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Helix Surfaces in Lorentzian ambient spaces

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Abstract. This survey describes the study of helix (or constant angle) surfaces in different ambient spaces equipped with a Lorentzian metric. We present the study of constant angle surfaces in the Minkowski space, the Lorentzian Heisenberg group, the Lorentzian Berger sphere and some new results the for the 3-dimensional anti-de Sitter space with Berger-like metrics. In every case, we give characterization theorems which describe such surfaces.

Keywords: Helix surfaces, constant angle surfaces, Lorentzian geometry

MSC 2022 classification: 53B25, 53B30, 53C50

1 Introduction

A *helix surface* (or *constant angle surface*) is an oriented surface with the property that the normal vector field forms a constant angle with a fixed field of directions in the ambient space. This surfaces were defined for the first time in this way in the work of Dillen, Fastenakels, Van der Veken and Vrancken ([4]).

The large interest towards this particular type of surfaces was motivated by the paper of Cermelli and Di Scala (see [3]) who analyzed the case of surfaces with constant angle in the three-dimensional Euclidean space. In their work they deduced a crucial relationship between the Hamilton-Jacobi equation and such a type of surfaces showing also some interesting applications in the physical field connected to the equilibrium configuration of liquid crystals. In particular, one must observe that their molecules present naturally a tendency to align themselves according to a direction given by a critical field of some energy functional.

In recent years, different ambient spaces have been considered by many authors while studying helix surfaces. Several examples of the study of helix surfaces in Riemannian settings may be found in [3]-[6],[9],[12],[15], [17]-[19] and references therein. Furthermore, the investigation of such surfaces also extended to other settings. On the one hand, higher codimensional Riemannian helix surfaces were studied (see for example [7], [8], [22]). On the other hand, Lorentzian ambient spaces were considered in [13], [14], [20] and [21].

When we take into account Lorentzian settings, we are allowed to consider more possibilities, as both spacelike and timelike surfaces can be studied. These possibilities and other fascinating properties of this type of surfaces motivate us in exploring further the description of helix surfaces in Lorentzian ambient spaces. For this reason, this survey presents the study of constant angle surfaces in the Minkowski space, the Lorentzian Heisenberg group, the Lorentzian Berger sphere and announces some original results on the 3-dimensional anti-de Sitter space with Berger-like metrics.

The paper is organized in the following way. In Section 2 we introduce the problem featuring the general settings to establish in these studies, giving some useful definitions and tools. Then, in Section 3 we treat the Minkowski space, following the presentation in [13]. Section 3 analyzes the case of the Lorentzian Heisenberg group $H_3(\tau)$ as it appears in [20] and in Section 5 we present the characterization of the surfaces in the Lorentzian Berger sphere $\mathbb{S}^3_\varepsilon$ whose unit normal vector field makes a constant angle with the unit Hopf vector field, following [21]. In the conclusive part of the paper we announce some original results towards the characterization of helix surfaces in the anti-de Sitter space $\mathbb{H}^3_1$ endowed with some left-invariant metrics, offering a generalization to the case proved in [14].

2 Preliminaries

Let $(\bar{M}, \bar{g})$ be a 3-dimensional Lorentzian manifold and M an oriented surface immersed into $\bar{M}$. We denote by $\bar{\nabla}$ the Levi-Civita connection of $\bar{M}$ and fix the convention:

$$\bar{R}(X,Y) = [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]} \tag{2.1}$$

for the Riemann curvature tensor.

Let N be the unit normal to M; then, if we denote by $\lambda = \bar{g}(N,N) = \pm 1$ the causal character of the normal, we can make a distinction between two cases:

- $\lambda = -1$, that means that the induced metric on M is Riemannian and so M is called *spacelike*;
- $\lambda = 1$, that means that the induced metric on M is Lorentzian and so M is called *timelike*.

At this point, we consider a Killing vector field $\tilde{V}$ in order to define the *angle function*

$$\nu := \bar{g}(N, \tilde{V})\bar{g}(N,N) = \lambda \bar{g}(N, \tilde{V}). \tag{2.2}$$

Therefore, we report the following.

Definition 1. M is called a *helix surface* (or *a constant angle surface*) if the angle function ν is constant on M.

The well known Gauss and Weingarten formulas, for all $X, Y \in \mathfrak{X}(M)$, read:

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(X,Y), \quad \bar{\nabla}_X N = -A(X), \tag{2.3}$$

where ∇ is the Levi–Civita connection on M and α the second fundamental form with respect to the immersion. In this way, we have

$$\alpha(X,Y) = \lambda \bar{g}(A(X),Y)N. \tag{2.4}$$

In addition, we can also write, with the conventions used above, the Gauss equation for pseudo-Riemannian surfaces:

$$K = \bar{K} + \lambda \frac{\bar{g}(A(X),X)\bar{g}(A(Y),Y) - \bar{g}(A(X),Y)^2}{\bar{g}(X,X)\bar{g}(Y,Y) - \bar{g}(X,Y)^2} \tag{2.5}$$

where

$$\frac{\bar{g}(A(X),X)\bar{g}(A(Y),Y) - \bar{g}(A(X),Y)^2}{\bar{g}(X,X)\bar{g}(Y,Y) - \bar{g}(X,Y)^2} = detA.$$

Moreover, the Codazzi equation for hypersurfaces yields:

$$\bar{g}(\bar{R}(X,Y)Z,N) = \bar{g}(\nabla_X A(Y) - \nabla_Y A(X) - A[X,Y],Z). \tag{2.6}$$

3 Helix surfaces in $\mathbb{E}^3_1$

Let $\langle , \rangle$ be the standard flat metric in $\mathbb{E}^3_1$, the three-dimensional Minkowski space, that is the Lorentzian metric

$$\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2.$$

In $\mathbb{E}^3_1$ one can define several types of angles depending on the causality of the two vectors considered (see [10], [11]) and, consequently, different types of constant angle surfaces may be studied. In particular, in this survey, we considered the notion of angle between two timelike vectors and the spacelike surfaces obtained from this choice, following the work of Lopez and Munteanu ([13]).

To begin, if $v, w \in \mathbb{E}^3_1$ are two timelike vectors, then $\langle v,w\rangle \neq 0$. In particular, in the case that both vectors lie in the same timelike cone, there exists a unique number $\theta \geq 0$, called the *hyperbolic angle* between v and w, such that

$$\langle v,w\rangle = -|v||w|\cosh\theta.$$

Now, let M be a surface immersed into $\mathbb{E}^3_1$. Our purpose is to describe a characterization for M to be of constant angle. It is well known that if the immersion is spacelike, then the surface is orientable. Then, let M be an oriented spacelike surface immersed into $\mathbb{E}^3_1$ with N its unit normal, which is clearly timelike. Therefore, since the notion of hyperbolic angle works between timelike vectors, let us take without loss of generality the Killing timelike vector field $e_3 = (0, 0, 1)$. We then specify the relation (2.2) defining the *angle function* between N and e_3 by:

$$\nu := -\langle N, e_3\rangle = \cosh\theta, \tag{3.1}$$

where θ is the hyperbolic angle between N and e_3.

In analogy with the Euclidean case (see [17]), we can recall the Gauss and Weingarten formulas given in (2.3) using ∇ for the Levi–Civita connection on M and α for the second fundamental form with respect to the immersion. In this way, one gets $\alpha(X, Y) = -g(A(X), Y)\, N$, where g is the restriction of $\langle,\rangle$ to M. We now consider the decomposition of e_3 as follows:

$$e_3 = T + \cosh\theta N,$$

where T is tangent to M. It is easy to determine that $||T|| = \sinh\theta$.

Remark 1. If $\theta = 0$, then e_3 is parallel to N and so M is a plane orthogonal to e_3. For this reason, we can assume, from now on, that $\theta \neq 0$.

Now, we can define a unit vector field on M, namely E_1, collinear to T and E_2 orthogonal to E_1 such that $\{E_1, E_2\}$ is an orthonormal basis for TM. From now on, in our discussion we will always consider M a spacelike helix surface (i.e. θ constant on M). At this point, one is able to calculate the associated Levi-Civita connection ∇ for a constant angle spacelike surface in $\mathbb{E}^3_1$:

$$\nabla_{E_1} E_1 = \nabla_{E_1} E_2 = 0, \qquad \nabla_{E_2} E_1 = -a\coth\theta\, E_2, \qquad \nabla_{E_2} E_2 = a\coth\theta\, E_1, \tag{3.2}$$

where a is a function. Moreover, with respect to $\{E_1, E_2\}$, the matrix describing the shape operator is given by

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}.$$

3.1 The characterization of helix surfaces in $\mathbb{E}^3_1$

Following the scheme used in [13], one can choose now a local coordinate system (u, v) such that:

$$\partial_u = E_1, \quad \partial_v = b\, E_2, \tag{3.3}$$

where b is a function on M. The condition $0 = [\partial_u, \partial_v]$ leads to:

$$b_u + ba \coth\theta = 0. \tag{3.4}$$

Now, let us consider a parametrization for the helix surface M: $F(u,v) = F(x(u,v), y(u,v), z(u,v))$ with the coordinate defined above.

Therefore, taking into account the relations for the Levi-Civita connection (3.2) one gets the following:

$$F_{uu} = 0; \tag{3.5}$$

$$F_{uv} = \frac{b_u}{b} F_v; \tag{3.6}$$

$$F_{vv} = -\frac{b_v}{b} F_v - bb_u\, F_u + ab^2 N. \tag{3.7}$$

Moreover, the Schwarz identity and the expressions of the partial derivatives of N leads to an equation for a:

$$a_u - a^2 \coth\theta = 0. \tag{3.8}$$

Also, combining (3.4) and (3.8) the authors obtained $a_u + a\dfrac{b_u}{b} = 0$, that is $(ab)_u = 0$ and then there exists a smooth function $\varphi = \varphi(v)$, depending only on v, such that:

$$ab = \varphi(v). \tag{3.9}$$

Our purpose is now to get a solution for a and b in order to obtain a parametrization for M.

Remark 2. We have already used that $N_u = 0$, but this fact also implies that the coefficients of the second fundamental form e and f vanish and so does the Gaussian curvature. Then, M is locally flat.

Integrating the equations (3.4) and (3.8), also using the (3.9), one gets.

Proposition 1. *[13] The functions a and b are given by the following expressions: either*

$$\begin{cases} a(u,v) = \frac{1}{-\coth\theta u + \alpha(v)}, \\ b(u,v) = \varphi(v)(-\coth\theta u + \alpha(v)), \end{cases} \tag{3.10}$$

where α and φ are smooth functions on M, or

$$\begin{cases} a(u,v) = 0, \\ b(u,v) = b(v). \end{cases}$$

When $a = 0$ leads to $b_u = 0$ and so F_u vanishes, we have the following.

Proposition 2. *[13] Let M be a spacelike helix surface in $\mathbb{E}^3_1$ parametrized by $F = F(u, v)$, where (u, v) are the coordinates given in* (3.3)*. If $a(u, v) = 0$ on M, then F describes an affine plane.*

Now we can consider the following.

Theorem 1. *[13] Let M be a spacelike helix surface in $\mathbb{E}^3_1$ with constant hyperbolic angle θ which is not totally geodesic. Up to a rigid motion of the ambient space, there exist local coordinates u and v such that M is given by the parametrization*

$$F(u, v) = (u \cosh\theta \cos v + \gamma_1(v), u \cosh\theta \sin v + \gamma_2(v), -u \sinh\theta), \qquad (3.11)$$

with

$$\gamma(v) = (\gamma_1(v), \gamma_2(v)) = \sinh\theta\Big(\int_0^v \alpha(\tau) \sin\tau d\tau, -\int_0^v \alpha(\tau) \cos\tau d\tau\Big)$$

where α is a smooth function on an interval I. Conversely, a parametrization $F(u, v)$ as above defines a helix surface in $\mathbb{E}^3_1$.

Proof. To begin, if we consider the parametrizations given in (3.11), we have:

$$F_u = (\cosh\theta \cos v, \cosh\theta \sin v, -\sinh\theta);$$
$$F_v = (-(u \cosh\theta - \alpha(v) \sinh\theta) \sin v, (u \cosh\theta - \alpha(v) \sinh\theta) \cos v, 0).$$

Thus, the unit normal is given by:

$$N = (\sinh\theta \sin v, \sinh\theta \cos v, -\cosh\theta)$$

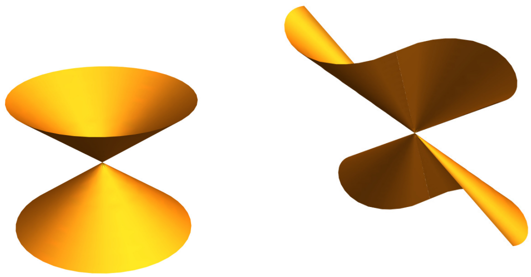
and hence the angle function $\nu = -\langle N, e_3\rangle = \cosh\theta$ is constant. Conversely, we have to prove that if M is a helix surface in $\mathbb{E}^3_1$ then it may be parametrized as in (3.11).

Since $F_u = E_1$, we can write $e_3 = \sinh\theta\, F_u + \cosh\theta\, N$ and so we get:

$$F(u, v) = (h(u, v), u \sin\theta) = (x(u, v), y(u, v), -u \sin\theta). \qquad (3.12)$$

Hence, in order to specify the expression for h we have to consider the expressions for a and b given in Proposition 1 observing that we have already given the characterization in the case for $a = 0$.At this point, the authors used the equations given in (3.10). From (3.5) they get $h_{uu} = 0$ and, since E_1 is a unit vector, $|h_u| = 1$ and so

$$h_u(u, v) = \cosh\theta f(v),$$



(a) $\alpha(y) = 0$ (b) $\alpha(y) = \frac{1}{\sin v}$

Figure 1: Constant angle spacelike surfaces for some values of α.

where $|f(v)| = 1$. Hence, integrating one obtains:

$$h(u, v) = u \cosh\theta f(v) + \gamma(v),$$

where γ is a smooth curve in $\mathbb{R}^2$. In addition one finds that

$$F_v = (h_v, 0) = (u \cosh\theta f'(v) + \gamma'(v), 0)$$

and from (3.6):

$$F_{uv} = \frac{\coth\theta}{u \coth\theta - \alpha(v)}(u \cosh\theta f'(v) + \gamma'(v), 0)$$

Moreover, since $h_{uv} = \cosh\theta f'(v)$, by comparing with the latter and since we can suppose that f is the natural parametrization of S^1 without loss of generality, one gets the parametrization in (3.11). QED

At this point we are able to introduce some examples of helix surfaces (see [13]) with the parametrization given in (1). In particular, in Fig.1 we plot some examples for $\theta = 1$ and different values for $\alpha(v)$.

3.2 Helix surfaces constructed on curves

Recalling Remark 2 and that, as in the Euclidean space (see [17]), all flat surfaces are characterized to be locally isometric to planes, cones, cylinders or tangent developable surfaces, we can consider the following.

Corollary 1. *[13] Any spacelike helix surface in $\mathbb{E}^3_1$ is isometric to either a plane, a cone, a cylinder or a tangent developable surface.*

The fact that a constant angle (spacelike) surface is a ruled surface appears clearly in Theorem 1. Exactly, the parametrization (3.11) writes as

$$F(u,v) = (\gamma(v),0) + u(\cosh\theta\cos v, \cosh\theta\sin v, -\sinh\theta)$$

which proves that these surfaces are ruled.

Consequently it is very natural for one to ask if the *viceversa* holds. In particular, in [13] the authors studied tangent developable surfaces, cones and cylinders that are helix surfaces.

In the first case, call $\gamma(s)$ the defining curve for the tangent surfaces; then we can express the tangent, normal and binormal vectors as follows:

$$\begin{cases} \mathbf{T}(s) = \gamma'(s), \\ \mathbf{N}(s) = \gamma''(s)/\kappa(s), \\ \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s), \end{cases}$$

where $\kappa(s) = |\gamma''(s)| > 0$ is the curvature of γ at s. The function $\tau(s) = -\langle \mathbf{N}'(s), \mathbf{B}(s)\rangle$ is called the *torsion* of γ at s. The principal result for tangent surface is the following.

Theorem 2. *[13] Let M be a tangent developable spacelike surface generated by γ. Then M is a helix surface if and only if γ is a helix with $\tau^2 < \kappa^2$. Moreover the direction U with which M makes a constant hyperbolic angle θ is given by*

$$U = \frac{1}{\sqrt{\kappa^2 - \tau^2}}\Big(-\tau(s)\mathbf{T}(s) + \kappa(s)\mathbf{B}(s)\Big) \tag{3.13}$$

and the angle θ is determined by the relation

$$\cosh\theta = \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}. \tag{3.14}$$

For the case of cones and cylinders we can report the following results.

Theorem 3. *[13] The following hold.*

- *The only constant angle (spacelike) cylinders are planes.*
- *A (spacelike) cone is a helix surface if and only if the generating curve is either a circle in a spacelike plane or a straight line (and the surface is a plane).*

4 Helix surfaces in the Lorentzian Heisenberg group

Let $H_3(\tau)$ ($\tau \neq 0$) denote the 3-dimensional Heisenberg group given by $\mathbb{R}^3$ equipped with the 1-parameter family of Lorentzian metrics

$$g_\tau = dx^2 + dy^2 - (dz - \tau(ydx - xdy))^2,$$

which makes the map $\pi : H_3(\tau) \to \mathbb{R}^2$ a Riemannian submersion. Now, let us consider the vector fields:

$$E_1 = \frac{\partial}{\partial x} + \tau y \frac{\partial}{\partial z}, \qquad E_2 = \frac{\partial}{\partial y} - \tau x \frac{\partial}{\partial z}, \qquad E_3 = \frac{\partial}{\partial z},$$

which form a Lorentzian orthonormal basis on $H_3(\tau)$. At this point, one can gets the associated Levi-Civita connection ∇^τ:

$$\begin{array}{ll} \nabla^\tau_{E_1} E_1 = \nabla^\tau_{E_2} E_2 = \nabla^\tau_{E_3} E_3 = 0, & \nabla^\tau_{E_2} E_1 = \tau E_3 = -\nabla^\tau_{E_1} E_2, \\ \nabla^\tau_{E_3} E_1 = -\tau E_2 = \nabla^\tau_{E_1} E_3, & \nabla^\tau_{E_3} E_2 = \tau E_1 = \nabla^\tau_{E_2} E_3. \end{array} \tag{4.1}$$

We may observe that E_3 is a unit timelike vector field tangent to the fibers of π. By (4.1), we have

$$\nabla^\tau_X E_3 = \tau X \wedge E_3 \quad \forall X \in \mathfrak{X}(H_3(\tau)), \tag{4.2}$$

where $\wedge$ is the cross product in $H_3(\tau)$ defined by the following relations

$$E_2 \wedge E_3 = E_1, \quad E_3 \wedge E_1 = E_2, \quad E_1 \wedge E_2 = -E_3$$

Also, using the fixed convention, one gets the non zero components of the Riemann curvature tensor, as follows:

$$\begin{aligned} R^\tau(E_1, E_2)E_1 = -3\tau^2 E_2, &\quad R^\tau(E_1, E_3)E_1 = \tau^2 E_3, \\ R^\tau(E_1, E_2)E_2 = 3\tau^2 E_1, &\quad R^\tau(E_1, E_3)E_3 = \tau^2 E_1, \\ R^\tau(E_2, E_3)E_3 = \tau^2 E_2, &\quad R^\tau(E_2, E_3)E_2 = \tau^2 E_3. \end{aligned} \tag{4.3}$$

Moreover, the tensor R^τ can be described as in the following result, obtained by accurate calculations.

Proposition 3. *[20] The Riemann curvature tensor R^τ of $H_3(\tau)$ is determined by*

$$\begin{aligned} R^\tau(X, Y)Z = &3\tau^2[g_\tau(Y, Z)X - g_\tau(X, Z)Y] \\ &+ 4\tau^2[g_\tau(Y, E_3)g_\tau(Z, E_3)X - g_\tau(X, E_3)g_\tau(Z, E_3)Y \\ &+ g_\tau(X, E_3)g_\tau(Y, Z)E_3 - g_\tau(Y, E_3)g_\tau(X, Z)E_3], \end{aligned}$$

for all vector fields X,Y,Z on $H_3(\tau)$.

4.1 The structure equations for surfaces in $H_3(\tau)$

Now, let us consider a pseudo-Riemannian oriented surface M immersed into $H_3(\tau)$ and, in analogy with previous section, we can specify the relation (2.2) defining the *angle function*, or simply *angle*, by:

$$\nu := g_\tau(N, E_3) g_\tau(N, N),$$

where N is the unit normal to M with $\lambda = g_\tau(N, N) = \pm 1$.

Then, if we consider the decomposition $E_3 = T + \nu N$ we have:

$$g_\tau(T, T) = -(1 + \lambda\nu^2). \tag{4.4}$$

that leads to:

$$\begin{aligned} \nabla^\tau_X E_3 &= \nabla^\tau_X T + X(\nu)N + \nu\nabla^\tau_X N = \\ &= \nabla_X T + \lambda g_\tau(A(X), T)N + X(\nu)N - \nu A(X), \end{aligned}$$

moreover, by (4.2) one gets:

$$\nabla^\tau_X E_3 = \tau X \wedge E_3 = \tau\lambda g_\tau(JX, T)N - \tau\nu JX$$

where we called $JX := N \wedge X$, the rotation of angle $\pi/2$ in TM, which satisfies the relations $g_\tau(JX, JX) = -\lambda g_\tau(X, X)$ and $J^2X = \lambda X$.

Comparing the two expressions one obtains:

$$\begin{cases} \nabla_X T = \nu(A(X) - \tau JX) \\ X(\nu) = -\lambda g_\tau(A(X) - \tau JX, T). \end{cases} \tag{4.5}$$

Now, using the equation (2.5) for a pseudo-Riemannian surface and recalling the Proposition 3 we will report the expressions of the Gauss and Codazzi equations for a pseudo-Riemannian surface M immersed into $H_3(\tau)$:

Proposition 4. *[20] Let X,Y denote vector fields tangent to M, K the Gaussian curvature of M and $\bar{K}$ the sectional curvature in $H_3(\tau)$ of the plane tangent to M. Then,*

$$K = \bar{K} + \lambda detA = -\tau^2 + \lambda\left[detA - 4\nu^2\tau^2\right] \tag{4.6}$$

and

$$\nabla_X A(Y) - \nabla_Y A(X) - A[X, Y] = 4\lambda\nu\tau^2\left[g_\tau(X, T)Y - g_\tau(Y, T)X\right].$$

4.2 Spacelike helix surfaces in $H_3(\tau)$

We now begin the presentation of the study of helix surfaces in $H_3(\tau)$ considering firstly the spacelike case, where $\lambda = -1$. Therefore, from the equation (4.4) it follows that (up to the orientation of N) we can write $\nu = \cosh\vartheta$, where $\vartheta \geq 0$ is called the *hyperbolic angle function* between N and E_3. Now, let us assume that ϑ is constant.

Remark 3. We observe that $\vartheta \neq 0$. In fact, if $\vartheta = 0$, then E_3 would be parallel to N and so E_1 and E_2 would be tangent to M, which is impossible as the horizontal distribution of π is not integrable.

Proposition 5. *[20] Let M denote a helix spacelike surface in $H_3(\tau)$ and N the unit vector field normal to M. Then:*

(i) with respect to the tangent basis $\{T, JT\}$, the matrix describing the shape operator is given by

$$A = \begin{pmatrix} 0 & -\tau \\ -\tau & \mu \end{pmatrix}$$

for some smooth function μ on M;

(ii) the Levi-Civita connection ∇ of M is described by

$$\nabla_T T = -2\tau\cosh\vartheta JT, \qquad \nabla_{JT}T = \mu\cosh\vartheta JT,$$
$$\nabla_T JT = 2\tau\cosh\vartheta T, \qquad \nabla_{JT}JT = -\mu\cosh\vartheta T;$$

(iii) the Gaussian curvature of M is constant and is given by

$$K = 4\tau^2\cosh^2\vartheta;$$

(iv) the function μ satisfies the equation

$$T(\mu) + \mu^2\cosh\vartheta + 4\tau^2\cosh^3\vartheta = 0. \tag{4.7}$$

Proof. Considering the tangent basis $\{T, JT\}$ and using (4.5), one gets *(i)* and *(ii)*.

One can now proceeds calculating the Gaussian curvature. By (4.6) one finds

$$K = 4\tau^2\nu^2 - [detA + \tau^2] = 4\tau^2\cosh^2\vartheta.$$

Finally, one can calculate

$$\begin{aligned}
&\nabla_T A(JT) - \nabla_{JT}A(T) - A[T, JT] = \\
&= \nabla_T(-\tau T + \mu JT) - \nabla_{JT}(-\tau JT) - A[2\tau\cosh\vartheta T - \mu\cosh\vartheta JT] = \\
&= [4\tau^2\cosh\vartheta + T(\mu) + \mu^2\cosh\vartheta]JT.
\end{aligned}$$

By Proposition 4, the authors obtained

$$\begin{aligned}\nabla_T A(JT) - \nabla_{JT} A(T) - A[T, JT] &= -4\tau^2 \cosh\vartheta [g_\tau(T,T)JT - g_\tau(JT,T)T] = \\ &= -4\tau^2 \cosh\vartheta \, \sinh^2\vartheta JT\end{aligned}$$

and so, by comparing, one gets (4.7). $\boxed{QED}$

As we know that $g_\tau(E_3, N) = -\cosh\vartheta$ and that E_3 is timelike, then there exists a smooth function φ on M such that $N = \sinh\vartheta\cos\varphi E_1 + \sinh\vartheta\sin\varphi E_2 + \cosh\vartheta E_3$, then:

$$\begin{aligned}&T = E_3 - \cosh\vartheta N = -\sinh\vartheta[\cosh\vartheta\cos\varphi E_1 + \cosh\vartheta\sin\varphi E_2 + \sinh\vartheta E_3],\\ &JT = \sinh\vartheta(\sin\varphi E_1 - \cos\varphi E_2).\end{aligned}$$

Moreover, we can consider the following

$$\begin{aligned}A(T) &= -\nabla^\tau_T N = [T(\varphi) + \tau\cosh^2\vartheta + \tau\sinh^2\vartheta]JT,\\ A(JT) &= -\nabla^\tau_{JT} N = JT(\varphi)JT - \tau T,\end{aligned}$$

and, comparing with *(i)* of Proposition 5, one gets:

$$\begin{cases} JT(\varphi) = \mu \\ T(\varphi) = -2\tau\cosh^2\vartheta, \end{cases}$$

whose compatibility is equivalent to (4.7). We now choose local coordinates (x, y) on M such that

$$\partial_x = T, \quad \partial_y = aT + bJT \tag{4.8}$$

where a, b are smooth functions on M. The condition $0 = [\partial_x, \partial_y]$ leads to:

$$\begin{cases} a_x = -2\tau b\cosh\vartheta \\ b_x = \mu\, b\, \cosh\vartheta. \end{cases}$$

In conclusion, integrating (4.7) one gets

$$\mu(x, y) = 2\tau\,\cosh\vartheta\,\tan\left(\eta(y) - 2\tau\cosh^2\vartheta\, x\right);$$

then, since we are searching just for one solution, let us take for example

$$\begin{cases} a(x, y) = \frac{\sin(\eta(y) - 2\tau\cosh^2\vartheta\, x)}{\cosh\vartheta} \\ b(x, y) = \cos(\eta(y) - 2\tau\cosh^2\vartheta\, x). \end{cases} \tag{4.9}$$

Therefore one gets $\varphi(x, y) = -2\tau\cosh^2\vartheta\, x + c$, where c is a real constant.

Theorem 4. *[20] Let M be a helix spacelike surface in $H_3(\tau)$ with constant hyperbolic angle ϑ. Then, with respect to the local coordinates (x, y) on M defined in (4.8) and (4.9), the position vector F of M in $\mathbb{R}^3$ is given by*

$$
\begin{aligned}
F(x,y) = \Big(&\frac{\tanh\vartheta}{2\tau}\sin x + f_1(y), -\frac{\tanh\vartheta}{2\tau}\cos x + f_2(y),\\
&-\frac{\sinh^2\vartheta}{2\tau}x + \frac{\tanh\vartheta}{2\tau}[f_1(y)\cos x + f_2(y)\sin x] + f_3(y)\Big),
\end{aligned}
\tag{4.10}
$$

where f_1, f_2, f_3 satisfy:

$$
f_1'(y)^2 + f_2'(y)^2 = \sinh^2\vartheta, \qquad f_3'(y) = \tau(f_2(y)f_1'(y) - f_1(y)f_2'(y)).
$$

Proof. By definition of position vector F in $\mathbb{R}^3$ one gets

$$
\partial_x F = T = -\sinh\vartheta[\cosh\vartheta\cos\varphi E_{1|F} + \cosh\vartheta\sin\varphi E_{2|F} + \sinh\vartheta E_{3|F}]
$$

and

$$
\begin{aligned}
\partial_y F &= aT + bJT =\\
&= \sinh\vartheta[(-a\cosh\vartheta\cos\varphi + b\sin\varphi)E_{1|F}\\
&\quad - (a\cosh\vartheta\sin\varphi + b\cos\varphi)E_{2|F} - a\sinh\vartheta E_{3|F}].
\end{aligned}
$$

Moreover, specifying the expressions for E_1, E_2 and E_3, in the equations above one can calculate explicitly F_1, F_2 and F_3. Therefore, using the map $\phi(x) \mapsto x$ one gets (4.10) and also the following conditions:

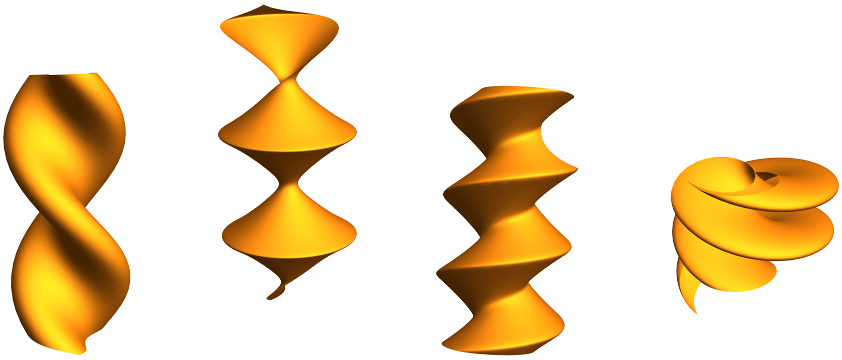
$$
\begin{cases}
f_1'(y) = -\sinh\vartheta\sin(\eta(y) - c),\\
f_2'(y) = -\sinh\vartheta\cos(\eta(y) - c),\\
f_3'(y) = \tau(f_2(y)f_1'(y) - f_1(y)f_2'(y)).
\end{cases}
\tag{4.11}
$$

QED

Example 1. [20] Choosing $\eta(y) = y + c$ in (4.11) one gets:

$$
f_1(y) = -\sinh\vartheta\sin(y), \quad f_2(y) = -\sinh\vartheta\cos(y), \quad f_3(y) = \tau y\sinh^2\vartheta.
$$

Therefore we have an explicit parametrization of helix spacelike surfaces depending only on the choice of ϑ. In Fig. 2 we observe plots for some values of ϑ.



(a) $\vartheta = \pi/3$ (b) $\vartheta = \pi/4$ (c) $\vartheta = \pi/6$ (d) $\vartheta = \pi/8$

Figure 2: Constant angle spacelike surfaces for some values of ϑ.

4.3 Timelike helix surfaces in $H_3(\tau)$

Following the same scheme as in the spacelike case, we now approach the study of timelike helix surfaces in $H_3(\tau)$, where $\lambda = 1$. In this case, from the equation (4.4) it follows that (up to the orientation of N) we can write $\nu = \sinh\vartheta$, where $\vartheta \geq 0$ is called, analogously, the *hyperbolic angle function* between N and E_3. Now, let us assume again that ϑ is constant.

Remark 4. From now on we can assume, also in this case, that $\vartheta \neq 0$. In fact, if $\vartheta = 0$, this time E_3 would be tangent to M and therefore M is a cylindrical surface.

Proceeding as in the previous section the authors proved an analogue of Proposition 5 determining the shape operator, the connection ∇ and the Gaussian curvature.

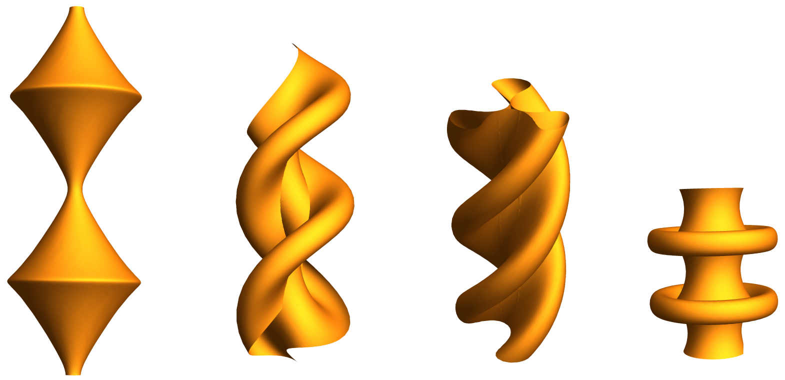
As we know that $g_\tau(E_3, N) = \sinh\vartheta$ and that E_3 is timelike, then, as above, there exists a smooth function φ on M such that $N = \cosh\vartheta\cos\varphi E_1 + \cosh\vartheta\sin\varphi E_2 + \sinh\vartheta E_3$.

We now choose local coordinates (x, y) on M as in 4.8, then the condition $0 = [\partial_x, \partial_y]$ and the analogue of (4.7) now leads to

$$\mu(x, y) = 2\tau\,\sinh\vartheta\,\tan(\eta(y) - 2\tau\cosh^2\vartheta\, x),$$

and

$$\begin{cases} a(x, y) = -\frac{\sin(\eta(y) - 2\tau\sinh^2\vartheta\, x)}{\sinh\vartheta} \\ b(x, y) = \cos(\eta(y) - 2\tau\sinh^2\vartheta\, x), \end{cases} \tag{4.12}$$



(a) $\vartheta = \pi/3$ (b) $\vartheta = \pi/4$ (c) $\vartheta = \pi/6$ (d) $\vartheta = \pi/8$

Figure 3: Constant angle timelike surfaces for some values of ϑ.

since we are searching again just for one solution for a and b. In addition, one gets $\varphi(x, y) = 2\tau \sinh^2 \vartheta\, x + c$, where $c \in \mathbb{R}$.

Again, similarly as in Proposition 4, one gets the following.

Theorem 5. *[20] Let M be a helix timelike surface in $H_3(\tau)$ with constant hyperbolic angle ϑ. Then, with respect to the local coordinates (x, y) on M defined in (4.8) and (4.12), the position vector F of M in $\mathbb{R}^3$ is given by*

$$\begin{aligned} F(x,y) = \Big(&\frac{-\coth\vartheta}{2\tau}\sin x + f_1(y), \frac{\coth\vartheta}{2\tau}\cos x + f_2(y), \\ &\frac{\cosh^2\vartheta}{2\tau}x - \frac{\coth\vartheta}{2\tau}[f_1(y)\cos x + f_2(y)\sin x] + f_3(y)\Big), \end{aligned} \tag{4.13}$$

where f_1, f_2, f_3 satisfy:

$$f_1'(y)^2 + f_2'(y)^2 = \cosh^2\vartheta, \qquad f_3'(y) = \tau(f_2(y)f_1'(y) - f_1(y)f_2'(y)).$$

Example 2. [20] Choosing $\eta(y) = y + c$ in (4.11) one gets:

$$f_1(y) = -\cosh\vartheta\cos(y), \quad f_2(y) = -\sinh\vartheta\sin(y), \quad f_3(y) = -\tau y\cosh^2\vartheta.$$

Therefore we have an explicit parametrization of helix timelike surfaces depending only on the choice of ϑ. In Fig. 3 we observe plots for some values of ϑ.

5 Helix surfaces in Lorentzian Berger Spheres

The *3-dimensional Lorentzian Berger spheres* are defined as follows, in terms of the Hopf fibration. Let us consider $\mathbb{S}^2(1/2) \subset \mathbb{C} \times \mathbb{R}$ and $\mathbb{S}^3 \subset \mathbb{C}^2$ in order to define the *Hopf map*:

$$\begin{aligned}\psi: \quad & \mathbb{S}^3 \to \mathbb{S}^2(1/2) \\ & \psi(z,w) = \frac{1}{2}(2z\bar{w}, |z|^2 - |w|^2).\end{aligned}$$

Such a map is a Riemannian submersion and the vector fields:

$$X_1(z,w) = (iz, iw), \quad X_2(z,w) = (-i\bar{w}, i\bar{z}), \quad X_3(z,w) = (-\bar{w}, \bar{z})$$

parallelize $\mathbb{S}^3$, where X_1 is vertical and X_2, X_3 are horizontal. In this way the *3-dimensional Lorentzian Berger spheres* $\mathbb{S}^3_\varepsilon$ are given by $\mathbb{S}^3$ endowed with the following 1-parameter family of Lorentzian metrics:

$$g_\varepsilon(X,Y) = \langle X, Y\rangle - (\varepsilon^2+1)\langle X, X_1\rangle\langle Y, X_1\rangle,$$

where $\langle,\rangle$ represents the canonical metric of $\mathbb{S}^3$. Now, $\{E_1 = \varepsilon^{-1}X_1, E_2 = X_2, E_3 = X_3\}$ is a pseudo-orthonormal basis for $\mathbb{S}^3_\varepsilon$. Computing the Lie brackets $[E_i, E_j]$ and using the Koszul formula, the authors obtained the description of the Levi-Civita connection of $\mathbb{S}^3_\varepsilon$ with respect to $\{E_1, E_2, E_3\}$:

$$\begin{array}{lll} \nabla^\varepsilon_{E_1}E_1 = 0, & \nabla^\varepsilon_{E_1}E_2 = \frac{2+\varepsilon^2}{\varepsilon}E_3, & \nabla^\varepsilon_{E_1}E_3 = -\frac{2+\varepsilon^2}{\varepsilon}E_2, \\ \nabla^\varepsilon_{E_2}E_1 = \varepsilon E_3, & \nabla^\varepsilon_{E_2}E_2 = 0, & \nabla^\varepsilon_{E_2}E_3 = \varepsilon E_1, \\ \nabla^\varepsilon_{E_3}E_1 = -\varepsilon E_2, & \nabla^\varepsilon_{E_3}E_2 = -\varepsilon E_1, & \nabla^\varepsilon_{E_3}E_3 = 0. \end{array} \tag{5.1}$$

We may observe that E_1, called *Hopf vector field*, is a unit timelike vector field tangent to the fibers of ψ. By (5.1), we have

$$\nabla^\varepsilon_X E_1 = -\varepsilon X \wedge E_1, \qquad X \in \mathfrak{X}(\mathbb{S}^3_\varepsilon), \tag{5.2}$$

where the cross product $\wedge$ is given by:

$$U \wedge V = \begin{vmatrix} -E_1 & E_2 & E_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \qquad U, V \in \mathfrak{X}(\mathbb{S}^3_\varepsilon).$$

We now consider the curvature tensor, taken with the convention (2.1) and, using (5.1), one gets

$$\begin{aligned} R^\varepsilon(E_1,E_2)E_1 &= -\varepsilon^2 E_2, \quad R^\varepsilon(E_1,E_3)E_1 = -\varepsilon^2 E_3, \\ R^\varepsilon(E_1,E_2)E_2 &= -\varepsilon^2 E_1, \quad R^\varepsilon(E_1,E_3)E_3 = -\varepsilon^2 E_1, \\ R^\varepsilon(E_2,E_3)E_3 &= (4+3\varepsilon^2)E_2, \quad R^\varepsilon(E_2,E_3)E_2 = -(4+3\varepsilon^2)E_3. \end{aligned} \tag{5.3}$$

The following result is obtained in a completely analogous way as in Proposition 3.

Proposition 6. *[21] The Riemann curvature tensor R^ε of $\mathbb{S}^3_\varepsilon$ is determined by:*

$$
\begin{aligned}
R^\varepsilon(X,Y)Z =&(4+\varepsilon^2)[g_\varepsilon(Y,Z)X - g_\varepsilon(X,Z)Y] \\
&+4(1+\varepsilon^2)[g_\varepsilon(Y,E_1)g_\varepsilon(Z,E_1)X - g_\varepsilon(X,E_1)g_\varepsilon(Z,E_1)Y \\
&+g_\varepsilon(X,E_1)g_\varepsilon(Y,Z)E_1 - g_\varepsilon(Y,E_1)g_\varepsilon(X,Z)E_1],
\end{aligned}
$$

for all vector field X, Y, Z on $\mathbb{S}^3_\varepsilon$.

Now we can conclude this subsection, recalling that the isometry group of $\mathbb{S}^3_\varepsilon$ can be identified with:

$$
\{Q \in O(4) : QJ_1 = \pm J_1 Q\} \quad \text{where} \quad J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
$$

Moreover, in order to describe a 1-parameter family $Q(y)$ of orthogonal matrices 4×4 in $Iso(\mathbb{S}^3_\varepsilon)$, one can use four functions: $\xi_1 = \xi_1(y)$, $\xi_2 = \xi_2(y)$, $\xi_3 = \xi_3(y)$ and $\xi = \xi(y)$ and consider:

$$
Q(\xi_1,\xi_2,\xi_3,\xi)(y) = \begin{pmatrix} \mathbf{r}_1(y) \\ \pm J_1\mathbf{r}_1(y) \\ \cos\xi(y)J_2\mathbf{r}_1(y) + \sin\xi J_3\mathbf{r}_1(y) \\ \mp\cos\xi(y)J_3\mathbf{r}_1(y) \pm \sin\xi J_2\mathbf{r}_1(y) \end{pmatrix}, \tag{5.4}
$$

where

$$
\mathbf{r}_1(y) = (\cos\xi_1(y)\cos\xi_2(y), -\cos\xi_1(y)\sin\xi_2(y), \sin\xi_1(y)\cos\xi_3(y), -\sin\xi_1(y)\sin\xi_3(y)),
$$

and

$$
J_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
$$

5.1 The structure equations for surfaces in $\mathbb{S}^3_\varepsilon$

Let us consider a pseudo-Riemannian oriented surface M immersed into $\mathbb{S}^3_\varepsilon$, and in analogy with previous cases we can specify (2.2), defining the *angle function*, or simply *angle*, by:

$$
\nu := g_\varepsilon(N,N)g_\varepsilon(N,E_1), \tag{5.5}
$$

where N is the unit normal to M.

Therefore, if we call again $\lambda := g_\varepsilon(N,N) = \pm 1$ and we specify the relations found in (2.3) and (2.4) for this case, all of them are still true. Then, considering the decomposition $E_1 = T + \nu N$ one gets:

$$g_\varepsilon(T,T) = -(1+\lambda\nu^2),$$

that leads to:

$$\begin{aligned}\nabla^\varepsilon_X E_1 &= \nabla^\varepsilon_X T + X(\nu)N + \nu\nabla^\varepsilon_X N = \\ &= \nabla_X T + \lambda g_\varepsilon(A(X),T)N + X(\nu)N - \nu A(X).\end{aligned}$$

Moreover, by (5.2) one gets:

$$\nabla^\varepsilon_X E_1 = -\varepsilon X \wedge E_1 = -\varepsilon\lambda g_\varepsilon(JX,T)N + \varepsilon\nu JX,$$

where we called $JX := N \wedge X$, $g_\varepsilon(JX,JY) = -\lambda g_\varepsilon(X,Y)$ and $J^2X = \lambda X$. Comparing the two expressions one obtains:

$$\begin{cases}\nabla_X T = \nu(A(X) + \varepsilon JX) \\ X(\nu) = -\lambda g_\varepsilon(A(X) + \varepsilon JX, T).\end{cases}$$

Now, using Proposition 6, the authors obtained the expressions of the Gauss and Codazzi equations for a pseudo-Riemannian surface M immersed into $\mathbb{S}^3_\varepsilon$:

Proposition 7. *[21] Let X, Y denote vector fields tangent to M, K the Gaussian curvature of M and $\bar{K}$ the sectional curvature in $\mathbb{S}^3_\varepsilon$ of the plane tangent to M. Then,*

$$K = \bar{K} + \lambda det A = -\varepsilon^2 + \lambda[det A - 4\nu^2(1+\varepsilon^2)]$$

and

$$\nabla_X A(Y) - \nabla_Y A(X) - A[X,Y] = 4\lambda\nu(1+\varepsilon^2)[g_\varepsilon(X,T)Y - g_\varepsilon(Y,T)X].$$

5.2 Helix surfaces in $\mathbb{S}^3_\varepsilon$

In the case of $\mathbb{S}^3_\varepsilon$, we consider the expression for the angle function given in 5.5. Then, one can study naturally helix surfaces in this ambient space.

We observe that if a helix surface M in $\mathbb{S}^3_\varepsilon$ is spacelike, then $|\nu| > 1$. Moreover, if M is timelike with $\nu = 0$, then E_1 is tangent to M and so M is an *Hopf tube*. For this reason, by now, we can assume $\nu \neq 0$. With same reasoning as in Proposition 5 the authors proved the following.

Proposition 8. *[21] Let M denote a helix surface in $\mathbb{S}^3_\varepsilon$ and N the unit vector field normal to M. Then:*

(i) with respect to the tangent basis $\{T, JT\}$, the matrix describing the shape operator is given by

$$A = \begin{pmatrix} 0 & -\lambda\varepsilon \\ \varepsilon & \mu \end{pmatrix}$$

for some smooth function μ on M;

(ii) the Levi-Civita connection ∇ of M is described by

$$\nabla_T T = 2\varepsilon\nu JT, \quad \nabla_{JT} T = \mu\nu JT, \quad \nabla_T JT = 2\lambda\varepsilon\nu T, \quad \nabla_{JT} JT = \lambda\mu\nu T;$$

(iii) the Gaussian curvature of M is constant and is given by

$$K = -4\lambda(1+\varepsilon^2)\nu^2;$$

(iv) function μ satisfies equation

$$T(\mu) + \nu\mu^2 + 4\lambda\nu B = 0, \tag{5.6}$$

where we put $B := 1 + \lambda\nu^2(1+\varepsilon^2)$.

Remark 5. We observe that if M is a spacelike (respectively, timelike) surface, then the constant B is negative (respectively, positive). Therefore, in both cases, we have that $\lambda B > 0$. Consequently, if a helix surface is minimal (i.e., $trA = 0$), from *(i)* of the Proposition 8 it follows that $\mu = 0$ and, so $\nu = 0$ and the surface is a timelike Hopf tube.

As we know that $g_\varepsilon(E_1, N) = \lambda\nu$ and that E_1 is timelike, then there exists a smooth function φ on M such that: $N = \lambda\nu E_1 + \sqrt{\lambda+\nu^2}\cos\varphi E_2 + \sqrt{\lambda+\nu^2}\sin\varphi E_3$, then:

$$\begin{aligned} T &= E_1 - \nu N = (1+\lambda\nu^2)E_1 - \nu\sqrt{\lambda+\nu^2}\cos\varphi E_2 - \nu\sqrt{\lambda+\nu^2}\sin\varphi E_3, \\ JT &= N \wedge T = \sqrt{\lambda+\nu^2}(\sin\varphi E_2 - \cos\varphi E_3). \end{aligned}$$

Moreover, one gets the following

$$\begin{aligned} A(T) &= -\nabla^\varepsilon_T N = [T(\varphi) + \varepsilon^{-1}(2+\varepsilon^2)(1+\lambda\nu^2) + \lambda\varepsilon\nu^2]JT, \\ A(JT) &= -\nabla^\varepsilon_{JT} N = JT(\varphi)JT - \lambda\varepsilon T \end{aligned}$$

and, comparing with *(i)* of Proposition 8, one obtains:

$$\begin{cases} JT(\varphi) = \mu \\ T(\varphi) = -2\varepsilon^{-1}B \end{cases}$$

whose compatibility is equivalent to (5.6). We now choose local coordinates (x, y) on M such that

$$\partial_x = T, \quad \partial_y = aT + bJT, \tag{5.7}$$

where a, b are smooth functions on M. Now, since we are searching just for one solution, by condition $0 = [\partial_x, \partial_y]$ and integrating (5.6) one gets, for example, $\mu(x, y) = 2\sqrt{\lambda B}\tan(\eta(y) - 2\nu\sqrt{\lambda B}x)$ and

$$\begin{cases} a(x, y) = \frac{\lambda\varepsilon}{\sqrt{\lambda B}}\sin(\eta(y) - 2\nu\sqrt{\lambda B}x) \\ b(x, y) = \cos(\eta(y) - 2\nu\sqrt{\lambda B}x). \end{cases} \tag{5.8}$$

Therefore one obtains $\varphi(x, y) = -2\varepsilon^{-1}Bx + c$ where $c \in \mathbb{R}$.

Now, using the definition of position vector F in $\mathbb{R}^4$ and clever calculations the authors proved the following.

Proposition 9. *[21] Let M be a helix surface in the Lorentzian Berger sphere $\mathbb{S}^3_\varepsilon$ with constant angle function ν. Then, with respect to the local coordinates (x, y) on M defined above, the position vector F of M in $\mathbb{R}^4$ satisfies the equation*

$$\frac{\partial^4 F}{\partial x^4} + (\tilde{b}^2 - 2\tilde{a})\frac{\partial^2 F}{\partial x^2} + \tilde{a}^2 F = 0, \tag{5.9}$$

where $\tilde{a} = \lambda\varepsilon^{-2}B(\lambda + \nu^2)$ and $\tilde{b} = -2\varepsilon^{-1}B$.

Remark 6. By $|F|^2 = 1$ and the relations given in Proposition 9, one gets:

$$\begin{aligned}
&\langle F, F\rangle = 1, && \langle F_x, F_x\rangle = \tilde{a}, && \langle F, F_x\rangle = 0,\\
&\langle F_x, F_{xx}\rangle = 0, && \langle F_{xx}, F_{xx}\rangle = D, && \langle F, F_{xx}\rangle = -\tilde{a},\\
&\langle F_x, F_{xxx}\rangle = -D, && \langle F_{xx}, F_{xxx}\rangle = 0, && \langle F, F_{xxx}\rangle = 0,\\
&\langle F_{xxx}, F_{xxx}\rangle = E,
\end{aligned} \tag{5.10}$$

where we put

$$D = \tilde{a}\tilde{b}^2 - 3\tilde{a}^2, \quad E = (\tilde{b}^2 - 2\tilde{a})D - \tilde{a}^3.$$

The following result is then obtained integrating equation (5.9).

Theorem 6. *[21] Let M be a helix surface in $\mathbb{S}^3_\varepsilon$ with constant angle function ν. Then, with respect to the local coordinates (x, y) defined above, the position vector F of M in $\mathbb{R}^4$ is explicitly given by*

$$F(x, y) = \cos(\alpha_1 x)g^1(y) + \sin(\alpha_1 x)g^2(y) + \cos(\alpha_2 x)g^3(y) + \sin(\alpha_2 x)g^4(y),$$

where

$$\alpha_{1,2} = \varepsilon^{-1}(\lambda B \pm \varepsilon|\nu|\sqrt{\lambda B})$$

are real constants and $g^i(y)$, $i = 1, 2, 3, 4$, are mutually orthogonal vector fields in $\mathbb{R}^4$, depending only on y, such that, setting $g_{ij} = \langle g^i(y), g^j(y)\rangle$ for all indices i,j, we have:

$$g_{11} = g_{22} = \frac{\lambda\varepsilon}{2B}\alpha_2, \qquad g_{33} = g_{44} = \frac{\lambda\varepsilon}{2B}\alpha_1.$$

Proof. The proof of this theorem consists in two parts. The first one is essentially the integration of (5.9) in terms of $\alpha_{1,2}$, the solutions of the associated characteristic equation, and $g^i(y)$, $i = 1, 2, 3, 4$, four vector fields in $\mathbb{R}^4$, depending only on y. This leads to:

$$F(x, y) = \cos(\alpha_1 x)g^1(y) + \sin(\alpha_1 x)g^2(y) + \cos(\alpha_2 x)g^3(y) + \sin(\alpha_2 x)g^4(y).$$

The second one consists in the computation of the norms $g_{ij} = \langle g^i(y), g^j(y)\rangle$ for all indices i,j by the solution of a system obtained evaluating $F(x, y)$ on $(0, y)$, in the relations found in (5.10). $\boxed{QED}$

5.3 The Characterization Theorem of the helix surfaces in $\mathbb{S}^3_\varepsilon$

We first observe that if F is the position vector of a helix surface in $\mathbb{S}^3_\varepsilon$, we have that:

$$J_1F(x, y) = X_{1|F(x,y)} = \varepsilon E_{1|F(x,y)} = \varepsilon(F_x + \nu N)$$

and thus, the conditions under which an immersion defines a helix surface in $\mathbb{S}^3_\varepsilon$ are given in the following proposition.

Proposition 10. *[21] Let $F : \Omega \to \mathbb{S}^3_\varepsilon$ be an immersion from an open set $\mathbb{R}^2$, with local coordinates (x, y). Then $F(\Omega)$ is a spacelike (respectively, timelike) helix surface and the projection of $E_1 = \varepsilon^{-1}J_1F$ to the tangent space of $F(\Omega) \subset \mathbb{S}^3_\varepsilon$ is F_x if and only if*

$$\begin{cases} g_\varepsilon(F_x, F_x) = g_\varepsilon(E_1, F_x) = -(1 + \lambda\nu^2) \\ g_\varepsilon(F_x, F_y) - g_\varepsilon(F_y, E_1) = 0, \end{cases} \tag{5.11}$$

where $\lambda = 1$ (respectively, $\lambda = -1$).

Now, we can consider the main result.

Theorem 7. *[21] Let M be a helix surface in $\mathbb{S}^3_\varepsilon$. Then, locally the position vector of M in $\mathbb{R}^4$ with respect to the local coordinates (x, y) on M defined in* (5.7) *and* (5.8), *is given by*

$$F(x, y) = Q(y)\beta(x),$$

where

$$\beta(x) = (\sqrt{g_{11}}\cos(\alpha_1 x), \sqrt{g_{11}}\sin(\alpha_1 x), \sqrt{g_{33}}\cos(\alpha_2 x), \sqrt{g_{33}}\sin(\alpha_2 x))$$

is a twisted geodesic in the torus $\mathbb{S}^1(\sqrt{g_{11}}) \times \mathbb{S}^1(\sqrt{g_{33}}) \subset \mathbb{R}^3$, the constants g_{11}, g_{33}, α_1 and α_2 are given in Theorem 6 and $Q(y)$ is a 1-parameter family of 4×4 orthogonal matrices such that $J_1 Q(y) = Q(y)J_1$, with ξ constant and

$$\cos^2(\xi_1(y))\xi_2'(y) - \sin^2(\xi_1(y))\xi_3'(y) = 0. \tag{5.12}$$

Conversely, a parametrization $F(x, y) = Q(y)\beta(x)$ as above defines a helix surface in $\mathbb{S}^3_\varepsilon$.

Proof. From the Theorem 6 we recover the expression for the position vector of M in $\mathbb{R}^4$ with respect to the local coordinates (x, y) on M defined in (5.7) and (5.8):

$$F(x, y) = \cos(\alpha_1 x)g^1(y) + \sin(\alpha_1 x)g^2(y) + \cos(\alpha_2 x)g^3(y) + \sin(\alpha_2 x)g^4(y)$$

Then, putting $e_i(y) = g^i(y)/||g^i(y)||, i = 1, 2, 3, 4$, one can write as follows:

$$F(x, y) = \sqrt{g_{11}}(\cos(\alpha_1 x)e_1(y) + \sin(\alpha_1 x)e_2(y)) + \sqrt{g_{33}}(\cos(\alpha_2 x)e_3(y) + \sin(\alpha_2 x)e_4(y)). \tag{5.13}$$

Now, if we consider the matrix $\bar{J}_{i,j} = \langle J_1 e_i, e_j\rangle$ in [21] the authors proved that $\bar{J} = \lambda(J_1)^T$. Then, if we fix the orthonormal basis of $\mathbb{R}^4$ defined by:

$$\tilde{E}_1 = (1, 0, 0, 0), \qquad \tilde{E}_2 = (0, \lambda, 0, 0), \qquad \tilde{E}_3 = (0, 0, 1, 0), \qquad \tilde{E}_4 = (0, 0, 0, \lambda),$$

there must exist a 1-parameter family of 4×4 orthogonal matrices such that $J_1 Q(y) = Q(y)J_1$ and that $e_i(y) = Q(y)\tilde{E}_i$. Consequently, from (5.13) we have

$$F(x, y) = Q(y)\beta(x)$$

with $\beta(x)$ and $Q(y)$ as in the statement.

Now, one has to show that the condition (5.12) is true. From (5.7) and (5.8) one gets that $\langle F_y, F_y\rangle = \lambda + \nu^2$, and so

$$\frac{\partial}{\partial x}\langle F_y, F_y\rangle_{|x=0} = 0. \tag{5.14}$$

Moreover if we denote $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ the columns of $Q(y)$ the latter leads to:

$$\langle \mathbf{q}_2', \mathbf{q}_3' \rangle = 0, \quad \langle \mathbf{q}_2', \mathbf{q}_4' \rangle = 0$$

where $'$ denotes the derivative with respect to y. Now, specifying the latter, using (5.4) one obtains:

$$\begin{cases} \xi' h(y) = 0 \\ \xi' k(y) = 0, \end{cases}$$

where $h(y)$ and $k(y)$ satisfy $h^2 + k^2 = 4(\xi_1)'^2 + \sin^2(2\xi_1)(-\xi' + \xi_2' + \xi_3')^2$. Consequently, two possibilities can occur:

(i) ξ is constant;

(ii) $4(\xi_1)'^2 + \sin^2(2\xi_1)(-\xi' + \xi_2' + \xi_3')^2 = 0$.

In [21] the authors proved that (ii) cannot occur since it is equivalent to the case of the timelike Hopf tube. So, (i) holds. Finally, in this case, the condition (5.12) is obtained by rewriting the second of (5.11).

For the converse, it suffices a direct calculation as follows from the Proposition 10.

$\boxed{QED}$

Considering the parametrization with arc length of the curve $\beta(x)$ the authors obtained the following.

Corollary 2. *[21] Let M be a helix spacelike (respectively, timelike) surface in $\mathbb{S}^3_\varepsilon$. Then, there exist local coordinates on M such that the position vector of M in $\mathbb{R}^4$ is given by:*

$$F(s, y) = Q(y)\beta(s),$$

where

$$\beta(s) = \frac{1}{\sqrt{1+d^2}}\Big(d\cos\Big(\frac{s}{d}\Big), \lambda d \sin\Big(\frac{s}{d}\Big), \cos(ds), \lambda\sin(ds)\Big) \tag{5.15}$$

is a twisted geodesic in the torus $\mathbb{S}^1(\frac{d}{\sqrt{1+d^2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{1+d^2}}) \subset \mathbb{R}^4$ parametrized by arc length, whose slope is given by

$$d = \frac{\sqrt{\lambda B} - \varepsilon|\nu|}{\sqrt{\lambda + \nu^2}} \in (0, 1),$$

where $\lambda = -1$ (respectively, $\lambda = 1$) and $Q(y)$ as in Theorem 7.

Conversely, a parametrization $F(s, y) = Q(y)\beta(s)$ as above defines a helix surface in $\mathbb{S}^3_\varepsilon$.

Figure 4: Stereographic projection in $\mathbb{R}^3$ of the helix surface spacelike and timelike with $\nu = 4$, $\varepsilon = 2$, $s \in (-4\pi, 4\pi)$, $y \in (-2\pi, 2\pi)$ e $\xi_2(y) = y$.

Remark 7. The curve $\beta : \mathbb{R} \to \mathbb{S}^3$ parametrized by (5.15) is a spherical helix in $\mathbb{S}^3$ with constant geodesic curvature and torsion given by

$$\kappa_g = \frac{\sqrt{1-d^2}}{d} = \frac{2\varepsilon|\nu|}{\sqrt{\lambda+\nu^2}}, \qquad |\tau_g| = 1.$$

Recalling that a non-null curve β in a Lorentzian manifold is called a *general helix* if there exist a Killing vector field $\tilde{V}$, called *axis* of β with constant length along β and such that the angle between $\tilde{V}$ and β' is a non-zero constant along β, a simple calculation give the following.

Proposition 11. *[21] The curve $\beta : \mathbb{R} \to \mathbb{S}^3_\varepsilon$ parametrized by* (5.15)*, used in Corollary 2 to characterize a constant angle spacelike (respectively, timelike) surface M is a spacelike (respectively, timelike) general helix in $\mathbb{S}^3_\varepsilon$ with axis E_1, i.e. it has a constant angle with the fibers of the Hopf fibration.*

As a consequence the authors obtained the following.

Corollary 3. *[21] Let M be a helix surface in $\mathbb{S}^3_\varepsilon$ parametrized by $F(s, y) = Q(y)\beta(s)$. Then, the hyperbolic angle between N and E_1 is the same that the one of general helix β makes with its axis E_1.*

Example 3. [21] Taking $\xi = \pi/2, \xi_1 = \pi/4, \xi_2(y) = \xi_3(y)$, specifying (5.4) one obtain

$$Q(y) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\xi_2(y) & -\sin\xi_2(y) & \cos\xi_2(y) & -\sin\xi_2(y) \\ \sin\xi_2(y) & \cos\xi_2(y) & \sin\xi_2(y) & \cos\xi_2(y) \\ -\cos\xi_2(y) & -\sin\xi_2(y) & \cos\xi_2(y) & \sin\xi_2(y) \\ \sin\xi_2(y) & -\cos\xi_2(y) & -\sin\xi_2(y) & \cos\xi_2(y) \end{pmatrix}.$$

In Fig. 4 and Fig. 5 we observe the plots of the stereographic projection in $\mathbb{R}^3$ of helix surfaces surfaces with $Q(y)$ as above.

Example 4. [21] We consider a constant angle surface $F(x, y) = Q(y)\beta(x)$. Following the proof of Theorem 7, it is easy to check that:

$$\langle F_y, F_y \rangle = \langle \mathbf{q}_i', \mathbf{q}_i' \rangle, \qquad i = 1, ...4.$$

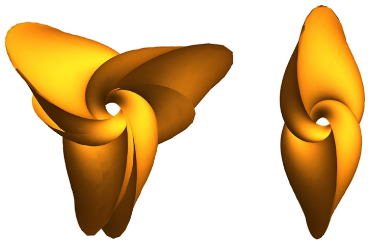


Figure 5: Stereographic projection in $\mathbb{R}^3$ of the helix surface spacelike and timelike with $\nu = 2$, $\varepsilon = 2$, $s \in (-2\pi, 2\pi)$, $y \in (-2, 2)$ e $\xi_2(y) = e^y$.

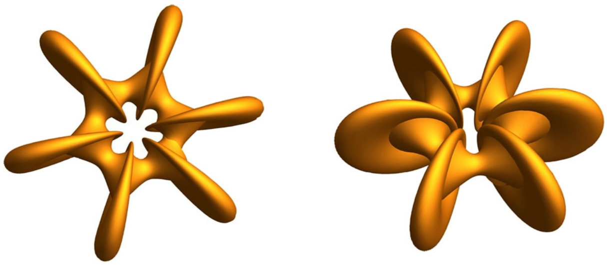


Figure 6: Stereographic projection in $\mathbb{R}^3$ of the helix surface spacelike and timelike with $\nu = \sqrt{5}$, obtained for $\varepsilon = 1$.

Then, one gets

$$\lambda + \nu^2 = \xi'(y)^2 + \cos^2(\xi_1(y))(\xi_2'(y))^2 + \sin^2(\xi_1(y))(\xi_3'(y))^2.$$

Assuming that ξ_1 is constant and such that $\cos(\xi_1(y)) \neq 0$ and $\sin(\xi_1(y)) \neq 0$, one obtains:

$$\xi_2(y) = \tan \xi_1 \sqrt{\lambda + \nu^2}\, y + d_2, \qquad \xi_3(y) = \cot \xi_1 \sqrt{\lambda + \nu^2}\, y + d_3$$

where d_2 and d_3 are real constants. In particular, choosing $d_2 = 0 = d_3$ and the constant $\xi_1 = 1/\sqrt{1 + d^2}$, where d is the constant given in the Corollary 2, the immersion $F(s, y)$ depends only on ν and λ. This permits us to show in Fig. 6 and Fig. 7 the plot of the stereographic projection in $\mathbb{R}^3$ of helix surfaces for some values of ν and λ.

6 Conclusions and announcements

This work gives a complete overview on the results present in literature on helix surfaces in Lorentzian ambient space. In [2], the author, in a joint work with G. Calvaruso, I. Onnis and D. Uccheddu, obtained a complete classification

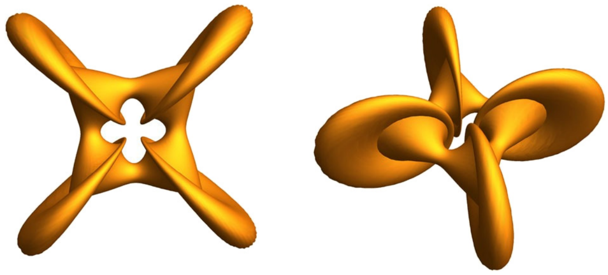


Figure 7: Stereographic projection in $\mathbb{R}^3$ of the helix surface spacelike and timelike with $\nu = 2$, obtained for $\varepsilon = 1$.

of helix surfaces in the anti-de Sitter space $\mathbb{H}^3_1$ endowed with a family of metrics which naturally extends the idea of Berger metrics introduced in [21].

We consider as a starting point the results given in [1], where the authors introduced and studied a new family of metrics $\tilde{g}_{\lambda\mu\nu}$ on the anti-de Sitter space $\mathbb{H}^3_1$. These metrics were induced in a natural way by corresponding metrics defined on the tangent sphere bundle $T_1\mathbb{H}^2(\kappa)$, after describing the covering map F from $\mathbb{H}^3_1(\kappa/4)$ to $T_1\mathbb{H}^2(\kappa)$ in terms of paraquaternions. A crucial role in this construction is played by the *hyperbolic Hopf map*:

$$\begin{array}{rcl} h : \mathbb{H}^3_1 & \to & \mathbb{H}^2(\kappa) \\ (z, w) & \mapsto & \dfrac{\sqrt{\kappa}}{4}(2z\bar{w}, |z|^2 + |w|^2). \end{array}$$

and the *hyperbolic Hopf vector field*:

$$X_1(z, w) = \frac{\sqrt{\kappa}}{2}(iz, iw),$$

that are, respectively, the hyperbolic counterparts of the Hopf map and Hopf vector field on $\mathbb{S}^3$, respectively. This fact leads us to investigate the description of surfaces whose normal vector field forms a constant angle with the hyperbolic Hopf vector field, similarly to the discussion made in [21] for $\mathbb{S}^3_\varepsilon$.

Some useful results are developed by Lucas and Ortega-Yagües on helix surfaces in $\mathbb{H}^3_1$, considering the canonical metric. In particular, in [14], they proved that such surfaces are flat and exhibited the distinction between Riemannian and Lorenztian helix surfaces giving explicit descriptions of such surfaces which involve general helices.

In a natural way, in our work, we focus on Berger-like metrics on $\mathbb{H}^3_1$, described by

$$g_\tau(X, Y) = \langle X, Y\rangle + (1 - \tau^2)\langle X, X_1\rangle\langle Y, X_1\rangle.$$

where $\langle , \rangle$ is the canonical metric of $\mathbb{R}^4_2$. By now, we denote by $\mathbb{H}^3_{1,\tau}$ the Lorentzian space $(\mathbb{H}^3_1(\kappa/4), g_\tau)$. In addition, we consider a pseudo-Riemannian oriented surface M immersed in $\mathbb{H}^3_{1,\tau}$ with N its λ-unit normal and we obtain the Gauss and Codazzi equations. Requiring the constant angle property (i.e. the angle ν is constant), we obtain the the Gaussian curvature of M:

$$K = \lambda\kappa\nu^2(1-\tau^2)$$

which recovers the flatness for the standard case ($\tau^2 = 1$) proved in [14]. Therefore, once we express the shape operator with respect to a tangent basis $\{T, JT\}$, constructed similarly as in Proposition 8, in the following way

$$A = \begin{pmatrix} 0 & -\dfrac{\sqrt{\kappa}}{2}\lambda\tau \\ \dfrac{\sqrt{\kappa}}{2}\tau & \mu \end{pmatrix},$$

we get

$$T(\mu) + \nu\mu^2 + \kappa\nu B = 0, \tag{6.1}$$

where $B := \nu^2(\tau^2-1) - \lambda$ and then we need to make a distinction between the cases where B is positive, null or negative. Now we choose local coordinates (x, y) on M, such that

$$\begin{cases} \partial_x = T, \\ \partial_y = a\,T + b\,JT, \end{cases} \tag{6.2}$$

for some smooth functions $a = a(x,y)$, $b = b(x,y)$ on M. Then we get a differential equation for the position vector F of M in $\mathbb{R}^4_2$ in the different cases:

(a) if $\mathbf{B} = \mathbf{0}$,

$$\frac{\partial^2 F}{\partial x^2} = 0, \tag{6.3}$$

(b) if $\mathbf{B} \neq \mathbf{0}$,

$$\frac{\partial^4 F}{\partial x^4} + (\tilde{b}^2 + 2\tilde{a})\frac{\partial^2 F}{\partial x^2} + \tilde{a}^2 F = 0, \tag{6.4}$$

where

$$\tilde{a} = \frac{\kappa}{4}\frac{B}{\tau^2}(\lambda+\nu^2), \qquad \tilde{b} = -\sqrt{\kappa}\frac{B}{\lambda\tau}.$$

In conclusion, after we integrate (6.3) and (6.4) and discuss some necessary and sufficient conditions to be satisfied in the case of helix surfaces in $\mathbb{H}^3_{1,\tau}$ as we made in Proposition 10, we give the characterization theorem.

Theorem 8. *[2] (of characterization) Let M be a helix surface in $\mathbb{H}^3_{1,\tau} \subset \mathbb{R}^4_2$ with constant angle function ν. Then, locally, the position vector of M in $\mathbb{R}^2_4$, with respect to the local coordinates (x, y) on M defined in* (6.2)*, is*

$$F(x, y) = A(y)\,\gamma(x),$$

where γ is a curve and $A(y) = A(\xi, \xi_1, \xi_2, \xi_3)(y)$ is a 1-parameter family of 4×4 pseudo-orthogonal matrices commuting with X_1. This curve γ is explicitly described depending on whether $\mathbf{B} > \mathbf{0}$, $\mathbf{B} = \mathbf{0}$ or $\mathbf{B} < \mathbf{0}$.

As in the previous section, we showed that the curves γ involved in the parametrization of helix surfaces in $\mathbb{H}^3_{1,\tau}$ are general helices.

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