Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **42** (2022) no. 2, 93–107. doi:10.1285/i15900932v42n2p93

(α, τ)-P-derivations on left near-rings

Samir Mouhssine
Department of Mathematics, Polydisciplinary Faculty, University Sidi Mohammed Ben Abdellah, LSI, P.O.Box 1223, Taza, Morocco.
samir.mouhssine@usmba.ac.ma or samirfes27@gmail.com

Abdelkarim Boua[i]
Department of Mathematics, Polydisciplinary Faculty, University Sidi Mohammed Ben Abdellah, LSI, P.O.Box 1223, Taza, Morocco.
abdelkarimboua@yahoo.fr or karimoun2006@yahoo.fr

Received: 28.8.2022; accepted: 21.3.2023.

Abstract. Suppose that $\mathcal{N}$ is a near-ring and P is a 3-prime ideal of $\mathcal{N}$. In this paper we introduce the notion of (α, τ)-P derivation in near-rings, we also study the structure of the quotient near-ring $\mathcal{N}/P$ which satisfies certain algebraic identities involving (α, τ)-P derivations.

Keywords: Near-rings, Prime ideals, (α, τ)-derivations, Commutativity.

MSC 2022 classification: primary 16U60, secondary, 16W25.

1 Introduction

Throughout this paper, $\mathcal{N}$ will denote a left near-ring with multiplicative center $Z(\mathcal{N})$ and additive center $C(\mathcal{N})$. A near-ring $\mathcal{N}$ is said to be zero-symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that a left distributivity in $\mathcal{N}$ yields that $x0 = 0$). Also, $\mathcal{N}$ is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. Recall that $\mathcal{N}$ is called a 3-prime near-ring, if for $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. For all $x, y \in \mathcal{N}$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ shall denote the Lie product and the Jordan products, respectively. The symbol (x, y) will denote the additive-group commutator $x + y - x - y$. A normal subgroup P of $(\mathcal{N}, +)$ is called a left ideal (resp. a right ideal) if $P\mathcal{N} \subseteq \mathcal{N}$ (resp. $(x + p)y - xy \in P$ for all $x, y \in \mathcal{N}$ and $p \in P$), and if P is both a left ideal and a right ideal, then P is said to be an ideal of $\mathcal{N}$. According to Groenewald [6], an ideal P is a 3-prime if for $a, b \in \mathcal{N}$, $a\mathcal{N}b \subseteq P \Rightarrow a \in P$ or $b \in P$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a (α, τ)-derivation if there exist automorphisms $\alpha, \tau : \mathcal{N} \to \mathcal{N}$ such that $d(xy) = \tau(x)d(y) + d(x)\alpha(y)$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [1], such that $d(xy) = d(x)\alpha(y) + \tau(x)d(y)$ for all $x, y \in \mathcal{N}$. A mapping $d : \mathcal{N} \to \mathcal{N}$ is said to be P-additive if $d(x + y) - (d(x) + d(y)) \in P$

[i]Corresponding author

for all $x, y \in \mathcal{N}$. A mapping $d : \mathcal{N} \to \mathcal{N}$ is P-trivial if $d(\mathcal{N}) \subseteq P$. Element x of $\mathcal{N}$ for which $d(x) \in P$ is called P constant. A mapping $d : \mathcal{N} \to \mathcal{N}$ is called $(\alpha, \tau) - P$-commuting if $[d(x), x]_{\alpha,\tau} \in P$ for all $x \in \mathcal{N}$.

Many results in the literature show how the global structure of a near-ring $\mathcal{N}$ is often closely related to the behavior of derivations defined on $\mathcal{N}$. Recently, a number of more general notions of derivations on near-rings have been introduced and studied (see for example [3], [4], [5], [7], [8] and [9]). In the following, we define the notion of (α, τ)-P-derivation in near rings, which generalizes the notion of (α, τ)-derivation, and we enrich this definition with an example that justifies the existence of this type of application:

Definition 1. Let $\mathcal{N}$ be a near-ring and P be a subgroup of $(\mathcal{N}, +)$. An P-additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called a (α, τ)-P-derivation of $\mathcal{N}$, if there exist maps $\alpha, \tau : \mathcal{N} \to \mathcal{N}$ such that $d(xy) - (\tau(x)d(y) + d(x)\alpha(y)) \in P$ for all $x, y \in \mathcal{N}$.

Definition 2. Let $\mathcal{N}$ be a near-ring and P be a subgroup of $(\mathcal{N}, +)$. An P-additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called a (α, τ)-P^+-derivation of $\mathcal{N}$, if d is a (α, τ)-P-derivation such that

(**a**) $d\left(d(xy) - (\tau(x)d(y) + d(x)\alpha(y))\right) \in P$ for all $x, y \in \mathcal{N}$,

(**b**) $d\left(d(xy) - (d(x)\alpha(y) + \tau(x)d(y))\right) \in P$ for all $x, y \in \mathcal{N}$.

In the case of $\alpha = \tau = I_{\mathcal{N}}$ we define the following notions:

Definition 3. Let $\mathcal{N}$ be a near-ring and P be a subset of $\mathcal{N}$. An P-additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called a P-derivation if $d(xy) - (xd(y) + d(x)y) \in P$ for all $x, y \in \mathcal{N}$.

Definition 4. Let $\mathcal{N}$ be a near-ring and P be a subset of $\mathcal{N}$. A map $d : \mathcal{N} \to \mathcal{N}$ is a P^+-derivation if d is a P-derivation such that

(**a**) $d^2(xy) - d\left(xd(y) + d(x)y\right) \in P$ for all $x, y \in \mathcal{N}$,

(**b**) $d^2(xy) - d\left(d(x)y + xd(y)\right) \in P$ for all $x, y \in \mathcal{N}$.

Definition 5. Let $\mathcal{N}$ be a near-ring. A normal subgroup P of $(\mathcal{N}, +)$ is called a symmetric ideal if

(**a**) P is an ideal of $\mathcal{N}$,

(**b**) $P\mathcal{N} \subseteq P$.

If $P = \{0\}$ is a symmetric ideal of a near-ring $\mathcal{N}$, we get the concept of a zero-symmetric near-ring $\mathcal{N}$.

Definition 6. A near-ring $\mathcal{N}$ is said to be symmetric if every ideal of $\mathcal{N}$ is symmetric.

It is easy to see that every (α,τ) derivation on $\mathcal{N}$ is a (α,τ)-P derivation on $\mathcal{N}$. The following example justifies the existence of a (α,τ)-P derivation that is not a (α,τ) derivation:

Example 1. Let S be a left near-ring. Define $\mathcal{N}$, P by:

$$\mathcal{N}=\left\{\begin{pmatrix}0&a&b\\0&0&0\\0&c&0\end{pmatrix}\mid a,b,c,0\in S\right\},\ P=\left\{\begin{pmatrix}0&u&0\\0&0&0\\0&0&0\end{pmatrix}\mid 0,u\in S\right\}$$

then $\mathcal{N}$ is a left near-ring, and P is an ideal of $\mathcal{N}$
Let us define d,α, and $\tau:\mathcal{N}\longrightarrow\mathcal{N}$ as follow:

$$d\begin{pmatrix}0&a&b\\0&0&0\\0&c&0\end{pmatrix}=\begin{pmatrix}0&a&0\\0&0&0\\0&0&0\end{pmatrix},\alpha\begin{pmatrix}0&a&b\\0&0&0\\0&c&0\end{pmatrix}=\begin{pmatrix}0&a&0\\0&0&0\\0&c&0\end{pmatrix}$$

$$\text{and } \tau\begin{pmatrix}0&a&b\\0&0&0\\0&c&0\end{pmatrix}=\begin{pmatrix}0&a&b\\0&0&0\\0&0&0\end{pmatrix}.$$

It's clear to see that d is a (α,τ)-P^{+}-derivation, but not a (α,τ)-derivation on $\mathcal{N}$.

Example 2. Let S be a left near-ring. Define $\mathcal{N}$, P by:

$$\mathcal{N}=\left\{\begin{pmatrix}0&a&b\\0&0&0\\0&c&0\end{pmatrix}\mid a,b,c,0\in S\right\},P=\left\{\begin{pmatrix}0&u&0\\0&0&0\\0&0&0\end{pmatrix}\mid u\in S\right\}$$

then $\mathcal{N}$ is a left near-ring, and P is a symmetric ideal of $\mathcal{N}$.
The map $d:\mathcal{N}\longrightarrow\mathcal{N}$ given by:

$$d\begin{pmatrix}0&a&b\\0&0&0\\0&c&0\end{pmatrix}=\begin{pmatrix}0&0&b\\0&0&0\\0&0&0\end{pmatrix}$$

is a P^{+}-derivation, but not a derivation on $\mathcal{N}$.

With these definitions, by using P derivations, where P is an ideal of a near-ring $\mathcal{N}$, we will study properties of the near-ring $\mathcal{N}/P$. The originality of this work is that we use a P-derivation on $\mathcal{N}$ (and not on $\mathcal{N}/P$), which satisfies some algebraic identities on $\mathcal{N}$ and on P, without the primeness (semi-primeness) assumption on the considered near-ring.

2 Some preliminaries

Lemma 1. *Let $\mathcal{N}$ be a near-ring and P be an ideal of $\mathcal{N}$.*

a. *If P is 3-prime, then $\mathcal{N}/P$ is a 3-prime near-ring.*

b. *If P is symmetric, then $\mathcal{N}/P$ is a zero-symmetric near-ring.*

Proof. Due to the ease of proof, we leave it to readers to enjoy. QED

Theorem 1. *Let $\mathcal{N}$ be a near-ring and P be an ideal of $\mathcal{N}$. If $d : \mathcal{N} \to \mathcal{N}$ is a P-derivation of $\mathcal{N}$ preserving P, then the mapping $\widetilde{d} : \mathcal{N}/P \to \mathcal{N}/P$ defined by $\widetilde{d}(\overline{x}) = \overline{d(x)}$ is a derivation on $\mathcal{N}/P$.*

Proof. $\widetilde{d}$ is well defined, indeed let $y \in \overline{x}$, then $y - x = p$ for some $p \in P$, so $d(y) - (d(x) + d(p)) \in P$, so $\widetilde{d}(\overline{y}) = \overline{d(y)} = \overline{d(x)} = \widetilde{d}(\overline{x})$.
Now let $\overline{x}, \overline{y} \in \mathcal{N}/P$, we have $\widetilde{d}(\overline{x}.\overline{y}) = \widetilde{d}(\overline{xy}) = \overline{d(xy)} = \overline{(xd(y) + d(x)y)} = \overline{x}\overline{d(y)} + \overline{d(x)}\overline{y} = \overline{x}\widetilde{d}(\overline{y}) + \widetilde{d}(\overline{x})\overline{y}$. Also, we have $\widetilde{d}(\overline{x} + \overline{y}) = \widetilde{d}(\overline{x+y}) = \overline{d(x+y)} = \overline{(d(x) + d(y))} = \overline{d(x)} + \overline{d(y)} = \widetilde{d}(\overline{x}) + \widetilde{d}(\overline{y})$, which completes the proof of our theorem. QED

Theorem 2. *Let $\mathcal{N}$ be a near-ring and P be an ideal of $\mathcal{N}$. An P-additive map d on a near-ring $\mathcal{N}$ is a P-derivation if and only if $d(xy) - (d(x)y + xd(y)) \in P$ for all $x, y \in \mathcal{N}$.*

Proof. Suppose that d is a P-derivation. Since $x(y + y) = xy + xy$, it follows that

$$\begin{aligned} \overline{d(x(y+y))} &= \overline{x}\overline{d(y+y)} + \overline{d(x)}(\overline{y} + \overline{y}) \\ &= \overline{x}\overline{d(y)} + \overline{x}\overline{d(y)} + \overline{d(x)}\overline{y} + \overline{d(x)}\overline{y} \text{ for all } x, y \in \mathcal{N}. \end{aligned} \tag{2.1}$$

Now

$$\begin{aligned} \overline{d(xy+xy)} &= \overline{d(xy)} + \overline{d(xy)} \\ &= \overline{x}\overline{d(y)} + \overline{d(x)}\overline{y} + \overline{x}\overline{d(y)} + \overline{d(x)}\overline{y} \text{ for all } x, y \in \mathcal{N}. \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), we get $\overline{xd(y)} + \overline{d(x)y} = \overline{d(x)y} + \overline{xd(y)}$, for all $x, y \in \mathcal{N}$. Hence, $d(xy) - (d(x)y + xd(y)) \in P$, for all $x, y \in \mathcal{N}$.
For the converse, assume that $d(xy) - (d(x)y + xd(y)) \in P$ for all $x, y \in \mathcal{N}$. Since $\overline{x(y+y)} = \overline{xy} + \overline{xy}$ for all $x, y \in \mathcal{N}$, we get

$$\begin{aligned} \overline{d(x(y+y))} &= \overline{d(x)}(\overline{y} + \overline{y}) + \overline{x}\overline{d(y+y)} \\ &= \overline{d(x)}\overline{y} + \overline{d(x)}\overline{y} + \overline{x}\overline{d(y)} + \overline{x}\overline{d(y)} \text{ for all } x, y \in \mathcal{N}. \end{aligned} \tag{2.3}$$

Also

$$\begin{aligned}\overline{d(xy+xy)} &= \overline{d(xy)}+\overline{d(xy)}\\ &= \overline{d(x)}\,\overline{y}+\overline{x}\,\overline{d(y)}+\overline{d(x)}\,\overline{y}+\overline{x}\,\overline{d(y)}, \text{ for all } x,y\in\mathcal{N}. \quad (2.4)\end{aligned}$$

In view of (2.3) and (2.4), we obtain $\overline{d(x)}\,\overline{y}+\overline{x}\,\overline{d(y)}=\overline{x}\,\overline{d(y)}+\overline{d(x)}\,\overline{y}$ for all $x,y\in\mathcal{N}$, which gives $d(xy)-(xd(y)+d(x)y)\in P$ for all $x,y\in\mathcal{N}$. So, d is a P-derivation. QED

If $\mathcal{N}$ is a 3-prime near-ring in the previous theorem, then $P=\{0\}$ is a 3-prime ideal of $\mathcal{N}$, in which case we get the following result:

Corollary 1 ([10] Proposition 1)**.** *Let $\mathcal{N}$ be a 3-prime near-ring. An additive endomorphism d on a near-ring $\mathcal{N}$ is a derivation if and only if $d(xy)=d(x)y+xd(y)$ for all $x,y\in\mathcal{N}$.*

Theorem 3. *Let $\mathcal{N}$ be a near-ring and P be an ideal of $\mathcal{N}$ and d an arbitrary P-derivation of a near-ring $\mathcal{N}$. Then $\mathcal{N}/P$ satisfies the following partial distributive laws.*

a. $(\overline{x}\,\overline{d(y)}+\overline{d(x)}\,\overline{y})\overline{z}=\overline{x}\,\overline{d(y)}\,\overline{z}+\overline{d(x)}\,\overline{y}\,\overline{z}$ *for all $x,y,z\in\mathcal{N}$.*

b. $(\overline{d(x)}\,\overline{y}+\overline{x}\,\overline{d(y)})\overline{z}=\overline{d(x)}\,\overline{y}\,\overline{z}+\overline{x}\,\overline{d(y)}\,\overline{z}$ *for all $x,y,z\in\mathcal{N}$.*

Proof. **a.** It is clear that $\overline{d(xy)}=\overline{x}\,\overline{d(y)}+\overline{d(x)}\,\overline{y}$, for all $x,y\in\mathcal{N}$. Then

$$\begin{aligned}\overline{d((xy)z)} &= \overline{x}\,\overline{y}\,\overline{d(z)}+\overline{d(xy)}\,\overline{z}\\ &= \overline{x}\,\overline{y}\,\overline{d(z)}+(\overline{x}\,\overline{d(y)}+\overline{d(x)}\,\overline{y})\overline{z}. \quad (2.5)\end{aligned}$$

Also,

$$\begin{aligned}\overline{d(x(yz))} &= \overline{x}\,\overline{d(yz)}+\overline{d(x)}\,\overline{y}\,\overline{z}\\ &= \overline{x}(\overline{y}\,\overline{d(z)}+\overline{d(y)}\,\overline{z})+\overline{d(x)}\,\overline{y}\,\overline{z}. \quad (2.6)\end{aligned}$$

It is clear that in a near-ring $\mathcal{N}$ the associative law holds, then $\overline{d((xy)z)}=\overline{d(x(yz))}$, for all $x,y,z\in\mathcal{N}$. From (2.5) and (2.6), we get $\overline{x}\,\overline{y}\,\overline{d(z)}+(\overline{x}\,\overline{d(y)}+\overline{d(x)}\,\overline{y})\overline{z}=\overline{x}\,\overline{y}\,\overline{d(z)}+\overline{x}\,\overline{d(y)}\,\overline{z}+\overline{d(x)}\,\overline{y}\,\overline{z}$, for all $x,y,z\in\mathcal{N}$, which forces that $(\overline{x}\,\overline{d(y)}+\overline{d(x)}\,\overline{y})\overline{z}=\overline{x}\,\overline{d(y)}\,\overline{z}+\overline{d(x)}\,\overline{y}\,\overline{z}$ for all $x,y,z\in\mathcal{N}$.

b. We know that $\overline{d(xy)}=\overline{d(x)}\,\overline{y}+\overline{x}\,\overline{d(y)}$, for all $x,y\in\mathcal{N}$. Then

$$\begin{aligned}\overline{d(x(yz))} &= \overline{d(x)}\,\overline{y}\,\overline{z}+\overline{x}\,\overline{d(yz)}\\ &= \overline{d(x)}\,\overline{y}\,\overline{z}+\overline{x}(\overline{d(y)}\,\overline{z}+\overline{y}\,\overline{d(z)}). \quad (2.7)\end{aligned}$$

Also,

$$\begin{aligned}\overline{d((xy)z)} &= \overline{d(xy)}\,\overline{z} + (\overline{xy})\overline{d(z)} \\ &= (\overline{d(x)}\,\overline{y} + \overline{x}\,\overline{d(y)})\overline{z} + \overline{xy}\,\overline{d(z)}. \end{aligned} \tag{2.8}$$

This implies that $\overline{d(x(yz))} = \overline{d((xy)z)}$ for all $x, y, z \in \mathcal{N}$. Applying (2.7) and (2.8) give $\overline{d(x)}\,\overline{yz}+\overline{x}\,\overline{d(y)}\,\overline{z}+\overline{xy}\,\overline{d(z)} = (\overline{d(x)}\,\overline{y}+\overline{x}\,\overline{d(y)})\overline{z}+\overline{xy}\,\overline{d(z)}$ for all $x, y, z \in \mathcal{N}$, which ensures that $(\overline{d(x)}\,\overline{y}+\overline{x}\,\overline{d(y)})\overline{z} = \overline{d(x)}\,\overline{yz}+\overline{x}\,\overline{d(y)}\,\overline{z}$ for all $x, y, z \in \mathcal{N}$. QED

Using the same reasoning as above, we get the following result:

Corollary 2 ([10] Lemma 1)**.** *Let $\mathcal{N}$ be a near-ring and d be an arbitrary P-derivation of $\mathcal{N}$. Then $\mathcal{N}$ satisfies the following partial distributive laws.*

- ***a.*** $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$ *for all* $x, y, z \in \mathcal{N}$.
- ***b.*** $(d(x)y + xd(y))z = d(x)yz + xd(y)z$ *for all* $x, y, z \in \mathcal{N}$.

Lemma 2. *Let $\mathcal{N}$ be a near-ring and P be a 3-prime ideal of $\mathcal{N}$.*

- ***a.*** *If $\overline{z} \in Z(\mathcal{N}/P) \setminus \{\overline{0}\}$, then $\overline{z}$ is not a zero divisor.*
- ***b.*** *If $Z(\mathcal{N}/P)$ contains a nonzero element $\overline{z}$ for which $\overline{z}+\overline{z} \in Z(\mathcal{N}/P)$, then $(\mathcal{N}/P, +)$ is abelian.*
- ***c.*** *If $\overline{z} \in Z(\mathcal{N}/P) \setminus \{\overline{0}\}$ and $\overline{x} \in \mathcal{N}/P$ such that $\overline{x}\,\overline{z} \in Z(\mathcal{N}/P)$ or $\overline{z}\,\overline{x} \in Z(\mathcal{N}/P)$, then $\overline{x} \in Z(\mathcal{N}/P)$.*

Proof. By hypothesis, we have P is a 3-prime ideal of $\mathcal{N}$. Thus $\mathcal{N}/P$ is 3-prime near-ring. Therefore, (**a**), (**b**) and (**c**) are consequences of [2, Lemmas 1.2(i), 1.2(iii) and 1.3(iii)]. QED

Corollary 3 ([2] Lemmas 1.2(i), 1.2(iii) and 1.3(iii))**.** *Let $\mathcal{N}$ be a 3-prime near-ring.*

- ***a.*** *If $z \in Z(\mathcal{N}) \setminus \{0\}$, then z is not a zero divisor.*
- ***b.*** *If $Z(\mathcal{N})$ contains a nonzero element z for which $z+z \in Z(\mathcal{N})$, then $(\mathcal{N}, +)$ is abelian.*
- ***c.*** *If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $x \in \mathcal{N}$ such that $xz \in Z(\mathcal{N})$ or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*

Lemma 3. *Let $\mathcal{N}$ be a near-ring and P be a 3-prime ideal of $\mathcal{N}$. Let d a non P-trivial P-derivation on $\mathcal{N}$. Then $\overline{x}\,\overline{d(\mathcal{N})} = \{\overline{0}\}$, implies $\overline{x} = \overline{0}$, and $\overline{d(\mathcal{N})}\,\overline{x} = \{\overline{0}\}$, implies $\overline{x} = \overline{0}$.*

Proof. Suppose that $\overline{x}\overline{d(\mathcal{N})} = \{\overline{0}\}$. Then $\overline{0} = \overline{x}\overline{d(yz)} = \overline{x}\overline{d(y)}\overline{z} + \overline{xy}\overline{d(z)} = \overline{xy}\overline{d(z)}$ for all $y, z \in \mathcal{N}$, which implies that $x\mathcal{N}d(z) \subseteq P$. In light of 3-primeness of P, we have $\overline{0} = \overline{x}$ or $\overline{0} = \overline{d(z)}$ for all $z \in \mathcal{N}$. Since $d(\mathcal{N}) \nsubseteq P$, we conclude that $\overline{0} = \overline{x}$.
A similar argument works if $\overline{d(\mathcal{N})}\overline{x} = \{\overline{0}\}$. QED

Lemma 4. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$ and d a P^+-derivation on $\mathcal{N}$. If $d^2(\mathcal{N}) \subseteq P$, then $d(\mathcal{N}) \subseteq P$ or $2(\mathcal{N}/P) = \{\overline{0}\}$.*

Proof. By hypothesis, we have

$$\begin{aligned}
\overline{0} &= \overline{d^2(xy)} \\
&= \overline{d(d(x)y + xd(y))} \\
&= \overline{d^2(x)}\overline{y} + \overline{d(x)}\,\overline{d(y)} + \overline{d(x)}\,\overline{d(y)} + \overline{x}\overline{d^2(y)} \\
&= 2\overline{d(x)}\,\overline{d(y)} \\
&= \overline{d(x)}\,\overline{d(2y)} \text{ for all } x, y \in \mathcal{N}.
\end{aligned}$$

Replacing y by ny in the last equation we get $\overline{d(x)}\overline{n}\overline{d(2y)} = \overline{0}$ for all $n, x, y \in \mathcal{N}$, which implies that $\overline{d(x)}\,(\mathcal{N}/P)\,\overline{d(2y)} = \{\overline{0}\}$ for all $x, y \in \mathcal{N}$. By primeness of P, we find $d(\mathcal{N}) \subseteq P$ or $\overline{d(2y)} = \overline{0}$ for all $y \in \mathcal{N}$.
Suppose $d(\mathcal{N}) \nsubseteq P$, so $\overline{d(2y)} = \overline{0}$ for all $y \in \mathcal{N}$, then

$$\begin{aligned}
\overline{0} &= \overline{d(2xy)} \\
&= \overline{d(xy)} + \overline{d(xy)} \\
&= \overline{d(x)}\overline{y} + \overline{x}\overline{d(y)} + \overline{x}\overline{d(y)} + \overline{d(x)}\overline{y} \\
&= \overline{d(x)}\overline{y} + \overline{x}\overline{d(2y)} + \overline{d(x)}\overline{y} \\
&= \overline{d(x)}\overline{y} + \overline{d(x)}\overline{y} \\
&= \overline{d(x)}\,(\overline{y} + \overline{y}) \text{ for all } x, y \in \mathcal{N}.
\end{aligned}$$

That is, $\overline{d(\mathcal{N})}\,(\overline{y} + \overline{y}) = \{\overline{0}\}$ for all $y \in \mathcal{N}$. By Lemma 3, we get $2(\mathcal{N}/P) = \{\overline{0}\}$. QED

Theorem 4. *Let $\mathcal{N}$ be a near-ring, P be a symmetric 3-prime ideal of $\mathcal{N}$, and d be a P-derivation of $\mathcal{N}$. If $\overline{u}$ is not left zero divisor on $\mathcal{N}/P$ and $[u, d(u)] \in P$, then $\overline{d((x,u))} = \overline{0}$ for all $x \in \mathcal{N}$.*

Proof. From $u(u+x) = u^2 + ux$, we get $\overline{u}\overline{d(u+x)} + \overline{d(u)}(\overline{u} + \overline{x}) = \overline{u}\overline{d(u)} + \overline{d(u)}\overline{u} + \overline{u}\overline{d(x)} + \overline{d(u)}\overline{x}$, which reduces to $\overline{u}\overline{d(x)} + \overline{d(u)}\overline{u} = \overline{d(u)}\overline{u} + \overline{u}\overline{d(x)}$. Since $\overline{d(u)}\overline{u} = \overline{u}\overline{d(u)}$, this equation can be expressed as $\overline{u}\overline{(d(x) + d(u) - d(x) - d(u))} = \overline{0} = \overline{u}\overline{d((x,u))}$. Thus, $\overline{d((x,u))} = \overline{0}$. QED

3 Commutativity of $\mathcal{N}/P$

Theorem 5. *Let $\mathcal{N}$ be a near-ring and P be a symmetric 3-prime ideal of $\mathcal{N}$. Suppose that $\mathcal{N}/P$ has no nonzero divisors of zero. If $\mathcal{N}$ admits a non P-trivial P-commuting P-derivation d, then $(\mathcal{N}/P,+)$ is abelian.*

Proof. Let $\overline{c}$ be any additive commutator of $\mathcal{N}/P$. Then $\overline{d(c)} = \overline{0}$ by Lemma 4. Moreover, for any $\overline{w} \in \mathcal{N}/P$, $\overline{wc}$ is an additive commutator, so it is also a P-constant. Thus, $\overline{0} = \overline{d(wc)} = \overline{w}\overline{d(c)} + \overline{d(w)}\overline{c}$ and $\overline{d(w)}\overline{c} = \overline{0}$. Since $\overline{d(w)} \neq \overline{0}$ for some $\overline{w} \in \mathcal{N}/P$, we conclude that $\overline{c} = \overline{0}$. QED

Theorem 6. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$. If $\mathcal{N}$ admits a non P-trivial P-derivation d such that $\overline{d(\mathcal{N})} \subseteq Z(\mathcal{N}/P)$, then $(\mathcal{N}/P,+)$ is abelian. Moreover, if $d^2(\mathcal{N}) \nsubseteq P$, then $\mathcal{N}/P$ is a commutative ring.*

Proof. Suppose that $\overline{0}$ is the only P-constant. Since d is P-commuting, by Lemma 4 we have $\overline{x} \in C(\mathcal{N}/P)$ for all $\overline{x} \in \mathcal{N}/P$, which are nonzero divisors. In particular, for $d(x) \notin P$, we have $\overline{d(x)} \in C(\mathcal{N}/P)$. Then for all $\overline{y} \in \mathcal{N}/P$ we get $\overline{0} = \overline{d(y)} + \overline{d(x)} - \overline{d(y)} - \overline{d(x)} = \overline{d((y,x))}$, so $(\overline{y},\overline{x}) = \overline{0}$; a contradiction.
Let $\overline{c} \neq \overline{0}$ be an arbitrary P constant, and $\overline{x}$ be a non P constant. So $\overline{d(xc)} = \overline{d(x)}\overline{c} + \overline{x}\overline{d(c)} = \overline{d(x)}\overline{c} \in Z(\mathcal{N}/P)$. By lemma 2 (iii) we get $\overline{c} \in Z(\mathcal{N}/P)$. Since $\overline{c}+\overline{c}$ is a P constant, we get $\overline{c}+\overline{c} \in Z(\mathcal{N}/P)$. Thus, by lemma 2 (ii), $(\mathcal{N}/P,+)$ is abelian.
Now supposing that $d^2(N) \nsubseteq P$, and proving that $\mathcal{N}/P$ is a commutative ring. We have $\left(\overline{d(x)}\overline{y} + \overline{x}\overline{d(y)}\right)\overline{z} = \overline{d(xy)}\overline{z} = \overline{z}\overline{d(xy)} = \overline{z}\left(\overline{d(x)}\overline{y} + \overline{x}\overline{d(y)}\right)$ for all $x,y,z \in \mathcal{N}$. That is $\overline{d(x)}\overline{yz} + \overline{x}\overline{d(y)}\overline{z} = \overline{z}\overline{d(x)}\overline{y} + \overline{zx}\overline{d(y)}$ for all $x,y,z \in \mathcal{N}$. Thus $\overline{d(x)}[\overline{y},\overline{z}] = \overline{d(y)}[\overline{z},\overline{x}]$ for all $x,y,z \in \mathcal{N}$. Replacing y by $d(y)$ in last expression and using it we get $\overline{d^2(y)}[\overline{z},\overline{x}] = \overline{0}$ for all $x,y,z \in \mathcal{N}$. Since $\overline{d^2(y)} \in Z(\mathcal{N}/P)$, we obtain $\overline{d^2(y)}(\mathcal{N}/P)[\overline{z},\overline{x}] = \{\overline{0}\}$ for all $x,y,z \in \mathcal{N}$. The 3-primeness of $\mathcal{N}/P$ gives $[\overline{z},\overline{x}] = \overline{0}$ for all $x,z \in \mathcal{N}$, therefore $\mathcal{N}/P$ is a commutative ring. QED

Corollary 4. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$. If $\mathcal{N}$ admits a non P-trivial P^+-derivation d such that $\overline{d(\mathcal{N})} \subseteq Z(\mathcal{N}/P)$, then $(\mathcal{N}/P,+)$ is abelian. Moreover, if $2(\mathcal{N}/P) \neq \{\overline{0}\}$, then $\mathcal{N}/P$ is a commutative ring.*

Proof. In the light of Lemma 4 and Theorem 6 we get the proof. QED

Theorem 7. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$, d a non P-trivial P-derivation and $a \in \mathcal{N}$. If $d^2(\mathcal{N}) \nsubseteq P$ and $[d(x),a] \in P$ for all $x \in \mathcal{N}$, then $\overline{a} \in Z(\mathcal{N}/P)$.*

Proof. Let $a \in \mathcal{N}$. We set $C(a) = \{x \in \mathcal{N} \mid [x,a] \in P\}$. Next we claim that

$$d(C(a))\mathcal{N} \subseteq C(a). \tag{3.9}$$

Indeed, let $y \in C(a)$ and $x \in \mathcal{N}$. By assumption, we have that $d(yx), d(x) \in d(\mathcal{N}) \subseteq C(a)$. Since $y, d(x) \in C(a)$, $yd(x) \in C(a)$ as well. Hence $\overline{y}\overline{d(x)}\overline{a} = \overline{ay}\overline{d(x)}$. It follows from Theorem 3 **(a)** that

$$\begin{aligned}
\overline{y}\overline{d(x)}\overline{a}+\overline{d(y)}\overline{xa} &= (\overline{y}\overline{d(x)}+\overline{d(y)}\overline{x})\overline{a}\\
&= \overline{d(yx)}\overline{a}\\
&= \overline{a}\overline{d(yx)}\\
&= \overline{a}(\overline{y}\overline{d(x)}+\overline{d(y)}\overline{x}).
\end{aligned}$$

Which implies that $\overline{y}\overline{d(x)}\overline{a}+\overline{d(y)}\overline{xa} = \overline{ay}\overline{d(x)}+\overline{a}\overline{d(y)}\overline{x}$.

Since $\overline{y}\overline{d(x)}\overline{a} = \overline{ay}\overline{d(x)}$, we see that $\overline{d(y)}\overline{xa} = \overline{a}\overline{d(y)}\overline{x}$, which proves our claim. Finally, by our assumption $d^2(\mathcal{N}) \nsubseteq P$. Hence $\overline{d^2(z)} \neq \overline{0}$, for some $z \in \mathcal{N}$. Set $y = d(z)$ and pick an arbitrary $x \in \mathcal{N}$. Since $y \in d(\mathcal{N}) \subseteq C(a), d(y)x \in C(a)$ by (3.9). In particular $d(y)u, d(y)uv \in C(a)$ for all $u, v \in \mathcal{N}$. Now it follows that $\overline{0} = [\overline{a}, \overline{d(y)}\overline{uv}] = \overline{a}\overline{d(y)}\overline{uv} - \overline{d(y)}\overline{uva} = \overline{d(y)}\overline{uav} - \overline{d(y)}\overline{uva} = \overline{d(y)}\overline{u}(\overline{av}-\overline{va})$ or $\overline{d(y)}\overline{u}[\overline{a},\overline{v}] = \overline{0}$, for all $u, v \in \mathcal{N}$. Since $\mathcal{N}/P$ is a 3-prime near-ring and $\overline{d(y)} \neq \overline{0}$, we conclude that $[\overline{a},\overline{v}] = \overline{0}$, for all $v \in \mathcal{N}$, which completes the proof. QED

Corollary 5. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$ and d be a non P-trivial P^+-derivation. If $2(\mathcal{N}/P) \neq \{\overline{0}\}$ and $[d(x), a] \in P$ for all $x \in \mathcal{N}$, then $\overline{a} \in Z(\mathcal{N}/P)$.*

Theorem 8. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$ and d_1, d_2 be non P-trivial P-derivations of $\mathcal{N}$ such that $[d_1(x), d_2(y)] \in P$ for all $x, y \in \mathcal{N}$, then one of the following assertions holds:*

- ***a.*** $d_1^2(\mathcal{N}) \subseteq P$.
- ***b.*** $d_2^2(\mathcal{N}) \subseteq P$.
- ***c.*** *$\mathcal{N}/P$ is a commutative ring.*

Proof. Assume that $d_1^2(\mathcal{N}) \nsubseteq P$, and $d_2^2(\mathcal{N}) \nsubseteq P$. It follows from Theorem 7, that $\overline{d_1(N)} \subseteq Z(\mathcal{N}/P)$ and so $[\mathcal{N}/P, \overline{d_1(N)}] = \{\overline{0}\}$. Again by Theorem 7, we conclude that $\mathcal{N}/P \subseteq Z(\mathcal{N}/P)$ and so $\mathcal{N}/P$ is a commutative near-ring. In particular $\mathcal{N}/P$ is distributive. Let $\overline{u}, \overline{x}, \overline{y} \in \mathcal{N}/P$. Then $(\overline{u}+\overline{u})(\overline{x}+\overline{y}) = (\overline{u}+\overline{u})\overline{x} + (\overline{u}+\overline{u})\overline{y} = \overline{ux}+\overline{ux}+\overline{uy}+\overline{uy}$, it follows that $\overline{uy}+\overline{ux} = \overline{ux}+\overline{uy}$ and $\overline{u}(\overline{y}+\overline{x}-\overline{y}|\overline{x}) = \overline{0}$ for all $\overline{u}, \overline{x}, \overline{y} \in \mathcal{N}/P$. Since $\mathcal{N}/P$ is 3-prime, we have $(\overline{y}+\overline{x}-\overline{y}-\overline{x}) = \overline{0}$ for all $\overline{x}, \overline{y} \in \mathcal{N}/P$, and so $\mathcal{N}/P$ is a commutative ring. The proof is complete. QED

Corollary 6. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$ and d_1, d_2 are P^+-derivations of $\mathcal{N}$. If $[d_1(x), d_2(y)] \in P$ for all $x, y \in \mathcal{N}$, then one of the following assertions holds:*

a. $2(\mathcal{N}/P) = \{\overline{0}\}$.

b. $d_1(\mathcal{N}) \subseteq P$.

c. $d_2(\mathcal{N}) \subseteq P$.

d. *$\mathcal{N}/P$ is a commutative ring.*

Corollary 7. *Let R be a ring, P be a prime ideal of R and d_1, d_2 are derivations of R such that $[d_1(x), d_2(y)] \in P$ for all $x, y \in R$, then we have one of the following assertions:*

a. $Char(R/P) = 2$.

b. $d_1(R) \subseteq P$.

c. $d_2(R) \subseteq P$.

d. *R/P is a commutative integral domain.*

Theorem 9. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$. If $\mathcal{N}$ admits P-derivations d_1 and d_2 such that $d_1(x)d_2(y) + d_2(x)d_1(y) \in P$, for all $x, y \in \mathcal{N}$, then one of the following assertions holds:*

a. $d_1(\mathcal{N}) \subseteq P$.

b. $d_2(\mathcal{N}) \subseteq P$.

c. $2(\mathcal{N}/P) = \{\overline{0}\}$

Proof. Suppose that $d_1(\mathcal{N}) \nsubseteq P$ and $d_2(\mathcal{N}) \nsubseteq P$. By hypothesis, we have

$$\begin{aligned}
\overline{0} &= \overline{d_1(x)}\,\overline{d_2(u+v)} + \overline{d_2(x)}\,\overline{d_1(u+v)} \\
&= \overline{d_1(x)}[\overline{d_2(u)} + \overline{d_2(v)}] + \overline{d_2(x)}[\overline{d_1(u)} + \overline{d_1(v)}] \\
&= \overline{d_1(x)}\,\overline{d_2(u)} + \overline{d_1(x)}\,\overline{d_2(v)} + \overline{d_2(x)}\,\overline{d_1(u)} + \overline{d_2(x)}\,\overline{d_1(v)} \\
&= \overline{d_1(x)}\,\overline{d_2(u)} + \overline{d_1(x)}\,\overline{d_2(v)} - \overline{d_1(x)}\,\overline{d_2(u)} - \overline{d_1(x)}\,\overline{d_2(v)} \\
&= \overline{d_1(x)}[\overline{d_2(u)} + \overline{d_2(v)} - \overline{d_2(u)} - \overline{d_2(v)}] = \overline{d_1(x)}\,\overline{d_2((u,v))}.
\end{aligned}$$

Thus $\overline{d_1(\mathcal{N})}\overline{d_2((u,v))} = \{\overline{0}\}$ for all $u, v \in \mathcal{N}$. Using Lemma 3 gives $\overline{d_2((u,v))} = \overline{0}$ for all $u, v \in \mathcal{N}$. Substituting wu and wv for u and v respectively, we have $\overline{0} = \overline{d_2((wu,wv))} = \overline{d_2(w(u,v))} = \overline{d_2(w)}(\overline{u}, \overline{v})$ for all $u, v, w \in \mathcal{N}$. That is

$\overline{d_2(\mathcal{N})}.(\overline{u},\overline{v}) = \{\overline{0}\}$. From Lemma 3, we get $(\overline{u},\overline{v}) = \overline{0}$, for all $u, v, w \in \mathcal{N}$. Thus $(\mathcal{N}/P, +)$ is abelian.
Substituting x by uv in the hypothesis, we get

$$\begin{aligned} \overline{0} &= [\overline{u}\overline{d_1(v)} + \overline{d_1(u)}\overline{v}]\overline{d_2(y)} + [\overline{u}\overline{d_2(v)} + \overline{d_2(u)}\overline{v}]\overline{d_1(y)} \\ &= \overline{u}\overline{d_1(v)d_2(y)} + \overline{d_1(u)}\overline{v}\overline{d_2(y)} + \overline{u}\overline{d_2(v)d_1(y)} + \overline{d_2(u)}\overline{v}\overline{d_1(y)} \\ &= \overline{u}[\overline{d_1(v)d_2(y)} + \overline{d_2(v)d_1(y)}] + \overline{d_1(u)}\overline{v}\overline{d_2(y)} + \overline{d_2(u)}\overline{v}\overline{d_1(y)} \\ &= \overline{d_1(u)}\overline{v}\overline{d_2(y)} + \overline{d_2(u)}\overline{v}\overline{d_1(y)} \text{ for all } u, v, y \in \mathcal{N}. \qquad (3.10) \end{aligned}$$

Taking yt instead of y in (3.10) to obtain

$$\begin{aligned} \overline{0} &= \overline{d_1(u)}\overline{v}\overline{d_2(yt)} + \overline{d_2(u)}\overline{v}\overline{d_1(yt)} \\ &= \overline{d_1(u)}\overline{v}\left[\overline{d_2(y)}\overline{t} + \overline{y}\overline{d_2(t)}\right] + \overline{d_2(u)}\overline{v}\left[\overline{d_1(y)}\overline{t} + \overline{y}\overline{d_1(t)}\right] \\ &= \overline{d_1(u)}\overline{v}\overline{d_2(y)}\overline{t} + \overline{d_1(u)}\overline{v}\overline{y}\overline{d_2(t)} + \overline{d_2(u)}\overline{v}\overline{d_1(y)}\overline{t} + \overline{d_2(u)}\overline{v}\overline{y}\overline{d_1(t)} \\ &= \left[\overline{d_1(u)}\overline{v}\overline{d_2(y)}\overline{t} + \overline{d_2(u)}\overline{v}\overline{d_2(y)}\overline{t}\right] + \left[\overline{d_1(u)}\overline{v}\overline{y}\overline{d_2(t)} + \overline{d_2(u)}\overline{v}\overline{y}\overline{d_2(t)}\right] \\ &= \overline{d_1(u)}\overline{v}\overline{d_2(y)}\overline{t} + \overline{d_2(u)}\overline{v}\overline{d_1(y)}\overline{t} \text{ for all } u, v, t, y \in \mathcal{N}. \qquad (3.11) \end{aligned}$$

Placing $d_1(t)$ instead of t, in (3.11), we get that

$$\overline{d_1(u)}\overline{v}\overline{d_2(y)d_1(t)} + \overline{d_2(u)}\overline{v}\overline{d_1(y)d_1(t)} = \overline{0} \text{ for all } u, v, t, y \in \mathcal{N}. \qquad (3.12)$$

Taking $vd_1(y)$ and t instead of v and y respectively in (3.10), we find that

$$\overline{d_1(u)}\overline{v}\overline{d_1(y)d_2(t)} + \overline{d_2(u)}\overline{v}\overline{d_1(y)d_1(t)} = \overline{0} \text{ for all } u, v, t, y \in \mathcal{N}. \qquad (3.13)$$

Subtraction of (3.13) from (3.12) yields that

$$\overline{d_1(u)}\overline{v}[\overline{d_2(y)d_1(t)} - \overline{d_1(y)d_2(t)}] = \overline{0}.$$

Using the hypothesis, we obtain $\overline{d_1(u)}\overline{v}\left[\overline{d_2(y)d_1(t)} + \overline{d_2(y)d_1(t)}\right] = \overline{0}$. Since $d_1(\mathcal{N}) \nsubseteq P$, it follows that $\overline{d_1(u)} \neq \overline{0}$ for some $u \in \mathcal{N}$. As

$$\overline{d_1(u)}(\mathcal{N}/P)\left[\overline{d_2(y)d_1(t)} + \overline{d_2(y)d_1(t)}\right] = \{0\}$$

and $\mathcal{N}/P$ is 3-prime, we conclude that

$$\overline{d_2(y)d_1(t)} + \overline{d_2(y)d_1(t)} = \overline{0} \text{ for all } t, y \in \mathcal{N}. \qquad (3.14)$$

Recall that $(\mathcal{N}/P, +)$ is abelian. Letting yu instead of y in (3.14), we obtain

$$\begin{aligned} \overline{0} &= \overline{d_2(y)}\overline{u}\overline{d_1(t)} + \overline{y}\overline{d_2(u)d_1(t)} + \overline{d_2(y)}\overline{u}\overline{d_1(t)} + \overline{y}\overline{d_2(u)d_1(t)} \\ &= \overline{y}\left[\overline{d_2(u)d_1(t)} + \overline{d_2(u)d_1(t)}\right] + \left[\overline{d_2(y)}\overline{u}\overline{d_1(t)} + \overline{d_2(y)}\overline{u}\overline{d_1(t)}\right] \\ &= \overline{d_2(y)}\overline{u}\overline{d_1(t)} + \overline{d_2(y)}\overline{u}\overline{d_1(t)} \text{ for all } u, t, y \in \mathcal{N}. \qquad (3.15) \end{aligned}$$

Now substituting ut instead of t in (3.14), we obtain

$$\begin{aligned} \overline{0} &= \overline{d_2(y)}\overline{u}\overline{d_1(t)} + \overline{d_2(y)d_1(u)}\overline{t} + \overline{d_2(y)}\overline{u}\overline{d_1(t)} + \overline{d_2(y)d_1(u)}\overline{t} \\ &= \overline{d_2(y)d_1(u)}\overline{t} + \overline{d_2(y)d_1(u)}\overline{t} \text{ for all } u,t,y \in \mathcal{N}. \qquad (3.16) \end{aligned}$$

Therefore, $\overline{d_2(\mathcal{N})d_1(u)}(\overline{t}+\overline{t}) = \{\overline{0}\}$ for all $u,t \in \mathcal{N}$ and so $\overline{d_1(\mathcal{N})}(\overline{t}+\overline{t}) = \{\overline{0}\}$ for all $t \in \mathcal{N}$ by Lemma 3. Again applying Lemma 3, we conclude that $2(\mathcal{N}/P) = \{\overline{0}\}$. QED

Corollary 8. *Let P be a prime ideal of a ring R. If R admits P-derivations d_1 and d_2 such that $d_1(x)d_2(y) + d_2(x)d_1(y) \in P$, for all $x,y \in R$, then one of the following assertions holds:*

- ***a.*** $d_1(R) \subseteq P$.
- ***b.*** $d_2(R) \subseteq P$.
- ***c.*** $char(\mathcal{N}/P) = 2$

Lemma 5. *Let $\mathcal{N}$ be an arbitrary near-ring. Let S and T be nonempty subsets of $\mathcal{N}$ such that $st = -ts$ for all $s \in S$ and $t \in T$. If $a,b \in S$ and $c \in T$ for which $-c \in T$, then $(ab)c = c(ab)$.*

Theorem 10. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$. If d_1 and d_2 are P^+-derivations on $\mathcal{N}$ such that $d_1(x) \circ d_2(y) \in P$ for all $x,y \in \mathcal{N}$, then one of the following assertions holds:*

- ***a.*** $2(\mathcal{N}/P) = \{\overline{0}\}$.
- ***b.*** $d_1(\mathcal{N}) \subseteq P$.
- ***c.*** $d_2(\mathcal{N}) \subseteq P$.

Proof. Suppose that $2(\mathcal{N}/P) \neq \{\overline{0}\}$. By Lemma 4, we may assume $d_1^2(N) \nsubseteq P$ and $d_2^2(\mathcal{N}) \nsubseteq P$. Let $w \in d_2(\mathcal{N})$ then $-w \in d_2(\mathcal{N})$. Therefore, by Lemma 5, if $u,v \in d_1(\mathcal{N})$, then $\overline{uv}$ centralizes $\overline{d_2(\mathcal{N})}$, hence $\overline{uv} \in Z(\mathcal{N}/P)$ by Theorem 7. It follows that $\overline{d_1(x)^2 d_1(y)} = \overline{d_1(x)d_1(y)d_1(x)}$ and $\overline{d_1(x)^2 d_1(y)^2} = \left(\overline{d_1(x)d_1(y)}\right)^2$ for all $x,y \in \mathcal{N}$. Hence $\overline{d_1(x)d_1(y)}\left(\overline{d_1(x)d_1(y)} - \overline{d_1(y)d_1(x)}\right) = \overline{0}$ and $\overline{d_1(y)d_1(x)}\left(\overline{d_1(x)d_1(y)} - \overline{d_1(y)d_1(x)}\right) = \overline{0}$. Since $\overline{d_1(x)d_1(y)}$ and $\overline{d_1(y)d_1(x)}$ are central, Lemma 2 (i) shows that for any $x,y \in \mathcal{N}$, either $\overline{d_1(x)d_1(y)} = \overline{d_1(y)d_1(x)} = \overline{0}$ or $\overline{d_1(x)d_1(y)} = \overline{d_1(y)d_1(x)}$. Then, $[d_1(\mathcal{N}), d_1(\mathcal{N})] \subseteq P$. By Theorem 8, $\mathcal{N}$ is commutative. However, this fact with our hypothesis shows that $\overline{0} = 2\overline{d_1(x)d_2(y)}$ for all $x,y \in U$.

Suppose $d_1(\mathcal{N}) \nsubseteq P$ and $d_2(\mathcal{N}) \nsubseteq P$. Using similar arguments as in the proof of lemma 4, we get $2(\mathcal{N}/P) = \{\overline{0}\}$; a contradiction. So $d_1(\mathcal{N}) \subseteq P$ or $d_1(\mathcal{N}) \subseteq P$. QED

Corollary 9. *Let P be a prime ideal of a ring R. If R admits P-derivations d_1 and d_2 such that $d_1(x) \circ d_2(y) \in P$ for all $x, y \in R$, then one of the following assertions holds:*

a. $d_1(R) \subseteq P$.

b. $d_2(R) \subseteq P$.

c. $Char(R/P) = 2$.

Theorem 11. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$, and let d_1 and d_2 P-derivations such that $d_1 d_2(xy) - d_1(xd_2(y) + d_2(x)y) \in P$ for all $x, y \in \mathcal{N}$. If $d_1 d_2$ is a P-derivation, then one of the following assertions holds:*

a. $d_1(\mathcal{N}) \subseteq P$.

b. $d_2(\mathcal{N}) \subseteq P$.

c. $2(\mathcal{N}/P) = \{\overline{0}\}$.

Proof. Since $d_1 d_2$ is a P-derivation, we have

$$\overline{d_1 d_2(xy)} = \overline{x}\overline{d_1 d_2(y)} + \overline{d_1 d_2(x)}\overline{y}, \text{ for all } x, y \in \mathcal{N}.$$

On the other hand,

$$\begin{aligned} \overline{d_1 d_2(xy)} &= \overline{d_1(xd_2(y) + d_2(x)y)} \\ &= \overline{x}\overline{d_1 d_2(y)} + \overline{d_1(x)d_2(y)} + \overline{d_2(x)d_1(y)} + \overline{d_1 d_2(x)}\overline{y}. \end{aligned}$$

Comparing these two expressions, we obviously obtain

$$\overline{d_1(x)d_2(y)} + \overline{d_2(x)d_1(y)} = \overline{0}, \text{ for all } x, y \in \mathcal{N}.$$

Now, our assertion follows from Theorem 9. QED

Corollary 10. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$, and d is a P^+-derivation. If d^2 is a P-derivation, then one of the following assertions holds:*

a. $d(\mathcal{N}) \subseteq P$.

b. $2(\mathcal{N}/P) = \{\overline{0}\}$.

4 Semiprime ideal and derivations

Theorem 12. *Let P be a semiprime ideal of a symmetric near-ring $\mathcal{N}$, where $\mathcal{N}/P$ is 2-torsion free. Let d be a derivation of $\mathcal{N}$ such that $[d(x), d(y)] \in P$ for all $x, y \in \mathcal{N}$, then one of the following assertions holds:*

- ***a.*** *There exists a prime ideal $P_\alpha \supseteq P$ such that $d(\mathcal{N}) \subseteq P_\alpha$.*
- ***b.*** *$\mathcal{N}/P$ is a commutative ring.*

Proof. Since P is semiprime, there exists a family $\mathcal{P}$ of 3-prime ideals P_α such that $\cap P_\alpha = P$. Therefore,

$$[d(x), d(y)] \in P_\alpha \text{ for all } x, y \in R, P_\alpha \in \mathcal{P}. \tag{4.17}$$

Since d is a derivation, we get d is P_α-derivations on $\mathcal{N}$ for all $P_\alpha \in \mathcal{P}$. Using (4.17) and the fact that $2(\mathcal{N}/P_\alpha) \neq \{\overline{0}\}$, the corrolary 6 gives

$$d(\mathcal{N}) \subseteq P_\alpha \text{ or } \mathcal{N}/P_\alpha \text{ is a commutative ring for all } P_\alpha \in \mathcal{P}. \tag{4.18}$$

Suppose that $d(\mathcal{N}) \nsubseteq P_\alpha$ for all $P_\alpha \in \mathcal{P}$. Thus (4.18) implies that $\mathcal{N}/P = \mathcal{N}/\cap P_\alpha$ is commutative ring. QED

Theorem 13. *Let P be a semiprime ideal of a symmetric near-ring $\mathcal{N}$, where $\mathcal{N}/P$ is 2-torsion free. If d is a derivation on $\mathcal{N}$ such that $2d(x)d(y) \in P$ for all $x, y \in \mathcal{N}$, then $d(\mathcal{N}) \subseteq P$.*

Proof. Since P is semiprime, there exists a family $\mathcal{P}$ of 3-prime ideals P_α such that $\cap P_\alpha = P$. Therefore,

$$2d(x)d(y) \in P_\alpha \text{ for all } x, y \in R, P_\alpha \in \mathcal{P}. \tag{4.19}$$

Since d is a derivation, we get d is P_α-derivation on $\mathcal{N}$ for all $P_\alpha \in \mathcal{P}$. Using (4.19) with $2(\mathcal{N}/P_\alpha) \neq \{\overline{0}\}$, then Theorem 10 gives $d(\mathcal{N}) \subseteq P_\alpha$ for all $P_\alpha \in \mathcal{P}$, which forces that $d(\mathcal{N}) \subseteq P$. QED

Theorem 14. *Let P be a semiprime ideal of a symmetric near-ring $\mathcal{N}$ and $\mathcal{N}/P$ is 2-torsion free. If d is a derivation on $\mathcal{N}$ such that $d(x) \circ d(y) \in P$ for all $x, y \in \mathcal{N}$, then $d(\mathcal{N}) \subseteq P$.*

Proof. Since P is semiprime, there exists a family $\mathcal{P}$ of 3-prime ideals P_α such that $\cap P_\alpha = P$. Therefore,

$$d(x) \circ d(y) \in P_\alpha \text{ for all } x, y \in \mathcal{N}, P_\alpha \in \mathcal{P}. \tag{4.20}$$

Since d is a derivation, we obtain d is P_α-derivations on $\mathcal{N}$ for all $P_\alpha \in \mathcal{P}$. By (4.20) and $2(\mathcal{N}/P_\alpha) \neq \{\overline{0}\}$, Theorem 10 gives $d(\mathcal{N}) \subseteq P_\alpha$ for all $P_\alpha \in \mathcal{P}$, which implies that $d(\mathcal{N}) \subseteq P$. QED

Theorem 15. *Let P be a semiprime ideal of a symmetric near-ring $\mathcal{N}$, and d be a derivation on $\mathcal{N}$. Then d^2 is a derivation if one of the following assertions holds:*

a. *There exists a prime ideal $P_\alpha \supseteq P$ such that $d(\mathcal{N}) \subseteq P_\alpha$.*

b. $2(\mathcal{N}/P) = \{\overline{0}\}$.

Proof. Since P is semiprime, there exists a family $\mathcal{P}$ of 3-prime ideals P_α such that $\cap P_\alpha = P$. Therefore, since d is a derivation, d is also P_α^+-derivation on $\mathcal{N}$ for all $P_\alpha \in \mathcal{P}$. Using the corollary 10, we get $2(\mathcal{N}/P_\alpha) = \{\overline{0}\}$ or $d(\mathcal{N}) \subseteq P_\alpha$ for all $P_\alpha \in \mathcal{P}$, which complete the proof of our theorem. QED

Acknowledgements. The authors thank the reviewer for valuable suggestions and comments.

References

[1] M. Ashraf, A. Ali and A. Shakir: (σ, τ)*-derivations on prime near-rings*, Arch. Math. (Brno), **40** (2004), 281–286.

[2] H. E. Bell: *On derivations in near-rings, Vol. II. In: Nearrings, Nearfields and K-loops*, Math. Appl., **426** (1997), 191–197.

[3] A. Boua and A. A. M. Kamal: *Some results on* 3*-Prime near-rings with derivations*, Indian J. Pure Appl. Math., **47** (2016), 705-716.

[4] A. Boua: *Structure of* 3*-prime near-rings satisfying some identities*, Commun. Korean Math. Soc. **34**(1) (2019), 17-26.

[5] A. Boua *Some identities in rings and near rings with derivation*, Kragujevac J. Math., **45**(1) (2021), 75-80.

[6] N. J. Groenewald: *Different prime ideals in near-rings*, Comm. Algebra, **19**(10) (1991), 2667–2675.

[7] S. Mouhssine, A. Boua and M. M. El-Soufi: *On two-sided α-generalized derivations of 3-prime near-rings*, Comm. Algebra., **50**(11) (2022), 4682-4699.

[8] S. Mouhssine and A. Boua: *Homoderivations and semigroup ideals in* 3*-prime near-rings*, Algebraic Structures and Their Applications, **8**(2) (2021), 177-194.

[9] S. Mouhssine and A. Boua: *Right multipliers and commutativity of 3-prime near-rings*, IJAM., **34**(1) (2021), 169-181.

[10] X. K. Wang, *Derivations in prime near-rings*, Proc. Amer. Math. Soc., **121** (1994), 361-366.