Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **40** (2020) no. 2, 111–127. doi:10.1285/i15900932v40n2p111

On subgroups normalized by the base group in monomial groups, and their centralizers

Sezen Bostan
Department of Mathematics, Middle East Technical University, Ankara, Turkey
bostansezen@gmail.com

Dario Esposito [i]
Dip. di Mat. Renato Caccioppoli, Universitá di Napoli Federico II, Napoli, Italia
dario.esposito2@unina.it

Received: 24.8.2020; accepted: 19.11.2020.

Abstract. A large family of examples for subgroups normalized by the base group in a complete monomial group of finite degree over any group H is given. This is then proven to be a complete characterization of such subgroups in the case of an abelian group H. Centralizer structure for this kind of subgroups, even in the non abelian case, is completely determined. Notably, separate study of the case of elementary abelian 2-group H is needed. In the last part, the results are extended to the case of limit monomial groups.

Keywords: Monomial group, Base group, Centralizer, Limit monomial group.

MSC 2020 classification: 20B99

1 Introduction

Let H be an arbitrary group and $n \in \mathbb{N}$. The *complete monomial group of degree n over H*, denoted by $\Sigma_n(H)$, is the subgroup of the group of linear automorphisms of the free module $[\mathbb{Z}H]^n$ consisting of all automorphisms represented by generalized permutation matrices with entries in H (we also refer to [5] for more on the topic). An element f of $\Sigma_n(H)$ is called a *monomial substitution* and will be represented by

$$f = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 x_{i_1} & h_2 x_{i_2} & \dots & h_n x_{i_n} \end{pmatrix}$$

where $\{x_1, \dots, x_n\}$ is a set of n variables and f changes each variable x_j into some other variable x_{i_j}, multiplied by an element of H. The elements $h_i \in H$ will be called *factors* or *multipliers* of f.

[i]This paper was written while D. Esposito was a visiting Ph.D. student at Middle East Technical University.

If

$$g = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ a_1 x_{j_1} & a_2 x_{j_2} & \dots & a_n x_{j_n} \end{pmatrix} \in \Sigma_n(H),$$

then

$$fg = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 a_{i_1} x_{j_{i_1}} & h_2 a_{i_2} x_{j_{i_2}} & \dots & h_n a_{i_n} x_{j_{i_n}} \end{pmatrix}$$

and

$$f^{-1} = \begin{pmatrix} x_{i_1} & x_{i_2} & \dots & x_{i_n} \\ h_1^{-1} x_1 & h_2^{-1} x_2 & \dots & h_n^{-1} x_n \end{pmatrix}$$

The set of monomial substitutions of the form

$$\mu = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 x_1 & h_2 x_2 & \dots & h_n x_n \end{pmatrix} = [h_1, h_2, \dots, h_n], \quad h_i \in H,$$

is called the *base group*, denoted by $B(n, H)$ and it is a normal subgroup of $\Sigma_n(H)$, which is isomorphic to the direct product $H \times \ldots \times H$ of n copies of H. Elements in the base group are called *multiplications*.

The complete monomial group $\Sigma_n(H)$ is known to split as a semidirect product over the base group by a subgroup isomorphic to S_n (see [6]). Hence, every monomial substitution can be written uniquely as a product of a multiplication and a permutation. For example, the above element f can be written as

$$[h_1, \dots, h_n] \begin{pmatrix} x_1 & \dots & x_n \\ x_{i_1} & \dots & x_{i_n} \end{pmatrix}.$$

Moreover one may observe that, when we take the conjugate of a multiplication $[h_1, \dots, h_n]$ by a permutation $\sigma \in \Sigma_n(\{e_H\})$, the coordinates of the multiplication are permuted according to σ^{-1}. Indeed

$$[h_1, \dots, h_n]^\sigma = [h_{\sigma^{-1}(1)}, \dots, h_{\sigma^{-1}(n)}],$$

hence we have

$$\Sigma_n(H) \cong B(n, H) \rtimes S_n \cong (H \times \dots H) \rtimes S_n \cong H \wr S_n$$

where the symbol $\wr$ denotes the so-called permutational wreath product.

Multiplications with all factors equal, such as

$$[h, h, \dots, h]$$

are called *scalars* and we will simply denote them by $[h]$.

Any element of H, when thought of as a diagonal element of $B(n, H)$, induces a scalar of $\Sigma_n(H)$. It has been shown by Ore in [6] that the center of $\Sigma_n(H)$ is the subgroup of scalars induced by central elements of H.

Moreover, if the group H is abelian, the set of all multiplications $[h_1, h_2, \ldots, h_n]$ in $B(n, H)$ such that the product $h_1 h_2 \ldots h_n = 1$ is a normal subgroup of $B(n, H)$, denoted by $B_0(n, H)$ and its properties in $B(n, H)$ have been studied in [1].

Any monomial substitution f that is the product of a multiplication and a cycle, where all non-trivial factors of f correspond to variables moved by the cycle, is called a *monomial cycle*, or simply *cycle* where ambiguity is not possible, and it is usually denoted omitting the variables it fixes, i.e.

$$f = \begin{pmatrix} x_{i_1} & x_{i_2} & \ldots & x_{i_m} \\ h_{i_1} x_{i_2} & h_{i_2} x_{i_3} & \ldots & h_{i_m} h_{i_1} \end{pmatrix}.$$

As stated in [6], in a way similar to symmetric groups, any monomial substitution can be written as a product of disjoint cycles.

For any monomial substitution f that is a cycle of length m, its power f^m is a multiplication with factors

$$\Delta_1 = h_{i_1} h_{i_2} \ldots h_{i_m},\ \Delta_2 = h_{i_2} h_{i_3} \ldots h_{i_m} h_{i_1},\ \ldots,\ \Delta_m = h_{i_m} h_{i_1} \ldots h_{i_{m-1}}.$$

Notice $\Delta_j^{h_{i_j}} = \Delta_{j+1}$ for any $j < m$ so that all of the Δ_j's are conjugate and their conjugacy class is called the *determinant class* of the cycle f.

It was proven by Ore in [6] that two monomial cycles are conjugate if and only if they have the same length the same determinant class and a similar result extends to all monomial substitutions by looking at their unique decomposition in cycles. It should also be noted that two monomial cycles are conjugate by a multiplications if and only if they move the same variables in the same way and have same determinant classes, and a similar result extends to all monomial substitutions as above.

This means that for any monomial cycle f there is a conjugate of f by a multiplication such that all factors are trivial except for one, lying in the determinant class. This conjugate of f is called a *normal form* of f.

Let $\xi = (n_1, n_2, \ldots)$ be a sequence of natural numbers. ξ is called a *divisible sequence* if $n_i | n_{i+1}$ for each $i \in \mathbb{N}$. Let $\mathcal{D}$ be the set of all divisible sequences. Define a relation on $\mathcal{D}$; that is, for two divisible sequences $\xi_1 = (n_1, n_2, \ldots)$ and $\xi_2 = (m_1, m_2, \ldots)$, $\xi_1 \sim \xi_2$ if and only if for each $n_i \in \xi_1$, $n_i | m_j$ for some $m_j \in \xi_2$ and for each $m_t \in \xi_2$, there exists $n_r \in \xi_1$ such that $m_t | n_r$. One immediately observes that $\sim$ is an equivalence relation on $\mathcal{D}$, partitioning it into equivalence classes.

For a divisible sequence $\xi = (n_1, n_2, \ldots)$ for each $i \in \mathbb{N}$ define $n_1 = r_1$ and for $i \geq 1$, $\frac{n_{i+1}}{n_i} = r_{i+1}$. The sequence $(r_1, r_2, \ldots, r_i, \ldots)$ is called a *factor sequence* of ξ.

We may refine the divisible sequence ξ so that the factors $\frac{n_{i+1}}{n_i}$ are prime numbers without changing equivalence class. Hence we will use, as a representative of each equivalence class, a sequence whose factors are prime numbers.

Let $\mathbb{P}$ be the set of all prime numbers and let $r_p, k_p \in \mathbb{N} \cup \{0, \infty\}$. A *Steinitz number* (or supernatural number) is any infinite formal product of the form

$$\prod_{p \in \mathbb{P}} p^{r_p}.$$

The set of Steinitz numbers, denoted by $\mathbb{SN}$, is a semigroup with identity when endowed with the product

$$\left(\prod_{p \in \mathbb{P}} p^{r_p}\right) \cdot \left(\prod_{p \in \mathbb{P}} p^{k_p}\right) = \left(\prod_{p \in \mathbb{P}} p^{r_p + k_p}\right)$$

with the understanding that $t + \infty = \infty + \infty = \infty$ for any $t \in \mathbb{N}$. The set $\mathbb{SN}$ contains an isomorphic copy of the semigroup $\mathbb{N}$ as the sub-semigroup of Steinitz numbers such only finitely many exponents $\{r_p\}_{p \in \mathbb{P}}$ are non-zero and $r_p \neq \infty$ for all $p \in \mathbb{P}$.

Stenitz numbers in $\mathbb{SN} \backslash \mathbb{N}$ are called *infinite Steinitz numbers.* The divisibility relation $|$ in $\mathbb{SN}$, defined by

$$u | w \iff \exists v \text{ such that } w = uv$$

makes $\mathbb{SN}$ into a complete lattice.

We may associate a Steinitz number to each divisible sequence ξ, namely

$$Char(\xi) = \lambda := \prod_{p_i \in \mathcal{P}} p_i^{l_i}$$

where l_i is the number of times that the prime p_i appears in the factor sequence of ξ. This number is called the *characteristic* of ξ.

Observe that $\xi_1 \sim \xi_2$ if and only if $Char(\xi_1) = Char(\xi_2)$.

Let $\prod$ be the set of sequences consisting of prime numbers. Let $\xi \in \prod$ and $\xi = (p_1, p_2, \ldots)$ be a sequence consisting of not necessarily distinct primes p_i. From the given sequence ξ, we may obtain a divisible sequence $(n_1, n_2, \ldots n_i, \ldots)$ where $n_1 = p_1$ and $n_{i+1} = p_{i+1} n_i$, then we have $n_i | n_{i+1}$ for all $i \in \mathbb{N}$.

We define an embedding of a complete monomial group $\Sigma_{n_i}(H)$ diagonally into $\Sigma_{n_{i+1}}(H)$

$$d^{p_{i+1}} : \Sigma_{n_i}(H) \to \Sigma_{n_{i+1}}(H)$$

as follows.

For any $f = \begin{pmatrix} x_1 & x_2 & \dots & x_{n_i} \\ h_1 x_{j_1} & h_2 x_{j_2} & \dots & h_{n_i} x_{j_{n_i}} \end{pmatrix} \in \Sigma_{n_i}(H)$ we define

$$d^{p_{i+1}}(f) \in \Sigma_{n_{i+1}}(H)$$

as the monomial substitution mapping

$$x_{qn_i+r} \mapsto h_r x_{qn_i+j_r}, \quad \forall q \in \{0, \dots, p_{i+1}-1\} \text{ and } \forall r \in \{1, \dots, n_i\}.$$

One can think of $d^{p_{i+1}}(f)$ as a monomial substitution on n_{i+1} variables acting exactly as f on the first set of n_i variables, the second set of n_i variables and so on. This embedding corresponds to a strictly diagonal embedding of $\Sigma_{n_i}(H)$ into $\Sigma_{n_{i+1}}(H)$ as seen in [5].

According to the given sequence of primes, we continue to embed

$$d^{p_{i+2}} : \Sigma_{n_{i+1}}(H) \to \Sigma_{n_{i+2}}(H).$$

Then we have the following direct system:

$$\{1\} \stackrel{d^{p_1}}{\to} \Sigma_{n_1}(H) \stackrel{d^{p_2}}{\to} \Sigma_{n_2}(H) \stackrel{d^{p_3}}{\to} \Sigma_{n_3}(H) \stackrel{d^{p_4}}{\to} \dots$$

The direct limit group obtained from the above construction is called *limit monomial group* over the group H associated to the prime sequence ξ and denoted by $\Sigma_\xi(H)$. In [5] it is proven that this construction only depends on the Steinitz number $\lambda = Char(\xi)$, so that we will make use of the notation $\Sigma_\lambda(H)$, and that

$$\Sigma_\lambda(H) \cong \bigcup_{i=1}^{\infty} \Sigma_{n_i}(H) \cong S(\lambda) \ltimes B(\lambda, H)$$

where $B(\lambda, H)$ is called *base group*.

2 Subgroups that are normalized by the base group

We start this section by providing a large family of examples of subgroups F of the complete monomial group $\Sigma_n(H)$ of finite degree n over H that are normalized by the base group, i.e. such that $N_{\Sigma_n(H)}(F) \geq B(n, H)$.

Throughout all of this paper we are going to use the notation

$$\pi^* : \Sigma_n(H) \to \frac{\Sigma_n(H)}{B(n, H)} \cong S_n$$

to represent the canonical epimorphism of $\Sigma_n(H)$ onto S_n and

$$p : [k_1, \dots, k_n]\sigma \in \Sigma_n(H) \mapsto k_1 \cdot \ldots \cdot k_n \in H$$

to represent the function mapping each monomial substitution to the ordered product of its factors. In general, p is not a homomorphism, but it is in the case of an abelian group H.

Lemma 1. *Let $A \leq S_n$ and choose $H' \leq K \leq H$ and $\varphi : A \to H/K$ to be any homomorphism from A to H/K.*

Then

$$F_{A,K,\varphi} = \Big\{[k_1, \dots, k_n]\sigma \mid \sigma \in A,\ k_1 k_2 \dots k_n \in \varphi(\sigma)\Big\}$$

is a subgroup of $\Sigma_n(H)$ normalized by the base group.

Proof. First we prove that $F_{A,K,\varphi}$ is a subgroup. If $f_1 = [h_1, \dots, h_n]\pi$ and $f_2 = [k_1, \dots, k_n]\sigma$ belong to $F_{A,K,\varphi}$ then $\pi = \pi^*(f_1)$ and $\sigma = \pi^*(f_2)$ belong to A and $p(f_1)$ and $p(f_2)$ belong to $\varphi(\pi)$ and $\varphi(\sigma)$ respectively, which are cosets of K.

Of course

$$\pi^*(f_1 f_2^{-1}) = \pi^*(f_1)[\pi^*(f_2)]^{-1} = \pi\sigma^{-1} \in A$$

because A is a subgroup and

$$p(f_1 f_2^{-1})K = p(f_1)[p(f_2)]^{-1}K$$

as $K \geq H'$.

This means that

$$p(f_1 f_2^{-1}) \in \varphi(\pi)\varphi(\sigma^{-1}) = \varphi(\pi\sigma^{-1})$$

hence $f_1 f_2^{-1}$ lies in $F_{A,K,\varphi}$.

Now we will prove that $F_{A,K,\varphi}$ is normalized by elements of the base group. For any $f \in F_{A,K,\varphi}$ and $\omega = [\omega_1, \dots, \omega_n] \in B(n, H)$, we have

$$\pi^*(f^\omega) = [\pi^*(f)]^{\pi^*(\omega)} = \pi^*(f) \in A,$$

and again using $K \geq H'$ we get

$$p(f^\omega)K = p(f)K$$

so that $p(f^\omega) \in \varphi(\pi^*(f^\omega))$ proving that f^ω lies in $F_{A,K,\varphi}$ and hence

$$N_{\Sigma_n(H)}(F_{A,K,\varphi}) \geq B(n, H).$$

$\boxed{QED}$

Corollary 1. *For any* $1 \leq i \leq r$ *let* $n_i \in \mathbb{N}$ *be positive integers such that*

$$\sum_{i=1}^{r} n_i = n$$

and each $1 \leq i \leq r$ *choose*

- *any subgroup* $A_i \leq S_{n_i}$,
- *any subgroup* $H' \leq K_i \leq H$ *and*
- *any homomorphism* $\varphi_i : A_i \to H/K_i$.

Then $\operatorname*{Dr}_{i=1}^{r} F_{A_i,k_i,\varphi_i}$ *can be naturally embedded into* $\Sigma_n(H)$ *as a subgroup normalized by* $B(n,H)$.

Remark 1. Excluding the special case in which $H = H'$, by restricting the attention to soluble or abelian groups, for example, the above construction gives a large family of examples of subgroups of the complete monomial group of degree n over H that are normalized by the base group. What we are going to show next is that in the case of an abelian group H, the examples given above are precisely all of the subgroups normalized by the base group.

Lemma 2. *Let* H *be an abelian group and* $F \leq \Sigma_n(H)$ *be any subgroup of the complete monomial group of degree* n *over* H *that is normalized by the base group. Let* $A := \pi^*(F) \leq S_n$ *and* $K := p(F \cap B(n,H)) \leq H$. *Then*

$$\varphi : \sigma \in A \mapsto p(F \cap {\pi^*}^{-1}(\sigma)) \in H/K$$

is a homomorphism from A *to* H/K *and* $F \leq F_{A,K,\varphi}$.

Proof. Notice that K is a subgroup because it is the image of the subgroup $F \cap B(n,H)$ under the homomorphism p. Since $F \cap {\pi^*}^{-1}(\sigma)$ are cosets of $\ker(\pi^*_{|F})$ and $K = p(\ker \pi^*_{|F})$, we have that

$$\varphi(\sigma) = p(F \cap {\pi^*}^{-1}(\sigma)) \in H/K.$$

This means that φ is a well defined homomorphism from A to H/K. Now, for any $f \in F$ we have that

- $\pi^*(f) \in \pi^*(F) = A$
- $p(f) \in p(F \cap {\pi^*}^{-1}(\pi^*(f))) = \varphi(\pi^*(f))$

so that $F \leq F_{A,K,\varphi}$. $\boxed{QED}$

The following technical lemma, needed to get a better understanding of the abelian case, is going to require some definitions regarding partitions of a set (see also [2]).

We say a partition $\mathcal{F}_1$ is a *refinement* of $\mathcal{F}_2$ if every element Ω of $\mathcal{F}_2$ can be written as union of elements of $\mathcal{F}_1$. We can also say $\mathcal{F}_1$ is *finer than* $\mathcal{F}_2$.

For any two partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ it is easily proven that there exists a partition $\mathcal{F}$ of which they are both refinements and $\mathcal{F}$ is the finest possible partition with this property. We will denote this by $\mathcal{F}_1 \wedge \mathcal{F}_2$.

Moreover, for every partition $\mathcal{F}$ on $\{1, \ldots, n\}$ we define

$$B_{\mathcal{F}}(n, H) = \left\{ [k_1, \ldots, k_n] \mid \forall \Omega \in \mathcal{F}, \prod_{i \in \Omega} k_i = 1 \right\}.$$

Lemma 3. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two partitions of $\{1, \ldots, n\}$. Then*

$$\langle B_{\mathcal{F}_1}(n, H), B_{\mathcal{F}_2}(n, H) \rangle \geq B_{\mathcal{F}_1 \wedge \mathcal{F}_2}(n, H).$$

Proof. Choose any element $[a_1, \ldots, a_n] \in B_{\mathcal{F}_1 \wedge \mathcal{F}_2}$.

To be able to express it as a product of elements $[h_1, \ldots, h_n] \in B_{\mathcal{F}_1}(n, H)$ and $[k_1, \ldots, k_n] \in B_{\mathcal{F}_2}(n, H)$ is equivalent to solving the following system of equation

$$\begin{cases} h_i k_i = a_i & \forall\, i \leq n \\ \prod_{i \in \Omega} h_i = 1 & \forall\, \Omega \in \mathcal{F}_1 \\ \prod_{i \in \Gamma} k_i = 1 & \forall\, \Gamma \in \mathcal{F}_2. \end{cases}$$

which leads to the system

$$\begin{cases} \prod_{i \in \Omega} h_i = 1 & \forall\, \Omega \in \mathcal{F}_1 \\ \prod_{i \in \Gamma} h_i = \prod_{i \in \Gamma} a_i & \forall\, \Gamma \in \mathcal{F}_2. \end{cases} \tag{2.1}$$

This system of equations can be split into completely independent subsystems, one for each block in the partition $\mathcal{F}_1 \wedge \mathcal{F}_2$. For this reason, without loss of generality we can assume that

$$\mathcal{F}_1 \wedge \mathcal{F}_2 = \{\{1, \ldots, n\}\} \quad \text{and} \quad \prod_{i=1}^{n} a_i = 1.$$

This means (2.1) is a system of $s+t$ linear equations in n variables in an abelian group, where $s = |\mathcal{F}_1|$ and $t = |\mathcal{F}_2|$. The matter of consistency of this system can be approached by looking at the augmented matrix $(A|b)$ associated to the system (of course one should think of the system in additive notation to do so). A is going to have only 1's and 0's as entries and every column is going to contain exactly two non-zero entries. In particular, A can be divided into an upper block of the first s equations and a lower block of the last t such that in each block there is only 1 non-zero entry for each column. These blocks correspond to the two different type of equations in system (2.1).

To show that the system is consistent we are going to show that for any linear combination of the rows of A giving the null vector, the corresponding linear combination of elements of the column b is 0.

Let $v_1, \dots, v_s$ be the first s rows of A and $w_1, \dots, w_t$ be the other t rows and consider a vanishing linear combination of those, i.e.

$$\sum_{j=1}^{s} \alpha_j v_j + \sum_{j=1}^{t} \beta_j w_j = 0.$$

This means

$$v := \sum_{j=1}^{s} \alpha_j v_j = -\sum_{j=1}^{t} \beta_j w_j =: w.$$

However, since v is a linear combination of the rows of the upper block, two of its components are going to be equal whenever they correspond to indices in the same element of $\mathcal{F}_1$ and something similar can be said for w and $\mathcal{F}_2$. In symbols:

$$\forall \Omega \in \mathcal{F}_1,\ \forall i, j \in \Omega, \quad v^i = v^j \tag{2.2}$$

and

$$\forall \Gamma \in \mathcal{F}_2,\ \forall i, j \in \Gamma, \quad v^i = v^j \tag{2.3}$$

where v^j denotes the j-th component of the vector v.

We claim, then, v is necessarily a multiple of the vector $(1, 1, \dots, 1)$. As a matter of fact, by contradiction, assume the set:

$$\Psi = \{ j \leq n \mid v^j = v^1 \}$$

is a proper subset of $\{1, \dots, n\}$. Then by (2.2) and (2.3), any $\Omega \in \mathcal{F}_1$ and $\Gamma \in \mathcal{F}_2$ is either contained in Ψ or its complement. This is a contradiction because it shows that $\mathcal{F}_1$ and $\mathcal{F}_1$ are refinements of $\big\{\Psi, \{1, \dots, n\} \setminus \Psi\big\}$ which contradicts the assumption that $\mathcal{F}_1 \wedge \mathcal{F}_2 = \{\{1, \dots, n\}\}$. For this reason, $v = -w = \alpha(1, \dots, 1)$.

This means the only way of combining rows of A to make them vanish is to take $\alpha_j = \alpha = -\beta_j$ for some α and for all j.

Combining the elements of the column b with the same scalars gives

$$\alpha\left(\sum_{i}^{n} a_i\right) = 0.$$

Hence the system is consistent and the lemma is proved. $\boxed{QED}$

Lemma 4. *Let H be an abelian group and $F \leq \Sigma_n(H)$ be a subgroup of the monomial group of degree n over H that is normalized by the base group and such that $\pi^*(F)$ has transitive action on $\{1,\dots,n\}$.*

Then $F \cap B(n,H)$ contains the subgroup $B_0(n,H)$ consisting of all multiplications the product of whose factors is 1.

Proof. If $n = 1$ the result is immediately seen to be true, so assume $n > 1$.

Choose any non-identical permutation $\pi \in \pi^*(F)$ and $f = [h_1,\dots,h_n]\pi \in F$ such that $\pi^*(f) = \pi$ and take a multiplication $\omega = [\omega_1,\dots,\omega_n] \in B(n,H)$. Since F is a subgroup normalized by the base group, then also $f^\omega f^{-1}$ should be an element of F. In particular, since $B(n,H)$ is normal in $\Sigma_n(H)$, it should be an element of $F \cap B(n,H)$, in other words a multiplication inside F.

Computing the i-th of $f^\omega f^{-1}$ one get

$$\omega_i^{-1} h_i \omega_{\pi(i)} h_i^{-1} = \omega_i^{-1}\omega_{\pi(i)}.$$

Let $\{a_i\}_{i\leq n}$ with $a_i \in H$ be a sequence of elements of H. We will now understand a necessary and sufficient condition to be able to solve the system of equations $a_i = \omega_i^{-1}\omega_{\pi(i)}$, i.e. $\omega_{\pi(i)} = \omega_i a_i$.

Now, π will have a unique decomposition as product of disjoint cycles, inducing a partition $\mathcal{F}_\pi$ of $\{1,\dots,n\}$.

Let $\Omega \in \mathcal{F}_\pi$ and consider equations $\omega_{\pi(i)} = \omega_i a_i$ for all $i \in \Omega$. By trying to solve them by substitution, it is easily seen that they have at least one solution if and only if $\prod_{i\in\Omega} a_i = 1$.

So, what we have proven is that

$$F \cap B(n,H) \geq B_{\mathcal{F}_\pi}(n,H), \quad \forall \pi \in \pi^*(F).$$

Now, using Lemma 3 and transitivity of $\pi^*(F)$ we conclude that

$$\langle B_{\mathcal{F}_\pi}(n,H) | \pi \in \pi^*(F)\rangle = B_0(n,H)$$

hence the thesis of the theorem. $\boxed{QED}$

Theorem 1. *Let H be an abelian group and $F \leq \Sigma_n(H)$ be a subgroup of the monomial group of degree n over H that is normalized by the base group and such that $\pi^*(F)$ has transitive action on $\{1, \ldots, n\}$.*

Let $A := \pi^*(F) \leq S_n$, $K := p(F \cap B(n, H)) \leq H$ and

$$\varphi : \sigma \in A \mapsto p(F \cap {\pi^*}^{-1}(\sigma)) \in H/K.$$

Then $F = F_{A,K,\varphi}$.

Proof. Lemma 2 shows that $F \leq F_{A,K,\varphi}$. What is left to show is that for any choice of $\pi \in A$, F contains the set of all monomial substitutions f such that $\pi^*(f) = \pi$ and $p(f) \in \varphi(\pi)$.

For any $k \in K$, there exists $f \in F \cap B(n, H)$ such that $p(f) = k$. By Lemma 4, we know F contains $B_0(n, H)$, hence it also contains the coset $fB_0(n, H)$, which is the set of all multiplications g such that $p(g) = k$. This means that:

$$F \cap B(n, H) = \{[h_1, \ldots, h_n] \in B(n, H) \mid h_1 \ldots h_n \in K\}.$$

For any $\pi \in A$, there exists $f \in F$ such that $\pi^*(f) = \pi$ and $p(f) \in \varphi(\pi)$. Since it also contains the coset $f[F \cap B(n, H)]$, which is the set of all monomial substitutions g such that $\pi^*(g) = \pi^*(f) = \pi$ and $p(g)K = p(f)K$, the proof is concluded. $\boxed{QED}$

3 Centralizers of subgroups that are normalized by the base group

In this section we will describe the structure of centralizers of subgroups F normalized by the base group of the monomial group $\Sigma_n(H)$ of degree n over an arbitrary group H. In the next section, this will then be used to understand the structure of centralizers of subgroups in limit monomial groups.

The structure of the centralizer of an element in a monomial group of finite degree was studied by Ore in [6]. This was then used to understand the structure of the centralizer of a monomial substitution in a limit monomial group of type $\Sigma_\lambda(H)$ by Kuzucuoğlu, Oliynyk and Sushchanskyy in [5].

However, the structure of centralizers of subgroups has only been studied in the context of symmetric groups and not in monomial groups both in the finite degree case and in the limit case.

We are going to start studying the centralizers of subgroups of monomial degrees starting with the additional assumption of transitive action of $\pi^*(F)$. Before that, we give an example to show that it is necessary to deal with the elementary abelian 2-group case separately.

Remark 2. Assume H is an elementary abelian 2-group, $n = 2$ and choose $A = S_2$, $K = \{1\}$ and φ to be the constant homomorphism from A to H/K.

Take then F to be $F_{A,K,\varphi} = \{[k_1, k_2]\sigma \mid \sigma \in S_2,\ k_1 = k_2\}$. F is abelian and it is an extension of degree two of H which means that for any choice of H having finite 2-rank, $C_{\Sigma_2(H)}(F)$ cannot be be embedded into H.

The following lemma, though, proves in particular that if H is not an elementary abelian 2-group and $\pi^*(F)$ is transitive, with any choice of a natural number n, $C_{\Sigma_n(H)}(F)$ can, in fact, be embedded into H, proving that the example given above is a needed exception in the statement of the following lemma.

Lemma 5. *Let H be a non-trivial group and n be any natural number. Let $F \leq \Sigma_n(H)$ be a subgroup of the complete monomial group $\Sigma_n(H)$ such that $N_{\Sigma_n(H)}(F) \geq B(n, H)$.*

If $\pi^\star(F)$ acts transitively on $\{1, \ldots, n\}$ then:

- *If $n = 1$, then*
$$C_{\Sigma_n(H)}(F) \cong C_H(F).$$
- *If $n \geq 2$ and H is not an elementary abelian 2-group then*
$$C_{\Sigma_n(H)}(F) \cong Z(H).$$

Proof. Take any $\gamma = \eta\sigma = [k_1, \ldots, k_n]\sigma \in C_{\Sigma_n(F)}(H)$.

Since $\pi^\star$ is a homomorphism, $\pi^\star(f^\gamma) = \pi^\star(f)^{\pi^\star(\gamma)}$, hence for γ to be in $C_{\Sigma_n(H)}(F)$ it has to satisfy $\sigma = \pi^\star(\gamma) \in C_{S_n}(\pi^\star(F))$.

Now, we know that f^γ must be equal to f for any element $f = \mu\pi = [h_1, \ldots, h_n]\pi$ of F. This means that

$$(\eta\mu\eta^{\pi^{-1}})^\sigma \pi = \mu\pi$$

since σ and π commute. From that we get the following necessary condition on factors:

$$k_i^{-1} h_i k_{\pi(i)} = h_{\sigma(i)}. \tag{$\star$}$$

We also remark that for any $f \in F$, the normal form of f is also in F as it can be obtained by conjugating f with a multiplication.

Case 1: $n = 1$ As $\Sigma_1(H)$ is trivially isomorphic to H, F can be embedded into H and the thesis is trivial.

Case 2: $n = 2$ and H is not an elementary abelian 2-group Consider any $\gamma = \eta\sigma = [k_1, k_2]\sigma \in C_{\Sigma_2(H)}(F)$. Now, we will prove by contradiction that σ is the identity.

If $\sigma = (12)$, take any $f \in F$ such that $\pi = \pi^\star(f) = (12)$. By substituting f with one of its conjugates we can assume f is in normal form, say:

$$f = \begin{pmatrix} x_1 & x_2 \\ x_2 & hx_1 \end{pmatrix}.$$

Then, by imposing equality of factors in the equation $f^\gamma = f$, by $(\star)$ we get:

$$\begin{cases} k_1^{-1}k_2 = h \\ k_2^{-1}hk_1 = 1 \end{cases} \tag{$\star\star$}$$

As γ is in $C_{\Sigma_2(H)}(F)$ it also has to commute with $f^{[\omega]}$ for any $\omega \in H$, and from that and $(\star)$ we get:

$$\begin{cases} k_1^{-1}k_2 = h^\omega \\ k_2^{-1}h^\omega k_1 = 1. \end{cases}$$

This, of course, only has solution compatible with $(\star\star)$ if $h^\omega = h$ for any $\omega \in H$. Since ω must be arbitrary then h must be central.

Again, as γ is in $C_{\Sigma_2(H)}(F)$ it also has to commute with $f^{[1,a]}$ for any $a \in H$ and from that, with $(\star)$, we get:

$$\begin{cases} k_1^{-1}ak_2 = a^{-1}h \\ k_2^{-1}a^{-1}hk_1 = a \end{cases}$$

Substituting $k_2 = k_1h$ (which comes from $(\star\star)$) in the first one, we get $a^{k_1} = a^{-1}$ for all $a \in H$. This can only happen in an elementary abelian 2-group, so it contradict our hypotheses.

Now we know σ has to be the identity. Then $(\star)$ will give:

$$\begin{cases} k_1^{-1}k_2 = 1 \\ k_2^{-1}hk_1 = h. \end{cases}$$

This implies that γ is a scalar. Imposing also that γ commutes with $f^{[1,a]}$ proves that the scalar has to commute with all elements a of H and so it is scalar induced by an element in $Z(H)$.

Viceversa, of course, any scalar induced by an element in $Z(H)$ is in $Z(\Sigma_n(H))$.

Case 3: $n \geq 3$ and H is not an elementary abelian 2-group. Let $\gamma = \eta\sigma = [k_1, \ldots, k_n]\sigma \in C_{\Sigma_n(H)}(F)$. We are going to prove by contradiction that σ has to be the identity map.

Assume that σ moves at least an element, say $\sigma(i) = j \neq i$. Since $n \geq 3$, we choose $l \neq i, j$ and since $\pi^\star(F)$ is transitive, there exists $f \in F$ with $\pi = \pi^\star(f)$ mapping i to l.

$f^{\gamma^{-1}} = f$ gives a condition on the i-th factor, namely:

$$k_i h_i k_l^{-1} = h_j. \tag{3.1}$$

Now, π moves i to l and it must also move j, because

$$\pi(j) = \pi(\sigma(i)) = \sigma(\pi(i)) = \sigma(l) \neq j$$

because σ is one-to-one.

This means that by considering a normal form of f we can assume $h_i = h_j = 1$ because i and j are moved by π and are not moved one to the other.

This means $k_i = k_l$ for any $l \neq i, \sigma(i)$, but this can be done for any i proving that η is a scalar. Now that we know η is a scalar, by using transitivity of the action of $\pi^*(F)$ there exists $f \in F$ such that $\pi = \pi^\star(f)$ moves i to j and without loss of generality we assume f in normal form.

In this case one can easily see that the cycle involving i and j is the same in the cycle decomposition of π and σ. Then for any index r in the cycle involving i and j we have:

$$k^{-1} h_{\sigma(r)} k = h_r.$$

This would mean that all factors pertaining to the indices in the cycle involving i and j in f must be conjugate. This is only possible if it has determinant class equal to $\{1\}$.

If this cycle is a 2-cycle then by conjugating it with multiplications we can make it of the form

$$f_a = \begin{pmatrix} \dots & x_i & \dots & x_j & \dots \\ \dots & ax_j & \dots & a^{-1}x_i & \dots \end{pmatrix}$$

for any $a \in H$, and for $\gamma = [k]\sigma$ to commute with all of those, k would have to be an element of H which acts by conjugation on all elements of the group H by inverting them. This is impossible since H is not an elementary abelian 2-group.

This implies the cycle of σ involving i and j is an m-cycle for $m > 2$.

This means that there is a conjugate of f in F with $h_i = 1$ and $h_j = h$ for any choice of $h \in H$. Since $\gamma = [k]\sigma$ commutes with it, $h^k = 1$ for any choice of $h \in H$, which contradicts the hypothesis that H is non-trivial and so σ is the identity.

Following similar steps to the case $n = 2$, we get that η has to be a scalar in the center.

Again, any scalar induced by an element of $Z(H)$ centralizes F as it is an element of $Z(\Sigma_n(H))$, which concludes the proof. $\boxed{QED}$

To approach the non transitive case, we first have to give a definition of equivalent orbits for a subgroup of a monomial group. If the action of $\pi^*(F)$ on $\{1,\dots,n\}$ is not transitive then it induces a partition of it into orbits.

Consider any two orbits Ω and Γ. For any monomial substitution $f \in F$, the restriction of f to $\Omega \cup \Gamma$ is well defined.

We say Ω and Γ are *equivalent* if there exists a bijection $\tau : \Omega \to \Gamma$ inducing a permutation

$$\tilde{\tau} : i \mapsto \begin{cases} \tau(i) & i \in \Omega \\ \tau^{-1}(i) & i \in \Gamma \end{cases}$$

on $\Omega \cup \Gamma$ and a monomial substitution ρ with $\pi^*(\rho) = \tilde{\tau}$ such that

$$f\rho = \rho f, \quad \forall f \in F.$$

Example 1. Choose a group H, elements $h_1, h_2, x \in H$ and define:

$$F = \langle [1, h_1, 1, 1, h_1^x, 1](12)(45), [1, 1, h_2, 1, 1, h_2^x](23)(56) \rangle.$$

$\pi^*(F)$ has $\{1,2,3\}$ and $\{4,5,6\}$ as orbits and, if we take

$$\tau : i \in \{1,2,3\} \to i+3 \in \{4,5,6\},$$

then $\bar{\tau} = (14)(25)(36)$ and the monomial substitution

$$\rho = [x, x, x, x^{-1}, x^{-1}, x^{-1}](14)(25)(36)$$

commutes with the generators of F, so the orbits $\{1,2,3\}$ and $\{4,5,6\}$ are equivalent.

Theorem 2. *Let H be a non-trivial group and n be any natural number. Let $F \leq \Sigma_n(H)$ be a subgroup of the complete monomial group $\Sigma_n(H)$ that is normalized by the base group.*

Let $\{\Gamma_i\}_{i \leq r}$ be the set of equivalence classes of orbits of $\pi^*(F)$ and define s_i to be the order of Γ_i and t_i to be the order of any orbit in Γ_i.

Then

$$C_{\Sigma_n(H)}(F) \cong \operatorname*{Dr}_{i=1}^{r} \mathfrak{C}_i \wr S_{s_i}$$

where

$$\mathfrak{C}_i \cong C_H(F_{|\Omega_i})$$

if $t_i = 1$ and Ω_i is any representative of Γ_i and

$$\mathfrak{C}_i \cong Z(H)$$

if $t_i > 1$.

Proof. The result easily follows from lemma 5 together with the definition of equivalent orbit. We only need to show that the choice of a representative in equivalence classes of orbits of size 1 is irrelevant. Assume, for simplicity of notation, $\Omega_1 = \{x_1\}$ and $\Omega_2 = \{x_2\}$ with Ω_1 and Ω_2 being equivalent. This means there is a monomial substitution $\rho = [k_1, k_2](12)$ in $\Sigma_2(H)$ commuting with any element of F restricted to $\Omega_1 \cup \Omega_2$. Now, of course for any element $h_1 \in F_{|\Omega_1}$ (which we think of as embedded into H), by definition, we will have some $f \in F$ such that $f_{|\Omega_1 \cup \Omega_2} = [h_1, h_2]$, where $h_2 \in F_{|\Omega_2}$. Doing the calculations to impose $f_{|\Omega_1 \cup \Omega_2}\rho = \rho f_{|\Omega_1 \cup \Omega_2}$ shows $F_{|\Omega_1} \leq F_{|\Omega_2}^{k_2}$ and $k_1 k_2 \in C_H(F_{|\Omega_1})$. Similarly one shows $F_{|\Omega_2} \leq F_{|\Omega_1}^{k_1}$, which allows to conclude $F_{|\Omega_2}^{k_2} \leq F_{|\Omega_1}^{k_1 k_2} = F_{|\Omega_1}$.

This means $F_{|\Omega_1}$ and $F_{|\Omega_2}$ are conjugate in H and hence their centralizer in H is isomorphic and the choice of a representative in their equivalence class is not important. $\boxed{QED}$

4 Centralizer of subgroups in limit monomial groups

In this section, we give a description of the structure of the centralizer of a subgroup F of $\Sigma_\lambda(H)$ that is normalized by the base group and such that $\pi^*(F)$ is finite.

As the group $\Sigma_\lambda(H)$ can be seen as the union of an infinite increasing sequence of monomial groups of degree n_j, each isomorphic to some $\Sigma_{n_j}(H)$, our subgroup F can be thought of as subgroup of $\Sigma_{n_k}(H)$ for some k and $\pi^*(F)$ as a subgroup of S_{n_k}. When thinking of F as a subgroup of $\Sigma_{n_k}(H)$, it is also trivial that it is normalized by the group.

In everything that follows, for any $i \leq r$, Γ_i is going to denote an equivalence class of orbits of $\pi^*(F)$ when thought of as a subgroup of S_{n_k} for some $k \in \mathbb{N}$.

Theorem 3. *Let λ be a Steinitz number, H an arbitrary group and $\Sigma_\lambda(H)$ be the limit monomial group over H associated to λ. Let F be a subgroup of $\Sigma_\lambda(H)$ normalized by the base group and such that $\pi^*(F)$ is finite.*

Let $\{\Gamma_i\}_{i \leq r}$ be the equivalence classes of orbits associated to F and let s_i denote the order of Γ_i and t_i denote the order of any orbit in Γ_i. Then

$$C_{\Sigma_\lambda(H)}(F) \cong \operatorname*{Dr}_{i=1}^{r} \Sigma_{\lambda_i}(\mathfrak{C}_i)$$

where $\lambda_i = s_i \frac{\lambda}{n_k}$ and $\mathfrak{C}_i$ is as in Theorem 2.

Proof. Using Theorem 2 we can find the structure of the centralizer of F as a subgroup of $\Sigma_{n_k}(H)$ as

$$C_{\Sigma_{n_k}(H)}(F) \cong \operatorname*{Dr}_{i=1}^{r} \mathfrak{C}_i \wr S_{s_i} \cong \operatorname*{Dr}_{i=1}^{r} \Sigma_{s_i}(\mathfrak{C}_i).$$

For any $j > k$ we can embed $C_{\Sigma_{n_j}(H)}(F)$ into $C_{\Sigma_{n_{j+1}}(H)}(F)$, inspired by the way it was done in ([4], Lemma 4.4) and ([3]), with strictly diagonal embedding.

To compute the centralizer in the direct limit, we take the limit of the direct system of centralizers of F in $\Sigma_{n_j}(H)$ with $j > k$, concluding

$$C_{\Sigma_\lambda(H)}(F) \cong \operatorname*{Dr}_{i=1}^{r} \Sigma_{\lambda_i}(\mathfrak{C}_i)$$

where $\lambda_i = s_i \frac{\lambda}{n_{j^*}}$ and the $\mathfrak{C}_i$ are as in Theorem 2. $\boxed{QED}$

Acknowledgements. The two authors are very grateful to Prof. Mahmut Kuzucuoglu of the Middle East Technical University of Ankara, Turkey, for posing the right question, for his precious suggestions and help and for the time dedicated to the two authors in discussing the topic.

References

[1] S. Bostan, M. Kuzucuoğlu: *Limit monomial groups over abelian groups.* South East Asian Bulletin of Mathematics, (2020) 44: 769–780.

[2] G. Grätzer: General Lattice Theory. doi: 10.1007/978-3-0348-9326-8, (1996).

[3] Ü. B. Güven, O. H. Kegel, M. Kuzucuoğlu: *Centralizers of Subgroups in Direct Limits of Symmetric Groups with Strictly Diagonal Embedding.* Communications in Algebra, 43(5), 1920–1934. doi: 10.1080/00927872.2013.879657, (2015).

[4] N. Kroshko, V. Sushchansky: *Direct limits of symmetric and alternating groups with strictly diagonal embeddings.* Archiv Der Mathematik, 71(3), 173–182. doi: 10.1007/s000130050249, (1998).

[5] M. Kuzucuoğlu, B. Oliynyk, V.I. Sushchanskyy: *Homogeneous monomial groups and centralizers.* Communications in Algebra, 46(2), 597–609. doi: 10.1080/00927872.2017.1324874, (2017).

[6] O. Ore: *Theory of Monomial Groups.* Transactions of the American Mathematical Society, 51(1), 15–64. doi: 10.2307/1989979, (1942).