Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **44** (2024) no. 2, 45–52. doi:10.1285/i15900932v44n2p45

Some geometric estimates of the first eigenvalue of quasilinear and (p, q)-Laplace operators

Sakineh Hajiaghasi
Department of Mathematics, Imam Khomeini International University
s4004196001@edu.ikiu.ac.ir

Shahroud Azami
Department of Mathematics, Imam Khomeini International University
azami@sci.ikiu.ac.ir

Received: 28.04.24; accepted: 09.10.24.

Abstract. In this paper, we use a particular smooth function $f : \Omega \to \mathbb{R}$ on a bounded domain Ω of a Riemannian manifold M to estimate the lower bound of the first eigenvalue for quasilinear operator $Lf = -\Delta_p f + V|f|^{p-2}f$. In this way, we also present a lower bound for the first eigenvalue of the (p, q)-Laplacian on compact manifolds.

Keywords: (p,q)-Laplacian, quasilinear operator, first eigenvalue

MSC 2020 classification: primary 53C60, secondary 53B40, 35P15

Introduction

It is well known that studying the eigenvalues and eigenfunctions of the Laplacian plays an important role in global differential geometry since they reveal important relations between the geometry of the manifold and analysis. So far, there has been some progress on the geometric operators as bi-Laplace, p-Laplace, and (p,q)-Laplace associated with a Riemannian metric g on a compact Riemannian manifold M^n. For instance, Lichnerowicz-type estimate had been studied in some research papers for the p-Laplace [10], p-Laplace with integral curvature condition [11], and recently investigated for the first eigenvalue of buckling and clamped plate problems in [8]. For more study about eigenvalue estimate see [2, 3, 7, 12]. In this paper, first we are going to study the first eigenvalue of the following quasilinear operator which was introduced in [1]. Then we are going to extend some results about the fundamental tone of the p-Laplacian from [7] to the (p, q)-Laplacian.
Let (M^n, g, dv) be a compact Riemannian manifold with volume element dv,

the quasilinear operator on M is defined as

$$Lf = -\Delta_p f + V|f|^{p-2}f. \tag{0.1}$$

Here V is a nonnegative smooth function on M, and for $p \in (1, \infty)$ the p-Laplace operator is defined as

$$\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f).$$

Corresponding to the p-Laplacian we have the following eigenvalue equation known as Dirichlet eigenvalue problem:

$$\begin{cases} Lf = \mu |f|^{p-2}f, & on \quad M \\ f = 0, & on \quad \partial M. \end{cases} \tag{0.2}$$

The first nontrivial Dirichlet eigenvalue for M is given by

$$\mu_{1,p}(M) = \inf_{f \in W_0^{1,p}(M), f \neq 0} \frac{\int_M (|\nabla f|^p + V|f|^p)dv}{\int_M |f|^p dv}.$$

The function f satisfies in (0.2) is called the eigenfunction of operator L corresponding to μ on M.

1 Main Results

We consider a bounded domain Ω in a n-dimensional Riemannian manifold M^n, $n \geq 2$. Under some additional assumption for $f : \Omega \to \mathbb{R}$, we will obtain a positive lower bound for $\mu_{1,p}$ on bounded domain Ω as follows:

Theorem 1. *Let Ω be a bounded domain on a Riemannian manifold M, and assume that there is a smooth function $f : \Omega \to \mathbb{R}$ such that $|\nabla f| \leq a$ and $\Delta_p f \geq b$ for some positive constants a, b, where $a > b$. Then the first Dirichlet eigenvalue of the quasilinear operator L satisfies*

$$\mu_{1,p}(\Omega) \geq \frac{b^p}{p^p a^{p(p-1)}}.$$

PROOF. We first note that by density we can use smooth functions in the variational characterization of $\mu_{1,p}(\Omega)$. So given $u \in C_0^\infty(\Omega)$, based on the fact that V is positive function, we have

$$\begin{aligned} b\int_\Omega |u|^p dv &\le \int_\Omega |u|^p(\Delta_p f + V)dv \\ &= -\int_\Omega < \nabla|u|^p, |\nabla f|^{p-2}\nabla f > dv + \int_\Omega |u|^p V dv \\ &= -p\int_\Omega |u|^{p-1} < \nabla|u|, |\nabla f|^{p-2}\nabla f > dv + \int_\Omega |u|^p V dv \\ &\le p\int_\Omega |u|^{p-1}|\nabla u||\nabla f|^{p-1}dv + \int_\Omega |u|^p V dv \\ &\le p\int_\Omega |u|^{p-1}a^{p-1}|\nabla u|dv + \int_\Omega |u|^p V dv. \end{aligned} \tag{1.3}$$

Now considering a constant $c > 0$ and using Young inequality, we obtain

$$\begin{aligned} |u|^{p-1}a^{p-1}|\nabla u| &\le \frac{c^q|u|^{q(p-1)}}{q} + \frac{a^{p(p-1)}|\nabla u|^p}{pc^p} \\ &= \frac{(p-1)c^{p/(p-1)}|u|^p}{p} + \frac{a^{p(p-1)}|\nabla u|^p}{pc^p}. \end{aligned}$$

Therefore

$$\begin{aligned} p\int_\Omega |u|^{p-1}a^{p-1}|\nabla u|dv + \int_\Omega |u|^p V dv \quad &\le \quad (p-1)c^{p/(p-1)}\int_\Omega |u|^p dv \\ &\quad + \frac{a^{p(p-1)}}{c^p}\int_\Omega |\nabla u|^p dv + \int_\Omega |u|^p V dv. \end{aligned} \tag{1.4}$$

We could choose c so that $b - (p-1)c^{p/(p-1)} = \frac{b}{p}$, that is $c^p = \frac{b^{p-1}}{p^{p-1}}$. Hence with the statement in theorem $a > b$, we know

$$\frac{p^{p-1}a^{p(p-1)}}{b^{p-1}} > 1,$$

so, (1.3) and (1.4) lead to

$$\frac{b}{p}\int_\Omega |u|^p dv \le \frac{p^{p-1}a^{p(p-1)}}{b^{p-1}}\Big(\int_\Omega |\nabla u|^p dv + \int_\Omega |u|^p V dv\Big).$$

Dividing both side to $\int_\Omega |u|^p$, completes the proof. QED

Let $M^n(k)$ denote the simply connected space form of constant sectional curvature k. The metric of $M^n(k)$ in polar coordinates is $g = dr^2 + f_k^2(r)dw^2$, where dw^2 is the standard metric on $\mathbb{S}^{n-1}$ and $f_k(r)$ defines as follows:

$$f_k(r) = \begin{cases} \frac{1}{\sqrt{-k}} sinh(\sqrt{-k}r), & if\ k < 0, \\ r, & if\ k = 0, \\ \frac{1}{\sqrt{k}} sin(\sqrt{k}r), & if\ k > 0. \end{cases} \tag{1.5}$$

So if we denote the distance function to the center of geodesic ball B_R of radius R, by r_k, then we know that $|\nabla r_k| = 1$, and $\Delta_p r_k = \mathrm{div}(|\nabla r_k|^{p-2}\nabla r_k) = \Delta r_k = (n-1)\frac{f_k^{'}}{f_k}$. In particular, we have

$$\Delta_p r_k \geq \begin{cases} (n-1)\sqrt{-k}coth(\sqrt{-k}R), & if\ \ k < 0, \\ \frac{(n-1)}{R}, & if\ \ k = 0, \\ (n-1)\sqrt{k}cot(\sqrt{k}R), & if\ \ k > 0. \end{cases}$$

Consequently, as a first application of Theorem 1, for the especial distance function on a geodesic ball B_R, we obtain:

Corollary 1. *Consider a bounded domain $\Omega \in M^n(k)$. If Ω is contained in a geodesic ball B_R, then*

$$\begin{aligned} \mu_{1,p}(\Omega) &\geq \frac{(n-1)^p(\sqrt{-k})^p}{p^p} coth^p(\sqrt{-k}R), \ \ if\ \ k < 0, \\ \mu_{1,p}(\Omega) &\geq \frac{(n-1)^p}{p^p R^p}, \qquad\qquad\qquad if\ \ k = 0, \\ \mu_{1,p}(\Omega) &\geq \frac{(n-1)^p(\sqrt{k})^p}{p^p} cot^p(\sqrt{k}R), \ \ if\ \ k > 0. \end{aligned}$$

We consider a warped product Riemannian manifold $M^n = \mathbb{R} \times N$, where (N, g_0) is an arbitrary Riemannian manifold, with corresponding warped metric $ds^2 = dt^2 + e^{2\rho(t)}g_0$. Given Busemann function associated to the geodesic ray $F : M \longrightarrow \mathbb{R}$, $F(s, x) = s$, a direct computation shows $|\nabla F| = 1$, and $\Delta_p F = \Delta F = (n-1)\rho^{'}(t)$. Subsequently, assuming $\rho^{'}(t) \geq \kappa > 0$, we could state the following corollary.

Corollary 2. *Let $M^n = \mathbb{R} \times N$ be a warped product Riemannian manifold endowed with the warped metric $ds^2 = dt^2 + e^{2\rho(t)}g_0$, such that the warped function satisfies $\rho^{'}(t) \geq \kappa > 0$, for some constant κ. Then the first Dirichlet eigenvalue of* (0.1)*, satisfies the following:*

$$\mu_{1,p}(M) \geq \frac{(n-1)^p}{p^p}\kappa^p.$$

Based on the studies in [5], this kind of estimate that we mentioned for warped product can be lifted for Riemannian manifolds which admit a Riemannian submersion over hyperbolic space.
Let $\pi : M^n \longrightarrow B^k$, be a surjective Riemannian submersion between two Riemannian manifolds M, B. It is obvious that $d\pi_x : T_xM \to T_{\pi(x)}B$ is surjective for all $x \in M$. Hence for each $b \in B$, $\pi^{-1}(b)$ is a submanifold of M of dimension $dimM - dimB$. In particular, the submanifolds $\pi^{-1}(b) = \mathcal{F}_b$ are called fibers, and a vector field on M is vertical if it is always tangent to fibers, and it is horizontal, whenever it is orthogonal to fibers. On the other hand, Riemannian metric on M gives the decomposition of a vector field $X \in TM$ as $X = X^{\mathcal{V}} + X^{\mathcal{H}}$, where $X^{\mathcal{V}}$ and $X^{\mathcal{H}}$ are the vertical and horizontal components respectively. Based on this notations, the second fundamental form of the fibers is a symmetric tensor $\alpha^{\mathcal{F}} : T\mathcal{F} \times T\mathcal{F} \to T^{\perp}\mathcal{F}$ defined by

$$\alpha^{\mathcal{F}}(v, w) = (\tilde{\nabla}_v W)^{\mathcal{H}},$$

where W is a vertical extension of w and $\tilde{\nabla}$ denotes the Levi-Civita connection on M. The mean curvature vector of the fiber is the horizontal vector field $H^{\mathcal{F}}$ defined by $H^{\mathcal{F}} = tr\alpha^{\mathcal{F}}$. Considering a local orthonormal frame $\{e_1, \cdots, e_{n-k}\}$ for a fiber $\mathcal{F}_b$, we have

$$H^{\mathcal{F}}(x) = \Sigma_{i=1}^{n-k}\alpha^{\mathcal{F}}(e_i, e_i) = \Sigma_{i=1}^{n-k}(\tilde{\nabla}_{e_i}e_i)^{\mathcal{H}}.$$

Now, given a smooth function g on B we set $\tilde{g} : M \to \mathbb{R}$, so that $\tilde{g}(x) = g \circ \pi(x)$ be the lift of g to M. See [6] for more details about Riemannain submersions. We may need the following Lemma from [5] for our main result:

Lemma 1. *Let $g : B \to \mathbb{R}$ be a smooth function and set $\tilde{g} = g \circ \pi$. Then $\forall b \in B$, and $\forall x \in \mathcal{F}_x$, we have*

$$\tilde{\Delta}\tilde{g}(x) = \Delta g(x) + < \tilde{\nabla}\tilde{g}(x), H^{\mathcal{F}}(x) > .$$

We just need to assume the function $g : B \to \mathbb{R}$ such that $|\nabla g| = 1$ and $\Delta g \geq \kappa > 0$. In this way, we also know that $\Delta_p g > \kappa$. Finally, we note that the gradient of $\tilde{g}$ is horizontal lifting of the gradient of g, and $|\tilde{\nabla}\tilde{g}| = 1$, and $\tilde{\Delta}_p\tilde{g} = \tilde{\Delta}\tilde{g}$. Here is our main result:

Theorem 2. *Let $\tilde{M}^m$ be a complete Riemannian manifold that admits a Riemannian submersion $\pi : \tilde{M}^m \longrightarrow M^n = \mathbb{R} \times N$, where π is a surjective map. If the mean curvature of the fibers satisfy $|H^{\mathcal{F}}| \leq \alpha$, for some $\alpha < (n-1)\kappa^{1/p}$, then for the first Dirichlet eigenvalue of* (0.1)*, we have*

$$\mu_{1,p}(M) \geq \frac{((n-1)^p\kappa - \alpha)^p}{p^p}.$$

Proof. From Lemma 1, we obtain

$$\tilde{\Delta}_p \tilde{g} \geq \kappa - \alpha,$$

so with the same way of Theorem 1, we reach our purpose. $\boxed{QED}$

1.1 Eigenvalue estimate of some (p,q)-Laplacian

Let Ω be a compact domain in a complete, simply connected Riemannian manifold (M,g) of constant sectional curvature k. We are going to study a class of (p,q)-Laplacian for $\forall u \in W = W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$ introduced in [4], as follows:

$$\Delta_p u + \Delta_q u = \operatorname{div}((|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u), \tag{1.6}$$

where $1 < q < p < \infty$, and W is the completion of $C_0^\infty(\Omega)$ with respect to the norm $||u|| = ||u||_{1,p} + ||u||_{1,q}$. Since W is without the boundary, the boundary condition is not needed. Here $\lambda_{1,p,q} \in \mathbb{R}$ is called an eigenvalue of (1.6) if there is a nontrivial solution for the following inequality:

$$-\Delta_p u - \Delta_q u = \lambda_{1,p,q} |u|^{p-2} u, \tag{1.7}$$

or equivalently for any $v \in W^{1,p}(\Omega) \cap W^{1,q}(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla v dv + \int_\Omega |\nabla u|^{q-2} \nabla u.\nabla v dv = \lambda_{1,p,q} \int_\Omega |u|^{p-2} uv dv. \tag{1.8}$$

Therefore the first positive eigenvalue $\lambda_{1,p,q}(\Omega)$ of (1.6) defines as

$$\lambda_{1,p,q}(\Omega) = \inf \left\{ \int_\Omega |\nabla u|^p dv + \int_\Omega |\nabla u|^q dv : u \in W, u \neq 0, \int_\Omega |u|^p dv = 1 \right\}. \tag{1.9}$$

Consider $u > 0$ as one of the eigenfunctions corresponding to $\lambda_{1,p,q}$, and a smooth function $f : \Omega \to \mathbb{R}$ such that $|\nabla f| \leq a, \Delta_p f \geq b$, and $\Delta_q f \geq b'$ for some constants $a, b, b' > 0$, where $a > b, a > b'$. From the proof of Theorem 1, we have

$$\begin{aligned}
b \int_\Omega |u|^p dv &\leq \int_\Omega |u|^p (\Delta_p f) dv \\
&= - \int_\Omega < \nabla |u|^p, |\nabla f|^{p-2} \nabla f > dv \\
&= -p \int_\Omega |u|^{p-1} < \nabla |u|, |\nabla f|^{p-2} \nabla f > dv \\
&\leq p \int_\Omega |u|^{p-1} |\nabla u| |\nabla f|^{p-1} dv \\
&\leq p \int_\Omega |u|^{p-1} a^{p-1} |\nabla u| dv.
\end{aligned}$$

So
$$\int_\Omega |\nabla u|^p dv \geq \left(\frac{b}{pa^{p-1}}\right)^p \int_\Omega |u|^p = \left(\frac{b}{pa^{p-1}}\right)^p,$$
and
$$\int_\Omega |\nabla u|^q dv \geq \left(\frac{b'}{qa^{q-1}}\right)^q \int_\Omega |u|^q = \left(\frac{b'}{qa^{q-1}}\right)^q,$$
so we achieve our next result as follows:

Theorem 3. *Let Ω be a compact domain with smooth boundary in a complete Riemannian manifold. Then for the first eigenvalue of (1.6), we obtain*
$$\lambda_{1,p,q}(\Omega) \geq \left(\frac{b}{pa^{p-1}}\right)^p + \left(\frac{b'}{qa^{q-1}}\right)^q.$$

Particularly, for distance function (1.5), we conclude

Corollary 3. *Consider a bounded domain $\Omega \in M^n(k)$ so that it is contained in a geodesic ball B_R, then we obtain*
$$\begin{aligned}
\lambda_{1,p,q}(\Omega) &\geq \frac{(n-1)^p(\sqrt{-k})^p}{p^p} coth^p(\sqrt{-k}R) + \frac{(n-1)^q(\sqrt{-k})^q}{q^q} coth^q(\sqrt{-k}R), \\
&\quad if\ \ k<0, \\
\lambda_{1,p,q}(\Omega) &\geq \frac{(n-1)^p}{p^pR^p} + \frac{(n-1)^q}{q^qR^q}, \qquad if\ \ k=0, \\
\lambda_{1,p,q}(\Omega) &\geq \frac{(n-1)^p(\sqrt{k})^p}{p^p} cot^p(\sqrt{k}R) + \frac{(n-1)^q(\sqrt{k})^q}{q^q} cot^q(\sqrt{k}R), \ \ if\ \ k>0.
\end{aligned}$$

Now we present the estimate for the first eigenvalue of (1.6) for a class of warped product metrics.

Corollary 4. *Consider a warped product Riemannian manifold $M^n = \mathbb{R} \times N$ with the warped metric $ds^2 = dt^2 + e^{2\rho(t)}g_0$, such that $\rho'(t) \geq \kappa > 0$, for some constant κ. Then the following estimate holds for the first eigenvalue of (1.6):*
$$\lambda_{1,p,q}(M) \geq \frac{(n-1)^p}{p^p}\kappa^p + \frac{(n-1)^q}{q^q}\kappa^q.$$

Let $\tilde{M}^m$ and M^n be Riemannian manifolds with $m > n$ and $\pi : \tilde{M} \to M$ as a surjective submersion on $\tilde{M}$, then with the same way that mentioned for the quasilinear operator (0.1), we can state the next result.

Theorem 4. *Let $\tilde{M}^m$ be a complete Riemannian manifold that admits a Riemannian submersion $\pi : \tilde{M}^m \longrightarrow M^n = \mathbb{R} \times N$, where π is a surjective map. If the mean curvature of the fibers satisfy $|H^{\mathcal{F}}| \leq \alpha$, for some $\alpha < (n-1)\kappa^{1/p}$, then we have*
$$\lambda_{1,p,q}(M) \geq \frac{((n-1)^p\kappa - \alpha)^p}{p^p} + \frac{((n-1)^q\kappa - \alpha)^q}{q^q}.$$

The proof of this theorem is just like Theorem 2. Note that in the last two results there is no difference between Δ_p and Δ_q.

References

[1] S. Azami: *Geometric estimates of the first eigenvalue of a quasilinear operator*, Math. Reports, 23(73), 1-2, 107–121, 2021.

[2] S. Azami: *The first eigenvalue of some (p, q)-Laplacian and geometric estimates*, Commun. Korean. Math. Soc. 33, No. **1**, pp. 317-323, 2018.

[3] S. Azami: *Evolution of the first eigenvalue of buckling problem on Riemannian manifold under Ricci flow*,Journal of new researches in mathematics, 6 (26), 81-92, 2020.

[4] N. Benouhiba, Z. Belyacine: *A class of eigenvaue problems for the (p,q)-Laplacian in* $\mathbb{R}^N$, Int. J. Pure Appl. Math. 80, no. **5**, 727-737, 2012.

[5] M.P. Cavalcante, F. Manfio: *On the fundamental tone of immersions and submersions*, Proc. Am. Math. Soc. 146 (7), 2963-2971, 2018.

[6] R.H. Escobales, Jr.: *Riemannian submersions with totally geodesic fibers*, J. Differential Geom. 10, 253-276, 1975.

[7] G. Francisco, S. Carvalho, P. Marcos, A. Cavalcante: *On the fundamental tone pf the p-Laplacian on Riemannian manifolds and applicatios*, J. Math. Anal. Appl. 506, 125703, 2022.

[8] M. J. Habibi Vosta Kolaei, S. Azami: *Lichnerowicz-type estimates for the first eigenvalue of biharmonic operator*, Complex Variable and Elliptic Equations, 1-8, 2021.

[9] A.E. Khalil, S.E. Manouni, M. Ouanan: *Simplicity and stability of the first eigenvalue of a nonlinear elliptic system*, Int. J. Math. Sci., no. **10**, 1555-1563, 2005.

[10] A. Matei: *First eigenvalue for the p-Laplace operator*, Nonlinear Anal Ser A., 39(8): 1051-1068, 2000.

[11] S. Seto, G. Wei: *First eigenvalue of the p-Laplacian under integral curvature condition*, Nonlinear Anal., 163: 60-70, 2017.

[12] L. Zhang,Y. Zhao, *The lower bounds of the first eigenvalues for the biharmonic operator on manifolds*, Journal of Inequalities and Applications, 5 (2016).