Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **43** (2023) no. 1, 15–40. doi:10.1285/i15900932v43n1p15

Kantorovich-type modifications of certain discrete-type operators on the positive real axis

Francesco Altomare, Mirella Cappelletti Montano
Department of Mathematics, University of Bari Aldo Moro, Italy
francesco.altomare@uniba.it, mirella.cappellettimontano@uniba.it

Vita Leonessa
Department of Mathematics, Computer Science and Economics, University of Basilicata, Italy
vita.leonessa@unibas.it

Dedicated to Professor Octavian Agratini on the occasion of his 65th anniversary

Received: 31.1.2023; accepted: 27.7.2023.

Abstract. The paper is concerned with the approximation properties of a modification of Kantorovich-type of a general class of operators of discrete-type. Such a modification was introduced by Agratini in 2015; in particular, we focus on extending its approximation properties in several function spaces, including polynomial weighted spaces of any degree as well as L^p-spaces. Some estimates of the rate of convergence are also obtained.

Keywords: positive linear operator, Kantorovich-type operator, weighted function space, L^p-space, rate of convergence

MSC 2022 classification: primary 41A35, 41A36, 41A81, secondary 41A25

Introduction

As it is well known, the classical Kantorovich operators on $[0,1]$ are obtained from Bernstein operators on $[0,1]$ by replacing the values of the given functions on the knots k/n with their mean values on the intervals $\left[\frac{k}{n}, \frac{k}{n+1}\right]$ $(n \geq 1, 0 \leq k \leq n)$.

Among other things, such modifications of Bernstein operators give rise to the possibility of expanding the space of functions which can be constructively approximated by polynomials by including, in particular, L^p-spaces $(1 \leq p < +\infty)$.

Over the years, many other approximation processes of discrete-type have been introduced, for functions acting both on compact and non compact intervals (for example, see [4, Section 5.3] and the reference quoted therein). Along with them, Kantorovich-type modifications were also introduced. As a matter of fact, there is a large literature on Kantorovich-type modifications of discrete operators, which have been studied on a case-by-case basis.

The possibility to study the properties of Kantorovich-type modifications starting from a rather general class of discrete operators was explored in [2]. In that paper, Agratini presented a general way to construct Kantorovich-type operators starting from an approximation process of discrete-type defined as

$$B_n(f)(x) = \sum_{k \in I_n} \alpha_{n,k}(x) f\left(\frac{k}{b_n}\right) \quad (n \geq 1, x \in J), \tag{1}$$

where J is a real interval, $(b_n)_{n\geq 1}$ is a sequence of strictly increasing real numbers such that $b_n \geq 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = +\infty$, I_n is a set of indices such that $\{k/b_n \,:\, k \in I_n\} \subset J$, $\alpha_{n,k}$ are positive continuous functions on J, $(n,k) \in \mathbb{N} \times I_n$, and, for every $n \geq 1$, $\sum_{k\in I_n} \alpha_{n,k}(x) = 1$ uniformly on compacts. Here $f \in C(J)$ if the cardinality of I_n is finite, otherwise f belongs to the space of all continuous functions on J such that the series in (1) is absolutely convergent.

In order to consider a more general Kantorovich-type modification of the operators B_n, Agratini fixed three sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ of positive real numbers such that $0 \leq \alpha_n \leq c_n \leq \beta_n \leq 1$, $\alpha_n \neq \beta_n$, for every $n \geq 1$. Then, for every $n \geq 1$ and $x \in J$, the integral operators object of investigation were defined as

$$C_n(f)(x) = \frac{b_n + c_n}{\beta_n - \alpha_n} \sum_{k \in I_n} \alpha_{n,k}(x) \int_{\frac{k+\alpha_n}{b_n+c_n}}^{\frac{k+\beta_n}{b_n+c_n}} f(\xi)\, d\xi. \tag{2}$$

Here f is assumed to be integrable on J in the case where I_n is a finite set; otherwise, f is assumed to be a locally integrable function on J such that the antiderivatives of f make the series in (2) absolutely convergent.

One of the possible advantages in considering such a general sequence of operators lies in the fact that, by means of them, it is possible to reconstruct a continuous or an integrable function by knowing its mean value on subintervals of $[0, +\infty[$ which do not need to be an equispaced subdivision of $[0, +\infty[$, as it happens in the context of Kantorovich operators. Similar results were also considered in [5].

In [2] Agratini showed that it is possible to transfer some approximation properties from the discrete class of operators to its integral counterpart. In particular, he studied the approximation properties of the operators K_n in the case of a compact interval and of an unbounded interval with a finite endpoint. Among other things, a convergence result in E_2^* is established, E_m^* $(m \geq 1)$

being the space of functions $g \in C([0,+\infty[)$ such that $\lim_{x\to+\infty} g(x)(1+x^m)^{-1}$ exists and it is finite.

The main objective of the present paper is to deepen the study of such Kantorovich-type modifications for those particular sequences of positive linear operators of discrete-type B_n on $[0,+\infty[$ such that the sequence $(B_n)_{n\geq 1}$ is an approximation process in the space $C_0([0,+\infty[)$ of all continuous functions in $[0,+\infty[$ which vanish at infinity.

Under such hypothesis, we are indeed able to extend Agratini's investigation by showing that the relevant modifications are an approximation process in every weighted space E_m^* as well as in L^p-spaces.

The paper is organized as follows. After fixing some notation, in Section 2 we recall the definition of the Kantorovich-type modifications C_n on the positive real axis, object of our study, collecting some concrete examples and showing some of their properties. In Sections 3 and 4 we investigate some approximation properties of the operators C_n, such as convergence results in continuous function spaces and in polynomial weighted spaces, respectively. In Section 5 we get approximation results in L^p-spaces. Estimates of the rate of convergence are also established.

1 Notation

Throughout the paper, the symbol $C([0,+\infty[)$ (resp., $C_b([0,+\infty[)$) stands for the space of all real valued continuous (resp., continuous and bounded) functions on $[0,+\infty[$. The space $C_b([0,+\infty[)$, endowed with the natural (pointwise) order and the sup-norm $\|\cdot\|_\infty$, is a Banach lattice.

We shall also consider the (closed) subspaces of $C_b([0,+\infty[)$

$$C_0([0,+\infty[) := \{f \in C([0,+\infty[) \mid \lim_{x\to+\infty} f(x) = 0\}$$

and

$$C_*([0,+\infty[) := \{f \in C([0,+\infty[) \mid \lim_{x\to+\infty} f(x) \in \mathbb{R}\}.$$

From now on, for $m \geq 1$, we consider the weight $w_m(x) = \frac{1}{1+x^m}$ $(x \geq 0)$ and the relevant Banach lattice

$$E_m := \{f \in C([0,+\infty[) \;:\; \sup_{x\geq 0} w_m(x)|f(x)| \in \mathbb{R}\}$$

endowed with the pointwise ordering and the weighted norm

$$\|f\|_m := \|w_m f\|_\infty = \sup_{x\geq 0} w_m(x)|f(x)| \quad (f \in E_m)\,.$$

Further, we shall consider the following Banach sublattices of E_m:

$$E_m^* := \{f \in E_m \, : \, \lim_{x\to+\infty} w_m(x)f(x) \in \mathbb{R}\}$$

and

$$E_m^0 := \{f \in E_m^* \, : \, \lim_{x\to+\infty} w_m(x)f(x) = 0\}\,.$$

Note that, by Stone-Weierstrass theorem, $C_0([0,+\infty[)$ is dense in each E_m^0, $m > 1$.

As usual, if $1 \le p < +\infty$, we shall denote by $L^p([0,+\infty[)$ the space of all (equivalence classes of) Borel measurable functions on $[0,+\infty[$ such that

$$\|f\|_p := \left(\int_0^{+\infty} |f(t)|^p \, dt\right)^{1/p} < +\infty.$$

2 Kantorovich-type modifications of discrete-type operators on the positive real axis

Let $(\alpha_{n,k})_{n,k\ge1}$ be a sequence of positive continuous functions on $[0,+\infty[$ such that, for every $n \ge 1$,

$$\sum_{k=0}^{\infty} \alpha_{n,k}(x) = 1 \quad (x \ge 0) \tag{2.1}$$

and the convergence is assumed to be uniform on each compact subinterval of $[0,+\infty[$.

Moreover, let $(b_n)_{n\ge1}$ be a sequence of strictly increasing real numbers such that

$$b_n \ge 1 \text{ for every } n \in \mathbb{N} \text{ and } \lim_{n\to\infty} b_n = +\infty. \tag{2.2}$$

Following Agratini [2], consider the positive linear operators of discrete-type defined as

$$B_n(f)(x) = \sum_{k=0}^{\infty} \alpha_{n,k}(x) f\left(\frac{k}{b_n}\right) \quad (n \ge 1, x \ge 0) \tag{2.3}$$

for all $f : [0,+\infty[\to \mathbb{R}$ for which the series at the right-hand side is absolutely convergent. Let us denote such a space by $C_a([0,+\infty[)$. Note that $C_b([0,+\infty[) \subset C_a([0,+\infty[)$.

As showed in [2], it is possible to construct an integral extension of the operators (2.3) that generalize the classical Kantorovich modification of such operators. More precisely, consider three sequences $(\alpha_n)_{n\ge1}$, $(\beta_n)_{n\ge1}$ and $(c_n)_{n\ge1}$

of positive real numbers such that $0 \leq \alpha_n \leq c_n \leq \beta_n \leq 1$, $\alpha_n \neq \beta_n$, for every $n \geq 1$. Then, for every $n \geq 1$ and $x \geq 0$, we consider the integral operators

$$C_n(f)(x) = \frac{b_n + c_n}{\beta_n - \alpha_n} \sum_{k=0}^{\infty} \alpha_{n,k}(x) \int_{\frac{k+\alpha_n}{b_n+c_n}}^{\frac{k+\beta_n}{b_n+c_n}} f(\xi)\, d\xi \tag{2.4}$$

defined for all functions f belonging to the space $L_a([0, +\infty[)$ consisting of all locally integrable functions f on $[0, +\infty[$ whose antiderivatives belong to $C_a([0, +\infty[)$. Observe that $L_a([0, +\infty[)$ contains $C_b([0, +\infty[)$ as well.

Clearly, if $\beta_n = 1$, $\alpha_n = c_n = 0$ and $b_n = n$ for every $n \geq 1$, we obtain the natural Kantorovich-type modification of operators (2.3).

In what follows we present some examples of operators (2.3) and (2.4). Other examples might be found in [2].

Examples 1. 1. Assume that $b_n = n$, $c_n = 0$, $\alpha_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}$ ($n \geq 1$, $k \in \mathbb{N}$). Then the operators (2.3) turn into the classical Szász-Mirakjan operators, defined by setting

$$M_n(f)(x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

whereas the operators (2.4) become a generalization of Szász-Mirakjan-Kantorovich operators first introduced in [5]. More precisely,

$$C_n(f)(x) = e^{-nx} \frac{n}{\beta_n - \alpha_n} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{\frac{k+\alpha_n}{n}}^{\frac{k+\beta_n}{n}} f(t)\, dt. \tag{2.5}$$

Clearly, for $\alpha_n = 0$ and $\beta_n = 1$ for all $n \geq 1$, we obtain the classical Szász-Mirakjan-Kantorovich operators (see [9]).

Other generalizations of Szász-Mirakjan and Szász-Mirakjan-Kantorovich operators can be also seen as particular cases of (2.3) and (2.4). For example, set

$$w_\beta(k, \alpha) = \frac{\alpha}{k!}(\alpha + k\beta)^{k-1} e^{-(\alpha+\beta k)},$$

($k \in \mathbb{N}$, $\alpha > 0$, $\beta \in [0, 1[$) and consider the operators

$$P_n^{[\beta]}(f)(x) = \sum_{k=0}^{\infty} w_\beta(k, nx) f\left(\frac{k}{n}\right) \quad (x \geq 0).$$

Such a class of operators was introduced in [13] and, clearly, for $\beta = 0$, they become the classical Szász-Mirakjan operators.

Their Kantorovich-type generalization (see [16]) is given by

$$\tilde{P}_n^{[\beta]}(f)(x) = n \sum_{k=0}^{\infty} w_\beta(k, nx) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt \quad (x \geq 0),$$

and, for $\beta = 0$, we obviously obtain the Szász-Mirakjan-Kantorovich operators.
2. Assume that $b_n = n$, $c_n = 0$, $\alpha_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$ $(n \geq 1,\, k \in \mathbb{N})$. Then the operators (2.3) turn into the classical Baskakov operators, defined by setting

$$B_n(f)(x) := \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k} f\left(\frac{k}{n}\right),$$

whereas the operators (2.4) become a generalization of Baskakov-Kantorovich operators in the same spirit of [5]. More precisely,

$$C_n(f)(x) = \frac{n}{\beta_n - \alpha_n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k} \int_{\frac{k+\alpha_n}{n}}^{\frac{k+\beta_n}{n}} f(\xi)\, d\xi. \qquad (2.6)$$

In the literature many further generalizations of Baskakov and Baskakov-Kantorovich operators are available and some of them can be recovered in the more general framework of operators (2.3) and (2.4), respectively.

For example, consider the operators

$$L_n(f)(x) = e^{-\frac{a_n x}{1+x}} \sum_{k=0}^{\infty} f\left(\frac{k}{b_n}\right) \frac{P_k(n, a_n)}{k!} x^k (1+x)^{-n-k},$$

where $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ are two sequences of positive real numbers such that

$$\lim_{n\to\infty} \frac{n}{b_n} = 1, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0, \quad \lim_{n\to\infty} b_n = +\infty,$$

and, for any $a \geq 0$,

$$P_k(n, a) = \sum_{i=0}^{k} \binom{k}{i} (n)_i a^{k-i},$$

with $(n)_0 = 1$, $(n)_i = n(n+1)\dots(n+i-1)$, $i \geq 1$ (see [11]).

In [12] the following Kantorovich-type modification was proposed:

$$C_n(f)(x) = \frac{b_n}{\beta_n - \alpha_n} e^{-\frac{anx}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n, a_n)}{k!} x^k (1+x)^{-n-k} \int_{\frac{k+\alpha_n}{b_n}}^{\frac{k+\beta_n}{b_n}} f(t)\, dt;$$

clearly, it is a particular case of operators (2.4).
3. Let $(\lambda_n)_{n\geq 1}$ be a strictly decreasing positive sequence such that $\lim_{n\to\infty} \lambda_n =$

0 and $\lim_{n\to\infty} n\lambda_n = +\infty$. Moreover, assume that $b_n = n\lambda_n$ and $\alpha_{n,k}(x) = \binom{n}{k}(\lambda_n x)^k(1+\lambda_n x)^{-n}$ for every $n \geq 1$, $h = 0,\dots,n$, and $x \geq 0$. Then the operators (2.3) become the positive linear operators of discrete-type introduced and studied by Agratini in [3], namely

$$B_n(f)(x) = \frac{1}{(1+\lambda_n x)^n}\sum_{k=0}^{n}\binom{n}{k}(\lambda_n x)^k f\left(\frac{k}{n\lambda_n}\right) \quad (n \geq 1\,, x \geq 0)\,. \tag{2.7}$$

Note that, for $\lambda_n = n^{\beta-1}$ ($n \geq 1$, $0 < \beta < 1$), the above operators are the ones studied by Balázs and Szabados in [8] (for $\beta = 2/3$ see [7]).
Then, for every $n \geq 1$ and $x \geq 0$, we can define the relevant integral modification of type (2.4) as follows:

$$C_n(f)(x) = \frac{n\lambda_n + c_n}{\beta_n - \alpha_n}(1+\lambda_n x)^{-n}\sum_{k=0}^{n}\binom{n}{k}(\lambda_n x)^k\int_{\frac{k+\alpha_n}{n\lambda_n+c_n}}^{\frac{k+\beta_n}{n\lambda_n+c_n}} f(\xi)\,d\xi.$$

As a particular case of the above operators we can obtain the Kantorovich-Balázs-Szabados ones considered in [1].

From now on, for every $\lambda > 0$, we denote by e_λ the function

$$e_\lambda(t) = t^\lambda \quad (t \geq 0) \tag{2.8}$$

and, for a fixed $x \geq 0$, by ψ_x the function defined as $\psi_x(t) = t - x$ $(t \geq 0)$.

Coming back to the general framework of operators (2.4), it is easy to see that $C_n(e_0) = e_0$ for every $n \geq 1$. As showed in [2, Lemma 1],

$$C_n(e_1) = \frac{b_n}{b_n + c_n}B_n(e_1) + \frac{\alpha_n + \beta_n}{2(b_n + c_n)} \tag{2.9}$$

and

$$C_n(e_2) = \left(\frac{b_n}{b_n + c_n}\right)^2 B_n(e_2) + \frac{b_n(\alpha_n + \beta_n)}{(b_n + c_n)^2}B_n(e_1) + \frac{\alpha_n^2 + \beta_n^2 + \alpha_n\beta_n}{3(b_n + c_n)^2}\,. \tag{2.10}$$

In the next result we evaluate $C_n(e_m)$ for every $n, m \geq 1$.

Proposition 1. *Fix $m \geq 1$ and assume that $e_h \in C_a([0,+\infty[)$ for every $h = 0,\dots m$. Then $e_m \in L_a([0,+\infty[)$ and*

$$C_n(e_m) = \frac{1}{(m+1)(b_n + c_n)^m}\sum_{h=0}^{m} b_n^h\binom{m+1}{h}\sum_{p=0}^{m-h}\beta_n^p\alpha_n^{m-h-p}B_n(e_h)\,. \tag{2.11}$$

Moreover, for every $n \geq 1$ and $x \geq 0$,

$$C_n(\psi_x^2)(x) \leq B_n(\psi_x^2)(x) + \frac{2}{b_n}|B_n(\psi_x)(x)| + \frac{2}{b_n}x^2 + \frac{1}{b_n}\,. \tag{2.12}$$

Proof. We begin to prove formula (2.11). For every $n \geq 1$ and $x \geq 0$,

$$C_n(e_m)(x) = \frac{b_n + c_n}{\beta_n - \alpha_n} \sum_{k=0}^{\infty} \alpha_{n,k}(x) \int_{\frac{k+\alpha_n}{b_n+c_n}}^{\frac{k+\beta_n}{b_n+c_n}} \xi^m \, d\xi$$

$$= \frac{1}{(m+1)(b_n+c_n)^m(\beta_n - \alpha_n)} \sum_{k=0}^{\infty} \alpha_{n,k}(x) \left[(k+\beta_n)^{m+1} - (k+\alpha_n)^{m+1} \right]$$

$$= \frac{1}{(m+1)(b_n+c_n)^m(\beta_n - \alpha_n)} \sum_{k=0}^{\infty} \alpha_{n,k}(x) \sum_{h=0}^{m} b_n^h \binom{m+1}{h}$$

$$\times \left[(\beta_n)^{m+1-h} - (\alpha_n)^{m+1-h} \right] \left(\frac{k}{b_n} \right)^h$$

$$= \frac{1}{(m+1)(b_n+c_n)^m} \sum_{h=0}^{m} b_n^h \binom{m+1}{h} \sum_{p=0}^{m-h} \beta_n^p \alpha_n^{m-h-p} B_n\left(e_h\right)(x)$$

Finally, for every $n \geq 1$ and $x \geq 0$,

$$C_n(\psi_x^2)(x) = \left(\frac{b_n}{b_n+c_n} \right)^2 B_n(\psi_x^2)(x) + 2x \left(\frac{b_n}{b_n+c_n} \right)^2 B_n(e_1)(x)$$

$$- \left(\frac{b_n}{b_n+c_n} \right)^2 x^2 + \frac{b_n(\alpha_n+\beta_n)}{(b_n+c_n)^2} B_n(e_1)(x)$$

$$+ \frac{\alpha_n^2 + \beta_n^2 + \alpha_n\beta_n}{3(b_n+c_n)^2} - 2x \frac{b_n}{b_n+c_n} B_n(e_1)(x) - x \frac{\alpha_n+\beta_n}{b_n+c_n} + x^2$$

$$\leq B_n(\psi_x^2)(x) - 2x \frac{b_n}{b_n+c_n} B_n(e_1)(x) \left(1 - \frac{b_n}{b_n+c_n} \right) + x^2 \left(1 - \left(\frac{b_n}{b_n+c_n} \right)^2 \right)$$

$$+ \frac{\alpha_n+\beta_n}{b_n+c_n} B_n(\psi_x)(x) + \frac{\alpha_n^2+\beta_n^2+\alpha_n\beta_n}{3(b_n+c_n)^2}$$

$$\leq B_n(\psi_x^2)(x) + \frac{2}{b_n} x^2 + \frac{2}{b_n} B_n(\psi_x)(x) + \frac{1}{b_n},$$

and hence (2.12). QED

For a given $\lambda > 0$, set

$$f_\lambda(x) = e^{-\lambda x} \quad (x \geq 0) \tag{2.13}$$

and note that $f_\lambda \in L_a([0,+\infty[) \cap C_a([0,+\infty[)$. We have the following result.

Lemma 1. *For every $n \geq 1$ and $\lambda > 0$,*

$$C_n(f_\lambda) = \frac{b_n+c_n}{\lambda(\beta_n - \alpha_n)} \left(e^{-\frac{\lambda\alpha_n}{b_n+c_n}} - e^{-\frac{\lambda\beta_n}{b_n+c_n}} \right) B_n\left(f_{\frac{b_n\lambda}{b_n+c_n}} \right). \tag{2.14}$$

Hence, if $B_{n+1}(f) \le B_n(f)$ *for every* $n \ge 1$ *and for every convex function* $f \in C_a([0,+\infty[)$*, then*

$$C_n(f_\lambda) \le B_n\left(f_{\frac{b_n\lambda}{b_n+c_n}}\right) \le B_1\left(f_{\frac{b_n\lambda}{b_n+c_n}}\right) \le B_1\left(f_{\frac{\lambda}{1+c_n}}\right) \le B_1\left(f_{\frac{\lambda}{2}}\right). \tag{2.15}$$

Proof. Formula (2.14) follows by direct calculations.

By using the well known inequality $1 - e^{-x} \le x$ $(x \ge 0)$, we get the first inequality in (2.15).

The second inequality in (2.15) is a consequence of the assumption on the behaviour of the operators B_n on convex functions.

Since the function $g(x) = \frac{x}{x+c_n}$ is increasing, $b_n \ge 1$, $c_n \le 1$ $(n \ge 1)$, we have that $f_\lambda \le f_{\frac{b_n\lambda}{b_n+c_n}} \le f_{\frac{\lambda}{1+c_n}} \le f_{\frac{\lambda}{2}}$; given the positivity of the operators B_n, we easily get the last two inequalities in (2.15). QED

3 Approximation results in continuous function spaces

In this section we investigate the approximation properties of operators $(C_n)_{n\ge1}$ defined by (2.4) in certain continuous function spaces.

As a matter of fact, for every $n \ge 1$, we have that C_n is a positive continuous operator from $C_b([0,+\infty[)$ into itself and $\|C_n\|_{C_b([0,+\infty[)} = 1$ (see [2, p. 684]). In [2, Theorem 2] it was also proved that, if $e_i \in C_a([0,+\infty[)$, $i = 1,2$, and for every $i = 0,1,2$, $\lim_{n\to\infty} B_n(e_i) = e_i$ uniformly on compact subsets of $[0,+\infty[$, then $\lim_{n\to\infty} C_n(f) = f$ for every $f \in C_b([0,+\infty[)$ uniformly on compact subsets of $[0,+\infty[$.

Agratini also presented some estimates of the rate of convergence (see [2, Theorem 4]).

In this section we study the approximation properties of the operators C_n on $C_0([0,+\infty[)$ and $C_*([0,+\infty[)$. We begin by stating the following result.

Proposition 2. *Consider the operators* $(C_n)_{n\ge1}$ *defined by* (2.4)*. Further, for every* $n,k \ge 1$*, assume that* $\alpha_{n,k} \in C_0([0,+\infty[)$*. Then, for every* $n \ge 1$*,* $C_n(C_0([0,+\infty[)) \subset C_0([0,+\infty[)$ *and* $C_n(C_*([0,+\infty[)) \subset C_*([0,+\infty[)$*.*

Proof. Fix $n \ge 1$, $f \in C_0([0,+\infty[))$, and $\varepsilon > 0$. Then there exists $x_1 > 0$ such that $|f(x)| \le \varepsilon$ for every $x > x_1$. Moreover, there exists $x_2 > x_1$ such that, for every $x > x_2$,

$$|\alpha_{n,k}(x)| \le \frac{\varepsilon}{2\|f\|_\infty(n[x_1]+1)}$$

for any $k = 0,\dots,n[x_1]$, $[x_1]$ being the integer part of x_1.

Then, for every $x > x_2$,

$$
\begin{aligned}
|C_n(f)(x)| &\le \frac{b_n+c_n}{\beta_n-\alpha_n}\sum_{k=0}^{\infty}\alpha_{n,k}(x)\int_{\frac{k+\alpha_n}{b_n+c_n}}^{\frac{k+\beta_n}{b_n+c_n}}|f(\xi)|\,d\xi \\
&= \frac{b_n+c_n}{\beta_n-\alpha_n}\sum_{k=0}^{n[x_1]}\alpha_{n,k}(x)\int_{\frac{k+\alpha_n}{b_n+c_n}}^{\frac{k+\beta_n}{b_n+c_n}}|f(\xi)|\,d\xi \\
&+ \frac{b_n+c_n}{\beta_n-\alpha_n}\sum_{k=n[x_1]+1}^{\infty}\alpha_{n,k}(x)\int_{\frac{k+\alpha_n}{b_n+c_n}}^{\frac{k+\beta_n}{b_n+c_n}}|f(\xi)|\,d\xi \\
&\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\sum_{k=n[x_1]+1}^{\infty}\alpha_{n,k}(x)\le\varepsilon\,.
\end{aligned}
$$

This completes the proof of the first part of the statement. The second one derives from the fact that $C_n(e_0) = e_0$ for every $n \ge 1$. QED

In order to state the next approximation results, we first prove the following inequality: for every $\lambda > 0$, $n \ge 1$,

$$
0 \le 1 - \frac{1}{\lambda}\frac{b_n+c_n}{\beta_n-\alpha_n}\left(e^{-\frac{\lambda\alpha_n}{b_n+c_n}} - e^{-\frac{\lambda\beta_n}{b_n+c_n}}\right) \le \frac{\lambda}{b_n}. \tag{3.16}
$$

In fact, by using the very well known inequalities $1 - e^{-x} \le x$, $1 - e^{-x} \ge x - x^2/2$ $(x \ge 0)$, we obtain

$$
\begin{aligned}
0 &\le 1 - \frac{b_n+c_n}{\lambda(\beta_n-\alpha_n)}\left(e^{-\frac{\lambda\alpha_n}{b_n+c_n}} - e^{-\frac{\lambda\beta_n}{b_n+c_n}}\right) \\
&= 1 - \frac{b_n+c_n}{\lambda(\beta_n-\alpha_n)}e^{-\frac{\lambda\alpha_n}{b_n+c_n}}\left(1 - e^{-\frac{\lambda(\beta_n-\alpha_n)}{b_n+c_n}}\right) \\
&\le 1 - \frac{b_n+c_n}{\lambda(\beta_n-\alpha_n)}e^{-\frac{\lambda\alpha_n}{b_n+c_n}}\left(\lambda\frac{\beta_n-\alpha_n}{b_n+c_n} - \frac{\lambda^2(\beta_n-\alpha_n)^2}{2(b_n+c_n)^2}\right) \\
&= 1 - e^{-\frac{\lambda\alpha_n}{b_n+c_n}} + \frac{\lambda(\beta_n-\alpha_n)}{2(b_n+c_n)} \le \frac{\lambda(\alpha_n+\beta_n)}{2(b_n+c_n)} \le \frac{\lambda}{b_n}.
\end{aligned}
$$

From now on, we assume that the sequence $(B_n)_{n\ge1}$ is an approximation process in $C_0([0,+\infty[)$, i.e., $B_n(C_0([0,+\infty[)) \subset C_0([0,+\infty[)$ and, for every $f \in C_0([0,+\infty[)$,

$$
\lim_{n\to\infty} B_n(f) = f \tag{3.17}
$$

uniformly on $[0,+\infty[$.

According to [4, Proposition 4.2.5, Part 2], (3.17) holds true if there exists $0 < \lambda_1 < \lambda_2 < \lambda_3$ such that

$$\lim_{n\to\infty} B_n(f_{\lambda_i}) = f_{\lambda_i} \tag{3.18}$$

uniformly on $[0, +\infty[$, for every $i = 1, 2, 3$.

We point out that, since $B_n(e_0) = e_0$, then from (3.17) it follows that for every $f \in C_*([0, +\infty[)$

$$\lim_{n\to\infty} B_n(f) = f \tag{3.19}$$

uniformly on $[0, +\infty[$.

Examples 2. 1. Szász-Mirakjan operators and Baskakov operators (see Example 1, 1 and 2) satisfy (3.18). For a proof, see [4, pp. 340–341 and p. 344], respectively.
2. For every $n \geq 1$, let B_n be the operators defined by (2.7) and considered in Example 1, 3. By simply calculations it is easy to see that, for a given $\lambda > 0$,

$$B_n(f_\lambda)(x) = \left(\frac{1 + \lambda_n x e^{-\lambda/(n\lambda_n)}}{1 + \lambda_n x}\right)^n = \left(1 + \frac{\lambda_n x}{1 + \lambda_n x}(e^{-\lambda/(n\lambda_n)} - 1)\right)^n,$$

and from this formula we immediately infer that these operators satisfy (3.18).

After these preliminaries, we are ready to prove the following approximation theorem.

Theorem 1. *Suppose that $\alpha_{n,k} \in C_0([0, +\infty[)$ for any $n, k \geq 1$. Under assumption (3.17), $\lim_{n\to\infty} C_n(f) = f$ uniformly on $[0, +\infty[$ for every $f \in C_*([0, +\infty[)$.*

Proof. In order to show the statement, it sufficies to show it in $C_0[(0, +\infty[)$ and, in particular, for each function $f_\lambda(x) = e^{-\lambda x}$, $\lambda > 0$, since the subspace generated by $(f_\lambda)_{\lambda>0}$ is dense in $C_0([0, +\infty[)$ and the sequence $(C_n)_{n\geq1}$ is equibounded on $C_0([0, +\infty[)$.

By means of (2.14) and (3.16), for every $x \geq 0$ and $n \geq 1$, we have

$$\begin{aligned}
|C_n(f_\lambda)(x) - f_\lambda(x)| &\leq \left|\frac{b_n + c_n}{\lambda(\beta_n - \alpha_n)}\left(e^{-\frac{\lambda\alpha_n}{b_n+c_n}} - e^{-\frac{\lambda\beta_n}{b_n+c_n}}\right) - 1\right| B_n\left(f_{\frac{b_n\lambda}{b_n+c_n}}\right)(x) \\
&+ \left|B_n\left(f_{\frac{b_n\lambda}{b_n+c_n}}\right)(x) - B_n(f_\lambda)(x)\right| + |B_n(f_\lambda)(x) - f_\lambda(x)| \\
&\leq \frac{\lambda}{b_n} + \|B_n\| e^{-\lambda x}\left|e^{\left(-\frac{\lambda b_n}{b_n+c_n}+\lambda\right)x} - 1\right| + \|B_n(f_\lambda) - f_\lambda\|_\infty \\
&\leq \frac{\lambda}{b_n} + e^{-\lambda x}\left(e^{\frac{\lambda c_n}{b_n+c_n}x} - 1\right) + ||B_n(f_\lambda) - f_\lambda||_\infty \leq \frac{\lambda+1}{b_n} + ||B_n(f_\lambda) - f_\lambda||_\infty
\end{aligned}$$

and this completes the proof. QED

We now provide estimates of the rate of convergence, by means of the first and second modulus of continuity $\omega(f,\delta)$ and $\omega_2(f,\delta)$ (for a definition see, e.g. [14]). These estimates are based on a similarity technique which consists in introducing a suitable isometric isomorphism between $C_*([0,+\infty[)$ and $C([0,1])$.

In general, let X and Y be two different Banach spaces and let $\Phi : X \to Y$ be an isometric isomorphism. Moreover, consider an approximation process $(L_n)_{n\geq 1}$ in X. Then the operators L_n^* defined by $L_n^* := \Phi \circ L_n \circ \Phi^{-1}$ $(n \geq 1)$ form an approximation process on Y and we have that

$$\|L_n(u) - u\|_X = \|L_n^*(\Phi(u)) - \Phi(u)\|_Y . \tag{3.20}$$

The above equality is the key to transfer the problem of estimating the rate of convergence for $(L_n)_{n\geq 1}$ in X to the sequence $(L_n^*)_{n\geq 1}$ in Y.

Coming back to the operators C_n, let us assume, for the sake of simplicity, that $b_n = n$ and $c_n = 0$.

Consider the isometric isomorphism $\Phi : C_*([0,+\infty[) \to C([0,1])$ defined by setting

$$\Phi(f)(t) = \begin{cases} f(-\log t) & \text{if } 0 < t \leq 1, \\ \lim\limits_{x\to+\infty} f(x) & \text{if } t = 0, \end{cases} \quad \text{for every } f \in C_*([0,+\infty[). \tag{3.21}$$

We observe that $\Phi^{-1} : C([0,1]) \to C_*([0,+\infty[)$ is defined as $\Phi^{-1}(g)(t) := g(e^{-t})$ for every $g \in C([0,1])$ and $t \geq 0$.

Moreover, for every $n \geq 1$ and $g \in C([0,1])$, set

$$C_n^*(g) := \Phi(C_n(\Phi^{-1}(g))). \tag{3.22}$$

In what follows, the next lemma will be useful.

Lemma 2. *For every $\lambda > 0$, $x \in]0,1]$, and $n \geq 1$,*

$$|C_n^*(e_\lambda)(x) - x^\lambda| = |C_n(f_\lambda)(-\log x) - x^\lambda| \leq \|B_n^*(e_\lambda) - e_\lambda\|_\infty + \frac{\lambda}{n}, \tag{3.23}$$

where $B_n^* = \Phi \circ B_n \circ \Phi^{-1}$, e_λ and Φ are defined, respectively, by (2.8) *and* (3.21).

Proof. Indeed, for a given $\lambda > 0$, $x \in]0,1]$, and $n \geq 1$, we have that

$$\begin{aligned} &|C_n^*(e_\lambda)(x) - x^\lambda| = |C_n(f_\lambda)(-\log x) - x^\lambda| \\ &= \left| \frac{n}{\lambda(\beta_n - \alpha_n)} \left(e^{-\frac{\lambda\alpha_n}{n}} - e^{-\frac{\lambda\beta_n}{n}} \right) B_n(f_\lambda)(-\log x) - x^\lambda \right| \\ &\leq \frac{n}{\lambda(\beta_n - \alpha_n)} \left(e^{-\frac{\lambda\alpha_n}{n}} - e^{-\frac{\lambda\beta_n}{n}} \right) |B_n(f_\lambda)(-\log x) - x^\lambda| \\ &+ \left(1 - \frac{n}{\lambda(\beta_n - \alpha_n)} \left(e^{-\frac{\lambda\alpha_n}{n}} - e^{-\frac{\lambda\beta_n}{n}} \right) \right) x^\lambda \leq \|B_n^*(e_\lambda) - e_\lambda\|_\infty + \frac{\lambda}{n}, \end{aligned}$$

because of (3.16). QED

Remark 1. We observe that if $(B_n)_{n\geq 1}$ is an approximation process in $C_*([0,+\infty[)$, then for every $\lambda > 0$,

$$\lim_{n\to\infty} \|B_n^*(e_\lambda) - e_\lambda\|_\infty = 0.$$

We have the following result.

Proposition 3. *Under the same assumptions of Theorem 1, for $n \geq 1$ and $f \in C_*([0,+\infty[)$,*

$$\|C_n(f) - f\|_\infty \leq \omega(\Phi(f), \delta_n) + \frac{3}{2}\omega_2(\Phi(f), \delta_n), \tag{3.24}$$

where $\delta_n = \sqrt{\|B_n^*(e_2) - e_2\|_\infty + \|B_n^*(e_1) - e_1\|_\infty + \frac{4}{n}}$ and $B_n^* = \Phi \circ B_n \circ \Phi^{-1}$.

Proof. According to (3.20) it eas enough to show (3.24) for $\|C_n^*(\Phi(f)) - \Phi(f)\|_\infty$. To this end we apply [14, Theorem 2.2.1] from which, for every $n \geq 1$, $f \in C_*([0,+\infty[)$, $0 \leq x \leq 1$ and $\delta > 0$,

$$\begin{aligned}&|C_n^*(\Phi(f))(x) - \Phi(f)(x)| \leq |C_n^*(e_0)(x) - 1||\Phi(f)(x)| \\ &+ \frac{1}{\delta}|C_n^*(\psi_x)(x)|\omega(\Phi(f), \delta) + \left(C_n^*(e_0)(x) + \frac{1}{2\delta^2}C_n^*(\psi_x^2)(x)\right)\omega_2(\Phi(f), \delta).\end{aligned}$$

It is easy to prove that $C_n^*(e_0) = e_0$,

$$C_n^*(\psi_x)(x) = \begin{cases} C_n(f_1 - xe_0)(-\log x) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0 \end{cases}$$

and

$$C_n^*(\psi_x^2)(x) = \begin{cases} C_n(f_2 - 2xf_1 + x^2 e_0)(-\log x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0, \end{cases} \tag{3.25}$$

where f_λ, $\lambda = 1, 2$, is defined by (2.13).

First of all, taking (3.23) into account, we notice that, for every $0 < x \leq 1$,

$$|C_n^*(\psi_x)(x)| = |C_n(f_1)(-\log x) - x| \leq \|B_n^*(e_1) - e_1\|_\infty + \frac{1}{n}.$$

Analogously,

$$\begin{aligned}&C_n^*(\psi_x^2)(x) = C_n(f_2)(-\log x) - x^2 - 2x(C_n(f_1)(-\log x) - x) \\ &\leq \|B_n^*(e_2) - e_2\|_\infty + \|B_n^*(e_1) - e_1\|_\infty + \frac{4}{n}.\end{aligned}$$

We note that, by means of the Cauchy-Schwarz inequality,

$$|C_n^*(\psi_x)(x)| \leq \sqrt{C_n^*(\psi_x^2)(x)}.$$

Now, set $\delta = \delta_n$; then we get the result. QED

4 Approximation properties in polynomial weighted function spaces

We pass now to study the approximation properties of the sequence $(C_n)_{n\geq 1}$ in polynomial weighted spaces of continuous functions.

Proposition 4. *Consider the operators* $(C_n)_{n\geq 1}$ *defined by* (2.4). *Further assume that* $B_n(E_m) \subset E_m$ *for every* $n, m \geq 1$. *Then* C_n *is a positive continuous operator from* E_m *into itself and*

$$\|C_n(f)\|_m \leq \|f\|_m(1 + \|C_n(e_m)\|_m).$$

Proof. First of all observe that $e_h \in E_m$, and hence $B_n(e_h) \in E_m$ for every $h = 0, \ldots, m$, thanks to the assumption on the B_n's. Hence, $C_n(e_m) \in E_m$ (see (2.11)). Fix now $f \in E_m$; then

$$w_m(x)|C_n(f)(x)| \leq \|f\|_m w_m(x) C_n(e_0 + e_m)(x) = \|f\|_m w_m(x)(1 + C_n(e_m)(x)) .$$

From this and formula (1) the claim follows. QED

From now on we assume that

$$\sup_{x\geq 0, n\geq 1} w_m(x) B_n(e_m)(x) < +\infty. \tag{4.26}$$

We note that (4.26) implies that, for every $k = 0, \ldots, m$,

$$\sup_{x\geq 0, n\geq 1} w_m(x) B_n(e_k)(x) < +\infty, \tag{4.27}$$

since $|e_h| \leq e_0 + e_k$, for every $1 \leq h \leq k$. Hence, on account of Proposition 1, (4.26), and (4.27),

$$M := \sup_{x\geq 0, n\geq 1} w_m(x) C_n(e_0 + e_m)(x) < +\infty$$

and, in particular, for every $n \geq 1$,

$$\|C_n\|_m \leq M \tag{4.28}$$

(see Proposition 4).

Proposition 5. *Consider the operators* $(C_n)_{n\geq 1}$ *defined by* (2.4). *Further assume that, for every* $n, k \geq 1$, $\alpha_{n,k} \in C_0([0, +\infty[)$. *Then, for every* $n, m \geq 1$, $C_n(E_m^0) \subset E_m^0$.

Proof. Let D be the subspace generated by the family $(f_\lambda)_{\lambda>0}$. By Stone–Weierstrass theorem D is dense in $C_0([0,+\infty[)$ and hence in E_m^0. In order to get the claim it is sufficient to note that $C_n(D) \subset C_0([0,+\infty[) \subset E_m^0$. QED

Theorem 2. *Assume that $\alpha_{n,k} \in C_0([0,+\infty[)$ for any $n,k \geq 1$. Moreover, suppose that (3.17) and (4.26) hold true. For a given $m \geq 1$, if $f \in E_m^*$ (and, in particular, if $f \in E_m^0$), then*

$$\lim_{n\to\infty} C_n(f) = f \text{ with respect to } \|\cdot\|_m.$$

Moreover, for every $f \in E_m$,

$$\lim_{n\to\infty} C_n(f) = f \tag{4.29}$$

uniformly on compact subsets of $[0,+\infty[$.

Proof. By means of Theorem 1, $\lim_{n\to\infty} C_n(f_\lambda) = f_\lambda$ with respect to $\|\cdot\|_\infty$ and hence with respect to $\|\cdot\|_m$. Since the sequence $(C_n)_{n\geq1}$ is equibounded on E_m^0 (see (4.28)) and the linear subspace generated by $(f_\lambda)_{\lambda>0}$ is dense in E_m^0, we have that $\lim_{n\to\infty} C_n(f) = f$ with respect to $\|\cdot\|_m$ for $f \in E_m^0$. On the other hand, if $f \in E_m^*$, then $f = g + \alpha_m(e_0 + e_m)$, where $\alpha_m := \lim_{x\to+\infty} w_m(x)f(x) \in \mathbb{R}$ and $g = f - \alpha_m(e_0 + e_m) \in E_m^0$. This completes the proof.

Taking that into account, since $E_m \subset E_{m+1}^0$ and the weight w_m is bounded from below, we get (4.29); in fact, if K is a compact subset of $[0,+\infty[$, then

$$w_m(x)|C_n(f)(x) - f(x)| \leq N\|C_n(f) - f\|_{m+1}$$

for every $x \in K$, where $N := \sup_{x\in J} \frac{w_m(x)}{w_{m+1}(x)}$. QED

We now want to provide some estimates of the rate of convergence in Theorem 2.

Proposition 6. *Under the same hypotheses of Theorem 2, assume that there exists $m_0 \geq 1$ such that*

$$\lim_{n\to\infty} \frac{\sqrt{B_n(\psi_x^2)(x)}}{1 + x^{m_0}} = 0$$

uniformly on $[0,+\infty[$. Then, for every $f \in E_m^0$, $m \geq m_0$, $n \geq 1$,

$$\|C_n(f) - f\|_m \leq 2\omega(f, \sigma_n),$$

where $\sigma_n = \sup_{x\geq0} \frac{\sqrt{B_n(\psi_x^2)(x)}}{1+x^m} + \frac{\sqrt{2}}{\sqrt{b_n}} \sup_{x\geq0} \frac{\sqrt{|B_n(\psi_x)(x)|}}{1+x^m} + \frac{\sqrt{2}+1}{\sqrt{b_n}}$.

Proof. It is known that (see [4, Theorem 5.1.2]), for every $n \geq 1$, $x \geq 0$, $f \in E_m^0$ $(m > m_0)$ and $\delta > 0$,

$$|C_n(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta}\sqrt{C_n(\psi_x^2)(x)}\right)\omega(f,\delta)\,.$$

Then

$$\|C_n(f) - f\|_m \leq \left(1 + \frac{1}{\delta}\sup_{x\geq 0}\frac{\sqrt{C_n(\psi_x^2)(x)}}{1+x^m}\right)\omega(f,\delta)\,,$$

from which the desired uniform weighted estimate could be derived, since, on account of (2.12),

$$\sup_{x\geq 0}\frac{\sqrt{C_n(\psi_x^2)(x)}}{1+x^m} \leq \sup_{x\geq 0}\frac{\sqrt{B_n(\psi_x^2)(x)}}{1+x^m} + \frac{\sqrt{2}}{\sqrt{b_n}}\sup_{x\geq 0}\frac{\sqrt{|B_n(\psi_x)(x)|}}{1+x^m} + \frac{\sqrt{2}+1}{\sqrt{b_n}}\,.$$

QED

The previous result applies, for example, in the context of Example 1, 1, since the classical Szász-Mirakjan operators M_n satisfy $M_n(\psi_x)(x) = 0$ and $M_n(\psi_x^2)(x) = x/n$.

We now proceed to present some estimates of the rate of convergence in E_m^* by using again the similarity technique illustrated at page 26.

From now on, for the sake of simplicity, we assume that $b_n = n$ and $c_n = 0$ for all $n \geq 1$. Moreover, we assume that, for all $n, h \geq 1$,

$$B_n(\mathbb{P}_h) \subset \mathbb{P}_h, \tag{4.30}$$

where $\mathbb{P}_h$ is the space of all polynomials of degree at most h; we also assume that, for every $h, n \geq 1$,

$$B_n(e_h) = e_h + \frac{1}{n}\,p_{h-1}, \tag{4.31}$$

where p_{h-1} is a polynomial of degree $h-1$.

Under these assumptions, $C_n(\mathbb{P}_h) \subset \mathbb{P}_h$ for every $h \geq 1$ and (see (2.11))

$$C_n(e_h) = e_h + \frac{1}{n}\,q_{h-1}, \tag{4.32}$$

where q_{h-1} is a polynomial of degree $h-1$.

We now consider the isometric isomorphism $\Phi_m : E_m^* \to C([0,1])$ defined by setting

$$\Phi_m(f)(t) = \begin{cases} (w_m f)(-\log t) & \text{if } 0 < t \leq 1, \\ \lim\limits_{x\to+\infty}(w_m f)(x) & \text{if } t = 0 \end{cases} \quad \text{for every } f \in E_m^*. \tag{4.33}$$

Note that $\Phi_m^{-1} : C([0,1]) \to E_m^*$ is defined as $\Phi_m^{-1}(g)(t) := w_m^{-1}(t)g(e^{-t})$ for every $g \in C([0,1])$ and $t \geq 0$.

Moreover, for every $n \geq 1$, we consider the similar positive linear operator $W_n^* : C([0,1]) \to C([0,1])$ defined by setting, for any $g \in C([0,1])$,

$$W_n^*(g) := \Phi_m(C_n(\Phi_m^{-1}(g))). \tag{4.34}$$

We have the following result.

Theorem 3. *Suppose that* (4.30) *and* (4.31) *hold true. Then, under the same assumptions of Theorem 2, for every $n \geq 1$ and $f \in E_m^*$,*

$$\|C_n(f) - f\|_m \leq \frac{H_{1,m}}{n}||\Phi_m(f)||_\infty + H_{2,m}\omega\left(\Phi_m(f), \sigma_n\right) + H_{3,m}\omega_2\left(\Phi_m(f), \sigma_n\right),$$

where

$$\sigma_n = (\|B_n^*(e_4) - e_4\|_\infty + 4\|B_n^*(e_3) - e_3\|_\infty + 6\|B_n^*(e_2) - e_2\|_\infty$$
$$+ 4\left\|B_n^*(e_1) - e_1\right\|_\infty + 32/n)^{1/4},$$

$B_n^* = \Phi \circ B_n \circ \Phi^{-1}$ *(see* (3.21)*), and $H_{1,m}, H_{2,m}, H_{3,m}$ are suitable positive constants which depend on m, only.*

Proof. We now establish a uniform estimate for $\|W_n^*(\Phi_m(f)) - \Phi_m(f)\|_\infty$, using again [14, Theorem 2.2.1]; in particular, for every $n \geq 1$, $f \in E_m^*$, $0 \leq x \leq 1$ and $\delta > 0$, we get

$$|W_n^*(\Phi_m(f))(x) - \Phi_m(f)(x)| \leq |W_n^*(e_0)(x) - 1||\Phi_m(f)(x)|$$
$$+ \frac{1}{\delta}|W_n^*(\psi_x)(x)|\omega(\Phi_m(f), \delta) + \left(W_n^*(e_0)(x) + \frac{1}{2\delta^2}W_n^*(\psi_x^2)(x)\right)\omega_2(\Phi_m(f), \delta).$$

From (4.33) and (4.34) it easily follows that

$$W_n^*(e_0)(x) = \begin{cases} (w_m C_n(e_0 + e_m))(-\log x) & \text{if } 0 < x \leq 1, \\ 1 & \text{if } x = 0, \end{cases}$$

$$W_n^*(\psi_x)(x) = \begin{cases} (w_m C_n((1 + e_m)(f_1 - xe_0)))(-\log x) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0 \end{cases}$$

and

$$W_n^*(\psi_x^2)(x) = \begin{cases} (w_m C_n((1 + e_m)(f_2 - 2xf_1 + x^2e_0)))(-\log x) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases}$$

with f_λ, $\lambda = 1, 2$, defined by (2.13).

In particular, for all $n \geq 1$ and $x \in]0,1]$,

$$|W_n^*(e_0)(x) - 1| = |w_m(-\log x) C_n(e_0 + e_m)(-\log x) - 1| \leq \frac{H_{1,m}}{n}$$

because of (4.30), (4.31), and (4.32).

Moreover, taking (3.22), (3.25), (4.32) and the Cauchy-Schwartz inequality into account,

$$\begin{aligned}
&|W_n^*(\psi_x)(x)| = |(w_m C_n((1+e_m)(f_1 - xe_0))(-\log x)| \\
&= w_m(-\log x)\sqrt{C_n((e_0+e_m)^2)(-\log x)}\sqrt{C_n((f_1 - xe_0)^2)(-\log x)} \\
&\leq H_{2,m}\sqrt{C_n^*(\psi_x^2)(x)} \leq H_{2,m}\sqrt[4]{C_n^*(\psi_x^4)(x)}.
\end{aligned}$$

Arguing in the same way,

$$\begin{aligned}
&W_n^*(\psi_x^2)(x) = (w_m C_n((1+e_m)(f_1 - xe_0)^2)(-\log x) \\
&= w_m(-\log x)\sqrt{C_n((e_0+e_m)^2)(-\log x)}\sqrt{C_n((f_1 - xe_0)^4)(-\log x)} \\
&\leq H_{2,m}\sqrt{C_n^*(\psi_x^4)(x)},
\end{aligned}$$

where C_n^* is defined by (3.22).

We now note that, by virtue of (3.23),

$$\begin{aligned}
&C_n^*(\psi_x^4)(x) = C_n(f_4)(-\log x) - x^4 - 4x(C_n(f_3)(-\log x) - x^3) \\
&+ 6x^2(C_n(f_2)(-\log x) - x^2) - 4x^3(C_n(f_1)(-\log x) - x) \\
&\leq \|B_n^*(e_4) - e_4\|_\infty + \frac{4}{n} + 4\|B_n^*(e_3) - e_3\|_\infty + \frac{12}{n} \\
&+ 6\|B_n^*(e_2) - e_2\|_\infty + \frac{12}{n} + 4\|B_n^*(e_1) - e_1\|_\infty + \frac{4}{n}.
\end{aligned}$$

Setting $\delta = \sigma_n$ we get the desired result. QED

5 Approximation properties in L^p-spaces

We now prove that, under suitable assumptions on $\alpha_{n,k}$, the operators C_n are well defined and are an approximation process on L^p-spaces, $p \geq 1$.

We first prove under which conditions the sequence $(C_n)_{n\geq 1}$ is well defined and equibounded from $L^p([0,+\infty[)$ into $L^p([0,+\infty[)$.

Lemma 3. *Let us assume that, for every $n \geq 1$,*

$$M_n := \sup_{k\geq 1} \int_0^{+\infty} \alpha_{n,k}(x)\, dx < +\infty.$$

Then $C_n(L^p([0,+\infty[)) \subset L^p([0,+\infty[)$ *and*

$$\|C_n\|_{L^p,L^p} \le M_n \frac{b_n + c_n}{\beta_n - \alpha_n}.$$

In particular, if there exists $M \ge 0$ *such that* $M_n \frac{b_n+c_n}{\beta_n-\alpha_n} \le M$ *for any* $n \ge 1$, *then the sequence* $(C_n)_{n\ge 1}$ *is equibounded in* $L^p([0,+\infty[)$.

Proof. Fix $n \ge 1$, $x \ge 0$ and $f \in L^p([0,+\infty[)$. By applying twice Jensen's inequality we get

$$|C_n(f)(x)|^p \le \frac{b_n + c_n}{\beta_n - \alpha_n} \sum_{k=0}^{\infty} \alpha_{n,k}(x) \int_{\frac{k+\alpha_n}{b_n+c_n}}^{\frac{k+\beta_n}{b_n+c_n}} |f(\xi)|^p \, d\xi.$$

Hence,

$$\int_0^{+\infty} |C_n(f)(x)|^p \, dx \le \sum_{k=0}^{\infty} \frac{b_n + c_n}{\beta_n - \alpha_n} \int_{\frac{k+\alpha_n}{b_n+c_n}}^{\frac{k+\beta_n}{b_n+c_n}} |f(\xi)|^p \, d\xi \int_0^{+\infty} \alpha_{n,k}(x) \, dx$$

$$\le M_n \frac{b_n + c_n}{\beta_n - \alpha_n} \|f\|_p^p$$

and this completes the proof. QED

Examples 3. 1. Since, for every $n \ge 1$,

$$\int_0^{+\infty} e^{-nx} \frac{(nx)^k}{k!} \, dx = \frac{1}{n},$$

if the operators C_n are defined as in (2.4), then

$$\|C_n\|_{L^p,L^p} \le \frac{b_n + c_n}{n(\beta_n - \alpha_n)},$$

so, in order for the sequence $(C_n)_{n\ge 1}$ to be equibounded in $L^p([0,+\infty[)$, it suffices that

$$\frac{b_n + c_n}{\beta_n - \alpha_n} \le M \qquad (n \ge 1). \tag{5.35}$$

Similar results were obtained in [5].

2. Since, for every $n > 1$,

$$\sup_{k\ge 0} \int_0^{+\infty} \frac{x^k}{(1+x)^{n+k}} \, dx = \frac{1}{n-1},$$

if the operators C_n are defined as in (2.4), then

$$\|C_n\|_{L^p,L^p} \le \frac{b_n + c_n}{(n-1)(\beta_n - \alpha_n)},$$

so the sequence $(C_n)_{n>1}$ is equibounded in $L^p([0,+\infty[)$ if (5.35) holds true.

Remark 2. We point out that results similar to Lemma 3 hold true also if $J=[0,1]$ and operators C_n are of the form

$$C_n(f)(x) = \frac{b_n + c_n}{\beta_n - \alpha_n} \frac{n+r}{r} \sum_{k=0}^{n} \alpha_{n,k}(x) \int_{\frac{k+\alpha_n}{b_n+c_n}}^{\frac{k+\beta_n}{b_n+c_n}} f(\xi)\, d\xi,$$

where, for every $n \geq 1$

$$\sum_{k=0}^{n} \alpha_{n,k}(x) = 1$$

and $\alpha_{n,k}(x) \geq 0$ for every $x \in [0,1]$.

Those operators can be seen as a generalization of the ones treated in [6, Section 5].

We can finally prove the following result.

Theorem 4. *Assume that* (3.17) *holds true and* $\alpha_{n,k} \in C_0([0,+\infty[)$ *for every* $n, k \geq 1$. *Further, suppose that, for every* $n \geq 1$, $B_{n+1}(f) \leq B_n(f)$ *for every convex function* $f \in C_a([0,+\infty[)$ *and that* $B_1(f_\lambda) \in L^p([0,+\infty[)$ *for every* $\lambda > 0$. *Then, for every* $f \in L^p([0,+\infty[)$

$$\lim_{n\to\infty} C_n(f) = f \qquad \text{in } L^p([0,+\infty[). \tag{5.36}$$

Proof. For any given $\lambda_1, \lambda_2, \lambda_3 > 0$, $\{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\}$ is a Korovkin subset in $L^p([0,+\infty[)$ (see [4, Proposition 4.2.5]) so to get (5.36) it suffices to prove that, for every $\lambda > 0$, $C_n(f_\lambda) \to f_\lambda$ in $L^p([0,+\infty[)$, where the function f_λ are defined by (2.13). Indeed, for every $\lambda > 0$, $C_n(f_\lambda) \to f_\lambda$ uniformly (see Theorem 1) and hence pointwise. Moreover (see (2.15)), for every $n \geq 1$,

$$C_n(f_\lambda)^p \leq B_1\left(f_{\lambda/2}\right)^p.$$

By applying the dominated convergence theorem, we conclude the proof. QED

Examples 4. 1. Let M_n be the n-th Szász-Mirakjan operator (see Example 1, 1). Then, for every $x \geq 0$, $n \geq 1$ and $\lambda > 0$,

$$M_n(f_\lambda)(x) = \exp\left(nx\left(e^{-\frac{\lambda}{n}} - 1\right)\right).$$

Hence $M_n(f_\lambda) \in L^p([0,+\infty[)$ for every $n \geq 1$, $p \geq 1$ and $\lambda > 0$ and Theorem 4 applies to the relevant operators (2.5).

These results can be compared with the ones proved in [5].

2. Let B_n be the Baskakov operators (see Example 1, 2). Then given $x \geq 0$, $n \geq 1$ and $\lambda > 0$,

$$B_n(f_\lambda)(x) = \frac{1}{\left(1 + x - xe^{-\frac{\lambda}{n}}\right)^n}.$$

Then, for every $p > 1$, $B_1(f_\lambda) \in L^p([0,+\infty[)$ and Theorem 4 can be used to prove that the relevant operators C_n defined by (2.6) are an approximation process in $L^p([0,+\infty[)$.

A way to get an estimate of the rate of convergence in Theorem 4 is to apply a similarity technique that allows to derive it by means a suitable isometric isomorphism between $L^p([0,+\infty[)$ and $L^p([0,1])$, as done in Sections 3 and 4.

For the sake of simplicity, let us assume that $b_n = n$ and $c_n = 0$.

Consider the isometric isomorphism $\Phi_p : L^p([0,+\infty[) \to L^p(]0,1])$ defined by setting, for every $f \in L^p([0,+\infty[)$, and for every $0 < t \leq 1$,

$$\Phi_p(f)(t) = t^{-1/p} f(-\log t). \tag{5.37}$$

Its inverse $\Phi_p^{-1} : L^p(]0,1]) \to L^p([0,+\infty[)$ is defined by $\Phi_p^{-1}(g)(t) = e^{-t/p} g(e^{-t})$ for every $g \in L^p(]0,1])$ and $t \geq 0$.

Now, for all $n \geq 1$, define the operators $W_n : L^p(]0,1]) \to L^p(]0,1])$ as follows

$$W_n(g) = \Phi_p(C_n(\Phi_p^{-1}(g))) \quad (g \in L^p(]0,1]))\,,$$

getting, as quoted before, an approximation process in $L^p(]0,1])$.

We point out that, for any $0 < x \leq 1$,

$$W_n(e_0 \,|_{]0,1]})(x) = x^{-\frac{1}{p}} C_n \left(f_{\frac{1}{p}} \right) (-\log x), \tag{5.38}$$

$$W_n(\psi_x \,|_{]0,1]})(x) = x^{-\frac{1}{p}} C_n \left(f_{\frac{1}{p}+1} - x f_{\frac{1}{p}} \right) (-\log x) \tag{5.39}$$

and

$$W_n(\psi_x^2 \,|_{]0,1]})(x) = x^{-\frac{1}{p}} C_n \left(f_{\frac{1}{p}+2} - 2x f_{\frac{1}{p}+1} + x^2 f_{\frac{1}{p}} \right) (-\log x) \tag{5.40}$$

(see (5.37)), where f_λ is defined by (2.13).

Consider now the positive linear operator $\sigma : L^p(]0,1]) \to L^p([0,1])$ defined by setting, for every $g \in L^p(]0,1])$,

$$\sigma(g)(t) = \begin{cases} g(t) & 0 < t \leq 1\,, \\ 0 & t = 0\,. \end{cases} \tag{5.41}$$

Then $\|g\|_p = \|\sigma(g)\|_p$ for all $g \in L^p(]0,1])$.

Now, for any $n \geq 1$, consider $W_n^* : L^p([0,1]) \to L^p([0,1])$ defined, for every $f \in L^p([0,1])$, by

$$W_n^*(f) = \sigma(W_n(f \,|_{]0,1]})).$$

We point out that, for any $f \in L^p([0,+\infty[)$ and $n \geq 1$,

$$W_n^*(\sigma(\Phi_p(f))) = \sigma(W_n(\sigma(\Phi_p(f)) \,|_{]0,1]}) = \sigma(W_n(\Phi_p(f))).$$

Moreover, for every $x \in]0,1]$ and $i \geq 0$, we set

$$V_n(e_i \mid_{]0,1]})(x) = \Phi_p(B_n(\Phi_p^{-1}(e_i \mid_{]0,1]})))(x) = x^{-\frac{1}{p}} B_n\left(f_{i+\frac{1}{p}}\right)(-\log x).$$

From now on, we shall assume that

$$V_n(e_i \mid_{]0,1]}) \in L^p(]0,1]) \text{ for every } i = 0,1,2; \tag{5.42}$$

moreover, we shall suppose that there exists $M \geq 0$ such that, for every $n \geq 1$ and $i = 0,1,2$,

$$\|V_n(e_i \mid_{]0,1]})\|_p \leq M.$$

This implies that

$$\|V_n^*(e_i)\|_p \leq M, \tag{5.43}$$

where $V_n^*(e_i) = \sigma(V_n(e_i \mid_{]0,1]}))$.

After these preliminaries, we recall that, in order to evaluate $\|C_n(f) - f\|_p$, by using a similarity technique, we have at our disposal in $L^p([0,1])$ a result due to Swetits and Wood (see [15, Theorem 1]), which involves the second-order integral modulus of smoothness in $L^p([0,1])$, denoted by $\omega_2(g,\delta)_p$ (see [10, Chapter 2, Section 7]).

The result runs as follows: if $L_n : L^p([0,1]) \to L^p([0,1])$ $(n \geq 1)$ is a positive linear operator then, for every $g \in L^p([0,1])$,

$$\|L_n(g) - g\|_p \leq K_p(\mu_{n,p}^2 \|g\|_p + \omega_2(g, \mu_{n,p})_p) \tag{5.44}$$

with $K_p > 0$ is independent on g and $n \geq 1$, provided that $\mu_{n,p} \to 0$ as $n \to \infty$, where

$$\mu_{n,p} := \sqrt{\max\{\|L_n(e_0) - e_0\|_p \, , \|\alpha_n\|_p \, , \|\beta_n\|_p^{2p/(2p+1)}\}}\,, \tag{5.45}$$

$$\alpha_n(x) = L_n(\psi_x)(x) \quad \text{and} \quad \beta_n(x) = L_n(\psi_x^2)(x) \quad (0 \leq x \leq 1).$$

After these preliminaries, we are now ready to state the next result.

Proposition 7. *Under the same assumptions of Theorem 4, let us assume that* (5.42) *and* (5.43) *hold true. Moreover let us suppose that*

$$\lim_{n\to\infty} \|V_n^*(e_0) - e_0\|_p = 0$$

and that, denoting by

$$a_n(x) = V_n^*(\psi_x)(x) \quad \textit{and} \quad b_n(x) = V_n^*(\psi_x^2)(x) \quad (0 \leq x \leq 1),$$

we have

$$\lim_{n\to\infty} \|a_n\|_p = \lim_{n\to\infty} \|b_n\|_p^{2p/(2p+1)} = 0.$$

Then

$$\lim_{n\to\infty} \mu_{np} \to 0$$

(see (5.45) *where the.operators* L_n *are exactly the operators* W_n^**) and estimate* (5.44) *applies to the operators* C_n*.*

Proof. By using (5.45), we have that, for every $f \in L^p([0,+\infty[)$,

$$\begin{aligned}
&\|C_n(f) - f\|_p = \|W_n(\Phi_p(f)) - \Phi_p(f)\|_p = \|\sigma(W_n(\Phi_p(f)) - \Phi_p(f))\|_p \\
&= \|\sigma(W_n(\Phi_p(f))) - \sigma(\Phi_p(f))\|_p = \|W_n^*(\sigma(\Phi_p(f))) - \sigma(\Phi_p(f))\|_p \\
&\le K_p(\mu_{n,p}^2 \|\sigma(\Phi_p(f))\|_p + \omega_2(\sigma(\Phi_p(f)), \mu_{n,p})_p) \\
&= K_p(\mu_{n,p}^2 \|\Phi_p(f)\|_p + \omega_2(\sigma(\Phi_p(f)), \mu_{n,p})_p),
\end{aligned}$$

provided that $\lim_{n\to\infty} \mu_{np} \to 0$.

We now proceed with some calculations.

First of all, for all $0 < x \le 1$ and taking (2.14), (3.16) and (5.41) into account,

$$\begin{aligned}
&|W_n^*(e_0)(x) - 1| = |W_n(e_0 \mid_{]0,1]}) - 1| \\
&= \left| \frac{pn}{\beta_n - \alpha_n} \left(e^{-\frac{\alpha_n}{pn}} - e^{-\frac{\beta_n}{pn}} \right) x^{-\frac{1}{p}} B_n \left(f_{\frac{1}{p}} \right) (-\log x) - 1 \right| \\
&\le \frac{pn}{\beta_n - \alpha_n} \left(e^{-\frac{\alpha_n}{pn}} - e^{-\frac{\beta_n}{pn}} \right) \left| x^{-1/p} B_n \left(f_{\frac{1}{p}} \right) (-\log x) - 1 \right| \\
&+ \left| \frac{pn}{\beta_n - \alpha_n} \left(e^{-\frac{\alpha_n}{pn}} - e^{-\frac{\beta_n}{pn}} \right) - 1 \right| \\
&\le \left| x^{-1/p} B_n \left(f_{\frac{1}{p}} \right) (-\log x) - 1 \right| + \frac{1}{pn}
\end{aligned}$$

Hence,

$$\begin{aligned}
&|W_n(e_0 \mid_{]0,1]})(x) - 1|^p \le 2^{p-1} \left(\left| x^{-1/p} B_n \left(f_{\frac{1}{p}} \right) (-\log x) - 1 \right|^p + \frac{1}{(pn)^p} \right) \\
&= 2^{p-1} \left(\left| V_n(e_0 \mid_{]0,1]})(x) - 1 \right|^p + \frac{1}{(pn)^p} \right);
\end{aligned}$$

accordingly,

$$\|W_n^*(e_0) - e_0\|_p^p \le 2^{p-1} \left(\|V_n^*(e_0) - e_0\|_p^p + \frac{1}{(pn)^p} \right). \tag{5.46}$$

Moreover, for a given $0 < x \le 1$, taking again (2.14) and (3.16) into account,

$$
\begin{aligned}
&|W_n^*(\psi_x)(x)| = |W_n(\psi_x\,|_{]0,1]})(x)| \\
&= x^{-1/p}\left|\frac{n}{(1+1/p)(\beta_n-\alpha_n)}\left(e^{-\frac{(1+1/p)\alpha_n}{n}} - e^{-\frac{(1+1/p)\beta_n}{n}}\right)B_n\left(f_{1+1/p}\right)(-\log x)\right. \\
&\left.-x\frac{pn}{\beta_n-\alpha_n}\left(e^{-\frac{\alpha_n}{pn}} - e^{-\frac{\beta_n}{pn}}\right)B_n\left(f_{1/p}\right)(-\log x)\right| \\
&\le x^{-1/p}\left[\left(1 - n\frac{e^{-\frac{(1+1/p)\alpha_n}{n}} - e^{-\frac{(1+1/p)\beta_n}{n}}}{(1+1/p)(\beta_n-\alpha_n)}\right)B_n\left(f_{1+1/p}\right)(-\log x)\right. \\
&+\left|B_n\left(f_{1+1/p}\right)(-\log x) - xB_n\left(f_{1/p}\right)(-\log x)\right| \\
&\left.+x\left(1-\frac{pn}{\beta_n-\alpha_n}\left(e^{-\frac{\alpha_n}{pn}} - e^{-\frac{\beta_n}{pn}}\right)\right)B_n\left(f_{1/p}\right)(-\log x)\right] \\
&\le x^{-1/p}\frac{1}{n(1+1/p)}B_n\left(f_{1/p+1}\right)(-\log x) + |V_n(\psi_x\,|_{]0,1]})(x)| \\
&+ x^{-1/p}\frac{1}{np}B_n\left(f_{1/p}\right)(-\log x) \\
&= \frac{1}{n(1+1/p)}V_n(e_1\,|_{]0,1]})(x) + |V_n(\psi_x\,|_{]0,1]})(x)| + \frac{1}{np}V_n(e_0\,|_{]0,1]})(x).
\end{aligned}
$$

Then

$$
\begin{aligned}
|W_n(\psi_x)(x)|^p \le 3^{p-1}&\left(\left(\frac{1}{n(1+1/p)}V_n(e_1\,|_{]0,1]})(x)\right)^p +\right. \\
&\left.|V_n(\psi_x\,|_{]0,1]})(x)|^p + \left(\frac{1}{np}V_n(e_0\,|_{]0,1]})(x)\right)^p\right),
\end{aligned}
$$

so that

$$
\|\alpha_n\|_p^p \le 3^{p-1}\left(\frac{1}{n^p(1+1/p)^p}M^p + \|V_n^*(\psi_x)\|_p^p + \frac{1}{(np)^p}M^p\right). \tag{5.47}
$$

Finally,

$$
\begin{aligned}
&W_n^*(\psi_x^2)(x) = x^{-1/p}\left[C_n(f_{2+1/p})(-\log x)\right. \\
&\left.-x^2C_n(f_{1/p})(-\log x) - 2x\left(C_n(f_{1+1/p})(-\log x) - xC_n(f_{1/p})(-\log x)\right)\right] \\
&\le V_n(\psi_x^2\,|_{]0,1]})(x) \\
&+ x^{-1/p}\left[\left(1 - n\frac{e^{-\frac{(2+1/p)\alpha_n}{n}} - e^{-\frac{(2+1/p)\beta_n}{n}}}{(2+1/p)(\beta_n-\alpha_n)}\right)B_n\left(f_{2+1/p}\right)(-\log x)\right. \\
&+ x^2\left(1-\frac{pn}{\beta_n-\alpha_n}\left(e^{-\frac{\alpha_n}{pn}} - e^{-\frac{\beta_n}{pn}}\right)\right)B_n\left(f_{1/p}\right)(-\log x)
\end{aligned}
$$

$$
\begin{aligned}
&+2x\left(1-n\frac{e^{-\frac{(1+1/p)\alpha_n}{n}}-e^{-\frac{(1+1/p)\beta_n}{n}}}{(1+1/p)(\beta_n-\alpha_n)}\right)B_n\left(f_{1+1/p}\right)(-\log x)\\
&+2x^2\left(1-\frac{pn}{(\beta_n-\alpha_n)}\left(e^{-\frac{\alpha_n}{pn}}-e^{-\frac{\beta_n}{pn}}\right)\right)B_n\left(f_{1/p}\right)(-\log x)\Bigg]\\
&\le V_n(\psi_x^2\mid_{]0,1]})(x)+\frac{1}{n(2+1/p)}V_n(e_2\mid_{]0,1]})(x)\\
&+\frac{2}{n(1+1/p)}V_n(e_1\mid_{]0,1]})(x)+\frac{3p}{n}V_n(e_0\mid_{]0,1]})(x),
\end{aligned}
$$

so that

$$
\|W_n(\psi_x^2)\|_p^p\le 4^{p-1}\left[\|V_n^*(\psi_x^2)\|_p^p+\left(\frac{1}{n(2+1/p)}+\frac{2}{n(1+1/p)}+\frac{3p}{n}\right)^p M^p\right]. \tag{5.48}
$$

The statement follows directly from (5.46), (5.47) and (5.48). QED

Acknowledgements. The paper has been performed within the activities of the Italian GNAMPA-INdAM Group, the UMI Group TAA “Approximation Theory and Applications” and, for the second and third author, of the network RITA (Research ITalian network on Approximation).

References

[1] O. Agratini: *On approximation properties of Balázs-Szabados operators and their Kantorovich extension*, Korean J. Comput. Appl. Math. **9** (2002), no. 2, 361–372.

[2] O. Agratini: *Kantorovich sequences associated to general approximation processes*, Positivity **19** (2015), 681–693.

[3] O. Agratini: *On a class of Bernstein-type rational functions*, Numer. Funct. Anal. Optim. **41** (2020), no. 4, 483–494.

[4] F. Altomare and M. Campiti: Korovkin-type approximation theory and its applications, de Gruyter Studies in Mathematics **17**, Walter de Gruyter & Co., Berlin, 1994.

[5] F. Altomare, M. Cappelletti Montano, V. Leonessa: *On a generalization of Szász-Mirakjan-Kantorovich operators*, Results Math. **63** (2013), no. 3-4, 837–863.

[6] F. Altomare, M. Cappelletti Montano, V. Leonessa, I. Raşa: *A generalization of Kantorovich operators for convex compact subsets*, Banach J. Math. Anal. **11** (3) (2017), 591–614.

[7] K. Balázs: *Approximation by Bernstein type rational functions*, Acta Math. Acad. Sci. Hung **26** (1975), 123–134.

[8] C. Balázs, J. Szabados: *Approximation by Bernstein type rational functions. II*, Acta Math. Acad. Sci. Hung. **40** (1982), 331–337.

[9] P. L. Butzer: *On the extensions of Bernstein polynomials to the infinite interval*, Proc. Amer. Math. Soc. **5** (1954), 547–553.

[10] R A. DeVore, G.G. Lorentz: Constructive Approximation, Grundlehren der mathematischen Wissenschaften **303**, Springer-Verlag, Berlin, 1993.

[11] A. Erençin, G. Başcanbaz-Tunca: *Approximation properties of a class of linear positive operators in weighted spaces*, C.R. Acad. Bulgare Sci., **63** (2010), no. 10, 1397–1404.

[12] A. Erençin, S. Büyükdurakoglu: *A modification of generalized Baskakov-Kantorovich operators*, Stud. Univ. Babeş-Bolyai Math. **59** (2014), no. 3, 351–364.

[13] G. C. Jain: *Approximation of functions by a new class of linear operators*, J. Austr. Math. Soc. **13** (1972), no. 3, 271–276.

[14] R. Păltănea: Approximation theory using positive linear operators, Birkhäuser, Boston, 2004.

[15] J.J. Swetits, B. Wood: *Quantitative estimates for L_p approximation with positive linear operators*, J. Approx. Theory **38** (1983), 81–89.

[16] S. Umar, Q. Razi: *Approximation of functions by a generalized Szász operators*, Comm. Fac. Sci. Univ. Ankara Sér. A1 Math. **34** (1985), 45–52.