}}\left(\hat{x}_1 + \imath\hat{x}_2 - \imath\theta\left(\hat{A}_1 + \imath\hat{A}_2\right)\right) = \hat{c} - \imath\sqrt{2\theta}\hat{A}_+ \quad (9)$$

and the corresponding adjoint $K^\dagger = \hat{c}^\dagger + \imath\sqrt{2\theta}\hat{A}_-$. The operator $\hat{c} = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + \imath\hat{x}_2)$ and its adjoint $\hat{c}^\dagger$ satisfy the canonical commutation relation $[\hat{c}, \ \hat{c}^\dagger] = 1$, furthermore in terms of complex variables $(z, \bar{z})$ one has the representation $[\hat{c}, \cdot] = \sqrt{2\theta}\partial_z$, $[\hat{c}^\dagger, \cdot] = -\sqrt{2\theta}\partial_{\bar{z}}$. The covariant derivatives act as

$$D_t\hat{\psi} = \partial_t\hat{\psi} - \imath\hat{A}_0\hat{\psi}, \qquad D_t\mathsf{O} = \partial_t\mathsf{O} - \imath\left[\hat{A}_0, \mathsf{O}\right], \quad (10)$$

$$D\hat{\psi} = \sqrt{\frac{\theta}{2}}\left(D_1 - \imath D_2\right)\hat{\psi} = \hat{\psi}\hat{c}^\dagger - K^\dagger\hat{\psi}, \qquad \bar{D}\hat{\psi} = \sqrt{\frac{\theta}{2}}\left(D_1 + \imath D_2\right)\hat{\psi} = K\hat{\psi} - \hat{\psi}\hat{c} \quad (11)$$

$$D\mathsf{O} = [\mathsf{O}, K^\dagger], \qquad \bar{D}\mathsf{O} = [K, \mathsf{O}], \tag{12}$$

for any vector operator O.

In [15] it was shown that for the (non gauged) Landau-Ginzburg fourth-order self-interacting model

$$\mathcal{L} = L_0 - V^\star = \left(i\bar{\psi} \star \partial_t \psi + \bar{\psi} \star \frac{\triangle \psi}{2} \right) - \frac{\lambda}{2} \bar{\psi} \star \bar{\psi} \star \psi \star \psi \tag{13}$$

explicit solutions can be found. In particular, it was found that bound states of two particles are characterized by a dipolar length, which is proportional to the transverse *total momentum* by the parameter θ and signals the breaking of the Galilean symmetry. This behaviour is analogous to what found in certain string models. So the question of the existence of a Galilei invariance in NC-theories is addressed in the articles [16, 17], we are going to review, and we will enlarge the results reported in [19, 20], concerning invariant solutions under boosts on the NC-plane.

In the attempt to perform a systematic symmetry analysis of field theories on the NC-plane, we start with observing that for the free version of (13) L_0 is quasi-invariant and acquires divergence-like terms (possibly proportional to θ) with respect to an "exotic " version of the 10-dimensional Schrödinger symmetry algebra. Note that because of the bilinear form of the Lagrangian and the integral property $\int f \star g\,(\vec{x})\, d^2\vec{x} = \int f\,(\vec{x})\, g\,(\vec{x})\, d^2\vec{x})$, the action coincides with the usual one in the commutative plane. The Euclidean subgroup is implemented by the inner automorphisms

$$\begin{aligned} \psi_{transl} = \psi\left(\vec{x} - \vec{h}, t\right) &= e^{-\imath \frac{\hat{\varepsilon}\vec{h}}{\theta}\cdot\vec{x}} \star \psi\,(\vec{x}, t) \star e^{\imath \frac{\hat{\varepsilon}\vec{h}}{\theta}\cdot\vec{x}}, \\ \psi_{rot} = \psi\left(\mathcal{R}_\varphi^{-1}\vec{x}, t\right) &= \left(1 + \theta^2\varphi^2\right) e^{-\imath \frac{\varphi}{2\theta}\vec{x}^2} \star \psi\,(\vec{x}, t) \star e^{\imath \frac{\varphi}{2\theta}\vec{x}^2}, \end{aligned} \tag{14}$$

where $\vec{h}$ and φ represent parameters of the translations $\vec{x} \to \vec{x} + \vec{h}$ and of the rotations $\vec{x} \to \mathcal{R}\vec{x}$, respectively. These relations express i) the nonlocality of the theory related to the scale of the momentum $\hat{\varepsilon}\vec{h}$, ii) the emergence of the space translations as particular gauge transformations. This allowed us to express covariant derivatives in terms of the gauge field in (5) in terms of the "covariant fields" (9) (see [1]).

The usual one-parameter centrally extended Galilei transformation, is replaced by the "exotic" two-parameters ones, which have the infinitesimal form

$$\delta^* \psi = (i\vec{b} \cdot \vec{x}) \star \psi - t\vec{b} \cdot \vec{\nabla} \psi = (i\vec{b} \cdot \vec{x})\psi - (\theta/2)\vec{b} \times \vec{\nabla}\psi - t\vec{b} \cdot \vec{\nabla}\psi, \tag{15}$$

and the finite expression (for $m = 1$)

$$\psi^\star_{\vec{b}}\,(\vec{x}, t) = e^{-\imath\left(\frac{\vec{b}^2}{2}t + \frac{\theta}{2} b_1 b_2\right)} e^{\imath \vec{b}\cdot\vec{x}} \star \psi\left(\vec{x} - \vec{b}t, t\right). \tag{16}$$

Analogously, a θ-deformed expansion symmetry is allowed, which in infinitesimal form is [16]

$$\delta^*_\eta \vec{x} = \eta t\vec{x}, \qquad \delta^*_\eta t = \eta t^2,$$
$$\delta^*_\eta \psi = -\eta\big[(-\frac{i}{2}x^2 + t)\psi + t\vec{x}\cdot\vec{\nabla}\psi + t^2\partial_t\psi\big] - \eta\left[\frac{\theta}{2}\vec{x}\times\vec{\nabla}\psi + \frac{\theta^2}{4}\partial_t\psi\right], \tag{17}$$

and the usual dilations [18]

$$\delta_\Delta \vec{x} = \Delta\vec{x}, \qquad \delta_\Delta t = 2\Delta t, \qquad \delta_\Delta\psi = -\Delta\big[\psi + \vec{x}\cdot\vec{\nabla}\psi + 2t\partial_t\psi\big]. \tag{18}$$

where η and $\Delta > 0$ are real parameters.

Since the Noether theorem still holds, possibly the associated conserved quantities get new terms, i.e.

$$M = \int d^2x|\psi|^2, \quad H_0 = \int d^2\vec{x}\,\frac{1}{2}|\vec{\nabla}\psi|^2,$$
$$P_i = -i\int d^2x\bar{\psi}\partial_i\psi, \quad J = -i\int d^2\vec{x}\epsilon_{ij}x_i\bar{\psi}\partial_j\psi, \tag{19}$$
$$G_i = -\int d^2\vec{x}\,x_i|\psi|^2 + tP_i + \frac{\theta}{2}\epsilon_{ij}\,P_j \tag{20}$$
$$D = -2tH_0 + \frac{1}{2i}\int d^2\vec{x}x_i\big(\bar{\psi}\partial_i\psi - (\partial_i\bar{\psi})\psi\big),$$
$$K = t^2H_0 + tD - \frac{1}{2}\int d^2\vec{x}\,\vec{x}^2|\psi|^2 + \frac{\theta}{2}J - \frac{\theta^2}{4}H_0. \tag{21}$$

Since we can use the Poisson brackets $\{\psi(\vec{x},t),\bar{\psi}(\vec{x}',t')\} = -i\delta(\vec{x}-\vec{x}')$, the symmetry algebra, expressed in terms of the above generators, contains additional θ-dependent terms, as in (1), where one establishes the relation

$$k = \theta M. \tag{22}$$

The other modified brackets are

$$\{K, G_i\} = \theta\epsilon_{ij}G_i,\ \{D,K\} = -2K + \theta J - \theta^2H_0,\ \{D, G_i\} = -G_i + \theta\epsilon_{ij}P_j. \tag{23}$$

All other commutation relations among the symmetry generators remain the same as in the commutative case [18].

Turning now to the theory in interaction, one can observe that all the self-interactions can be written in terms of the "chiral" densities $\rho_+ = \bar{\psi}\star\psi$, and $\rho_- = \psi\star\bar{\psi}$.

Now, if the usual implementation of the Galilei is applied, the densities infinitesimally transform as $\delta^0_b\rho_\pm = \pm\frac{\theta}{2}\vec{b}\times\vec{\nabla}\rho_\pm - t\vec{b}\cdot\vec{\nabla}\rho_\pm$. Analogously, resorting to the "exotic" boost (16), they change according to $\delta_b^\star \rho_+ = -t\vec{b}\cdot\vec{\nabla}\rho_+$, $\delta_b^\star \rho_- = -t\vec{b}\cdot\vec{\nabla}\rho_- - \theta\vec{b}\times\vec{\nabla}\rho_-$. Consequently, the variation of a generic potential V becomes exact w.r.t. such a kind of transformation only if it depends on one type of $\rho_\pm$:

$$\delta_b^* \widetilde{V}_+ = -t\vec{b}\cdot\vec{\nabla}\widetilde{V}_+,\ \delta_b^* \widetilde{V}_- = -t\vec{b}\cdot\vec{\nabla}\widetilde{V}_- - \theta\vec{b}\times\vec{\nabla}\widetilde{V}_-. \tag{24}$$

In conclusion, any "pure" expression $V_\pm = V(\rho_\pm)$ provides a theory, which is *Galilei-invariant* both in the the conventional implementation and in the "$\star$-implementation". This kind of chiral potentials was considered by several authors [21].

Finally, concerning now the conformal symmetry, by direct computation, one can verify that any potential made of products of $\rho_\pm$ necessarily breaks the conformal invariance. This is a consequence of the non local character of the theory and it could be related to the so-called UV/IR mixing [1].

Analyzing now the symmetry properties of the gauged model (5), with the pure fourth order self-interaction $V = \lambda(\psi\star\bar{\psi})^2$ [12], we see that the equations of motion (in Moyal and operatorial form)

$$\begin{aligned}
&\imath D_t\psi + \frac{1}{2}\vec{D}^2\psi + \left(2\lambda - \frac{1}{2\kappa}\right)\psi\star\bar{\psi}\star\psi = 0,\\
&\kappa E_i - \varepsilon_{ik}\jmath_{-,k} = 0,\\
&\kappa B + \rho_- = 0,
\end{aligned} \tag{25}$$

or

$$\begin{aligned}
&\imath D_t\hat{\psi} + \frac{1}{\theta}D\bar{D}\hat{\psi} + \left(2\lambda - \frac{1}{2\kappa}\right)\hat{\psi}\hat{\psi}^\dagger\hat{\psi} = 0,\\
&2\kappa\imath D_t K = \left\{K, \hat{\psi}\hat{\psi}^\dagger\right\} - 2\hat{\psi}\hat{c}\hat{\psi}^\dagger,\\
&\left[K, K^\dagger\right] = 1 - \frac{\theta}{\kappa}\hat{\psi}\hat{\psi}^\dagger
\end{aligned} \tag{26}$$

with

$$B = \epsilon_{ij}F_{ij}, \quad E_i = F_{i0}\text{and}\vec{\jmath}_- = \frac{1}{2i}\left(\vec{D}\psi\star\bar{\psi} - \psi\star(\overline{\vec{D}\psi})\right), \tag{27}$$

possesses an explicit "chirality" in the Hall and in the Gauss law, respectively. In particular, it turns out that both the usual implementation and the δ^* (16 -15) break the invariance of the Gauss law under a Galilean boost. However, Galilean symmetry is restored if we consider the *antifundamental representation*

$$\delta_*\psi = \psi\star(i\vec{b}\cdot\vec{x}) - t\vec{b}\cdot\vec{\nabla}\psi = (i\vec{b}\cdot\vec{x})\psi + \frac{\theta}{2}\vec{b}\times\vec{\nabla}\psi - t\vec{b}\cdot\vec{\nabla}\psi, \tag{28}$$

which is obtained as a right action of the $U(1)_*$ group, or more simply by changing the sign of θ in (15). At the same time the gauge field A_μ transforms as usual by

$$\delta A_i = -t\vec{b}\cdot\vec{\nabla}A_i,\ \delta A_0 = -\vec{b}\cdot\vec{A} - t\vec{b}\cdot\vec{\nabla}A_0. \tag{29}$$

Finally, it straightforward to prove that "pure" chiral self-interactions $V_\pm$ are also invariant under the δ_* transformations. Thus, the required Galilean symmetry strongly selects the type of interaction, as we considered in (25).

On the other hand, any potential breaks the conformal invariance.

A remarkable consequence is that one can build up the new conserved Noether charges associated to the boost charge

$$\vec{G}^r = t\vec{P} - \int \vec{x}\rho_+ \, d^2\vec{x}. \tag{30}$$

Their Poisson brackets differ from (1) only for the sign of θ.

Actually, one could define the family of conserved quantities $G_i^{(\alpha)} = G_i^r + \frac{\alpha}{2}\epsilon_{ij}P_j$, parametrized by a real α. This leads to new transformation rules $\delta^{(\alpha)}\psi = \vec{b}\cdot\{\psi,\vec{G}^\alpha\}$ and $\delta^{(\alpha)}A_i = \vec{b}\cdot\{A_i,\vec{G}^\alpha\}$, depending on α. The Poisson brackets for the boost charges are

$$\{G_i^\alpha, G_j^\alpha\} = \epsilon_{ij}(\alpha-\theta)\int|\psi|^2 d^2x \tag{31}$$

So that, for $\alpha = 0$ we recover the *-implementation (1), for $\alpha = \theta$ instead, we act on the matter field as in the commutative case, because of vanishing the second central charge. But the gauge potential changes non-conventionally. However, following [22], the gauge fields in the non commutative ($\theta \neq 0$) domain must be related to the commutative ($\theta = 0$) case by a differential relation in θ, precisely by

$$\frac{\partial}{\partial\theta}A_i(\theta) = -\frac{1}{4}\epsilon_{kl}\Big(A_k \star (\partial_l A_i + F_{li}) + (\partial_l A_i + F_{li}) \star A_k\Big). \tag{32}$$

This equation is manifestly form-invariant w. r. t. the boost transformations, provided α does not depend on θ. On the other hand, the boost transformation for the gauge fields on the ordinary plane ($\theta = 0$) holds only for $\alpha = 0$. In conclusion, the boost generator (30) is the only one admissible, because it is continuous for $\theta \to 0$. Hence, the non trivial second charge is dynamically defined by (31) for $\alpha = 0$.

However (28), (29), the proposed infinitesimal transformations are not gauge covariant. This fact may provide difficulties in deriving correct gauge invariant conserved quantities and invariant configurations under specific symmetry subgroups, as it is well known in dealing with gauge field theories. But, resorting to this analogy, in correspondence to an infinitesimal linear coordinates transformation $\delta_f x^\mu = -f^\mu$ of the NC-plane, in [23] it was proposed to associate an infinitesimal gauge-covariant transformation on a vectorfield $\hat{A}_\mu$ by $\hat{\delta}_f \hat{A}_\mu = \frac{1}{2}\{\hat{f}^\nu, \hat{F}_{\nu\mu}\}_*$, where the Moyal - anti-commutator $\{\cdot,\cdot\}_*$ has been introduced. To be specific, we modify the expressions (28)-(29) in the gauge-covariant way by

$$\hat{\delta}_{\vec{b}}\psi = \psi \star (\imath \vec{b}\cdot\vec{x}) - t\vec{b}\cdot\vec{D}\psi, \quad \hat{\delta}_{\vec{b}}A_i = -tb_j F_{ji} = t\varepsilon_{ij}b_j B,\ \hat{\delta}_{\vec{b}}A_0 = -t\vec{b}\cdot\vec{E}. \tag{33}$$

Of course, these transformations can be expressed in terms of ρ_- and $\vec{\jmath}_-$, accordingly to (25). It is remarkable that their algebra closes up to a gauge transformations. Specifically, noticing that

$$\hat{\delta}_{\vec{b}}B = -t\vec{b}\cdot\vec{D}B,\ \hat{\delta}_{\vec{b}}E_i = -t\vec{b}\cdot\vec{D}E_i - \varepsilon_{ij}b_j B, \tag{34}$$

one can find

$$\begin{gathered}\left[\hat{\delta}_{\vec{b}}, \hat{\delta}_{\vec{b}'}\right]\psi = \imath\vec{b}\times\vec{b}'\left(\theta - t^2 B\right)\psi, \qquad (35)\\ \left[\hat{\delta}_{\vec{b}}, \hat{\delta}_{\vec{b}'}\right]A_i = -t^2\vec{b}\times\vec{b}' D_i B, \quad \left[\hat{\delta}_{\vec{b}}, \hat{\delta}_{\vec{b}'}\right]A_0 = -\vec{b}\times\vec{b}' D_0\left(t^2 B\right).\end{gathered}$$

Thus the algebra of the covariant boosts closes up to a gauge transformation generated by $-\vec{b}\times\vec{b}'\ t^2 B$. Moreover, the wave function of the exotic particle acquires a time independent phase factor, given by the flux of θ, plus the gauge transformation contribution, if it is moved in a closed loop and in a finite time. Because of this result, in general the integration of such infinitesimal transformations is prevented.

However, the integrability can be obtained in the special case $D_\mu B = 0$. Because of the Gauss law, also the covariant time derivative of the chiral density ρ_- is vanishing, Thus, since a covariant version of the continuity equation holds, the chiral current $\vec{\jmath}_-$ is covariantly solenoidal, i.e. $\vec{D}\cdot\vec{\jmath}_- = 0$. Finally, taking into account the field-current identity, we end up with the covariant irrotational condition for the electric field $\vec{D}\times\vec{E} = 0$, consistently with the Bianchi identity for the connection A_μ.

Even if the above conditions restrict us to special field configurations, however we will show that non trivial solutions are still allowed. In fact, from (33) and introducing the notation $\zeta_b + tv_b = \frac{(\imath\theta - t)}{\sqrt{2\theta}}(b_1 + \imath b_2)$, the zero field covariant boost operator $\hat{\delta}^{(0)}_{\vec{b}} = (\zeta_b + tv_b)\,\hat{c}^\dagger - \left(\bar{\zeta}_b + t\bar{v}_b\right)\hat{c}$, applied from the right on the field $\hat{\psi}$, is the infinitesimal generator of a unitary transformation, which builds up a coherent state (for instance, see [24])

$$|\zeta_b + tv_b\rangle = e^{\hat{\delta}^{(0)}_{\vec{b}}}|0\rangle = U(\zeta_b, v_b)|0\rangle \tag{36}$$

from the ground state $|0\rangle$ of the Fock space. In position representation the coherent state is centered in a uniformly moving point of the plane of coordinates $\vec{x} = \vec{\zeta}_b + t\vec{v}_b = \sqrt{\frac{\theta}{2}}\,(-b_2, b_1) - \frac{t}{\sqrt{2\theta}}\,(b_1, b_2)$. Notice that the two component group parameter $\vec{b}$ provides simultaneously a drift (given by $\vec{v}_b$) and an orthogonal non vanishing translation $\vec{\zeta}_b$. Such operators form a subgroup of whole set of the unitary transformations, providing a realization of the Weyl group, and satisfy the composition rule

$$U\left(\zeta, v\right) U\left(\zeta', v'\right) = e^{-\imath\left[\vec{\zeta}\times\vec{\zeta}' + t\left(\vec{\zeta}\times\vec{v}' + \vec{v}\times\vec{\zeta}'\right) + t^2\vec{v}\times\vec{v}'\right]} U\left(\zeta + \zeta', v + v'\right), \tag{37}$$

for all ζ and v.

Via these operators it is possible to write down a 4-parameter family of uniformly moving solutions for the system (26), i.e.

$$\hat{\psi} = \sqrt{\frac{\kappa}{\theta}}|0\rangle\langle 0|U\left(\zeta, v\right), \quad K = -\left(\zeta + tv + \imath\theta v\right)|0\rangle\langle 0| + S_1\hat{c}S_1^\dagger, \quad A_0 = a_0|0\rangle\langle 0|, \tag{38}$$

where $U\left(\zeta, v\right)$ is a generic creator operator of coherent states as introduced in (36), $S_1 = \sum_{i=0}^{\infty}|i\rangle\langle i+1|$ and $a_0 = \imath\left(v\bar{\zeta} - \bar{v}\zeta\right) + \theta|v|^2 + \frac{\kappa}{\theta}\left(\frac{1}{2\kappa} - 2\lambda\right)$. Notice that this solution holds for any coupling λ, not only in the so called BPS limit [12], i.e for $2\lambda\kappa = 1$. Solutions of the kind (38) were found in [19]. They represent a uniformly moving particle of mass $M = \frac{2\pi\kappa}{\theta}$. This can be seen resorting to the conserved mass density $\rho_+ = \frac{\kappa}{\theta}U\left(\zeta, v\right)^\dagger|0\rangle\langle 0|U\left(\zeta, v\right)$. Resorting to the Wigner de-quantization formula, one has the representation $\langle w|\rho_+|w\rangle = \frac{\kappa}{\theta}\exp\left[-|\zeta + tv + w|\right]$, where $|w\rangle$ is an arbitrary coherent state of parameter w. Of course, this is a time dependent function, concentrated around the point $\zeta + tv + w = 0$ in the w complex plane. These solutions satisfy the restriction on the vanishing of the covariant derivatives of the field $B = -\frac{1}{\theta}|0\rangle\langle 0|$, which is a static coherent state centered in the origin of the reference frame on the plane. Covariant boosts do not change this expression, as one can see from (34). The CS-electric field takes the value $E_i = -\sqrt{\frac{2}{\theta}}\varepsilon_{ij}v_j|0\rangle\langle 0|$. Then, from (34) we see that the covariant boost transformation acts identically on the B component. Notice that this field configuration is similar as in the Hall motion. In particular, the relation $\frac{|E|}{|B|} = \sqrt{2\theta}v$ holds, while the particle is moving with velocity $|v|$ in the direction orthogonal to $\vec{E}$.

Concerning the infinitesimal covariant boost on the matter field ψ, from the relation (33), one distinguishes the usual free boost generator $\hat{\delta}_{\vec{b}}^{(0)}$ and the gauge field contribution $t\left(bK^\dagger - \bar{b}K\right)$. However, substituting the expressions in (38), we find that the new contribution is simply given by $i\alpha\left(t\right)\hat{\psi} =$

$-2\imath t\left(\vec{v}_b\times\vec{\zeta}+\vec{\zeta}_b\times\vec{v}+t\vec{v}_b\times\vec{v}\right)\hat{\psi}$. Thus, the existence of a non trivial gauge field has the effect of introducing a phase factor:

$$\begin{aligned}\hat{\psi}_{boost} &= e^{i\alpha(t)}\hat{\psi}U\left(\zeta_b, v_b\right)\\ &= \sqrt{\frac{\kappa}{\theta}}e^{i\left[t^2\vec{v}\times\vec{v}_b+t\left(\vec{\zeta}\times\vec{v}_b+\vec{v}\times\vec{\zeta}_b\right)+\vec{\zeta}_b\times\vec{\zeta}\right]}|0\rangle\langle 0|U\left(\zeta+\zeta_b, v+v_b\right), \quad (39)\end{aligned}$$

then the coherent state takes a new velocity and suffers a position shift.

Finally in the BPS limit, one obtains a simplest expression for energy of the particle, which interacts with the Chern–Simons field and with itself for $2\lambda\kappa = 1$. In fact, one has $\mathcal{E} = -2\pi\mathrm{Tr}[\hat{\psi}^\dagger\bar{D}D\hat{\psi}]$. For the solution (38), one sees that $\mathcal{E} = 2\pi\kappa\theta|v|^2 = \frac{1}{2}\theta M|\sqrt{2\theta}v|^2$, which confirms the continuous spectrum of the vortex energy. More interesting it is its expression as kinetic energy for a particle possessing the second central charge (22) as mass and moving at the selected Hall velocity, found by the ratio $|E|/|B|$.

In conclusion, we have discussed the symmetries of a field theory on the non commutative plane. The first result is that we can recover Galilean invariant theories, but only "chiral" interactions are admissible. However, the boost generators no longer commute, but they close up the second central extension parameter κ, which is dynamically determined. Finally, classes of uniformly moving solutions can be build up, just resorting to freely moving coherent states. However, the associated CS-magnetic and electric field are not trivial at all, even if they result to be static. More general type of moving solutions can be found [20] and boosted. In particular one can show that two vortex solutions can be boosted, without their relative positions change, in contrast with the dipolar solutions found by [15].

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