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Second cohomology of multiplicative Lie rings and Schreier's extension theory

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Abstract. Our goal of this paper is to develop an analogue of the theory of group extensions for multiplicative Lie rings. We first define a factor system of pair of multiplicative Lie rings (L, A), which use to construct an extension of A by L. Then we state Schreier's theorem for multiplicative Lie rings. We also use this notation to introduce second cohomology group of multiplicative Lie rings. Finally, we show that the equivalence classes of multiplicative Lie ring extensions can be identified with second cohomology group $H^2(L, A, \rho, h)$, where L acts on A by ρ and h.

Keywords: Multiplicative Lie ring; extension of multiplicative Lie ring; second cohomology group of multiplicative Lie ring and factor system

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1 Introduction

Ellis [4] introduced the notion of multiplicative Lie algebra, which is called the multiplicative Lie ring in the body of this paper. This concept contains both notations of groups and Lie rings. Ellis used this structure to investigate an interesting combinatorial problem on group commutators (see also [3]).

Point and Wantiez [8] studied algebraic structural properties of multiplicative Lie algebras. In particular, They defined nilpotency and proved several nilpotency results generalizing known ones for groups and Lie algebras. In [1], further structural properties of multiplicative Lie rings were investigated and two homology theories were introduced and compared with the usual homology theories of groups and Lie rings.

In [2], nonabelion tensor and exterior products of multiplicative Lie rings were introduced. They also proved an analogue of Miller's theorem for multiplicative Lie rings. The theory of group extensions and their interpretation in terms of cohomology is well known.

Fouladi, Jamali, and Orfi [6] developed an analogue of the theory of group extensions for Lie rings. They proved Schreier's extension theorem for Lie rings. They also used the notion of factor triples to define the second cohomology group of a Lie ring. The aim of this paper is to develop an analogue of the theory of group extensions for multiplicative Lie rings. The treatment is parallel to the known theory of the extensions of Lie groups and Lie algebras (see, for example, [5, 7]) and Lie rings (see, [6]).

The paper is organized as follows: In section 2, we recall some necessary notions about multiplicative Lie rings, which are needed in the rest of the paper.

In section 3, we define the notation of factor system of pair of multiplicative Lie ring (L, A) and use it to construct an extension of A by L. Also we prove Schreier's theorem for multiplicative Lie rings.

Finally in section 4, we introduce the concept of second cohomology group of a pair of multiplicative Lie ring (L, A) with L acting on the abelian multiplicative Lie ring A. We also see that the definition given in [6] for factor triples and the second cohomology group arise from our definition when the group of multiplicative Lie ring is abelian.

2 Basic concepts

In this section, we recall the notion of multiplicative Lie ring due to [1, 8, 3, 4].

A *multiplicative Lie ring* consists of a multiplicative (possibly nonabelian) group L together with a binary function $\{,\} : L \times L \to L$, which we call it the *Lie product*, satisfying the following conditions:

$$\{x, x\} = 1, \tag{2.1}$$
$$\{x, yy'\} = \{x, y\}\, {}^{y}\{x, y'\}, \tag{2.2}$$
$$\{xx', y\} = {}^{x}\{x', y\}\, \{x, y\}, \tag{2.3}$$
$$\{\{x, y\}, {}^{y}z\}\, \{\{y, z\}, {}^{z}x\}\, \{\{z, x\}, {}^{x}y\} = 1, \tag{2.4}$$
$${}^{z}\{x, y\} = \{{}^{z}x, {}^{z}y\}, \tag{2.5}$$

for all $x, x', y, y', z \in L$ in which ${}^{x}y = xyx^{-1}$. The following identities are deduced from (2.1)–(2.5) in [1, 4]:

$$\{y, x\} = \{x, y\}^{-1} \tag{2.6}$$
$${}^{\{x,y\}}\{x', y'\} = {}^{[x,y]}\{x', y'\} \tag{2.7}$$

for all $x, x'y, y' \in L$, in which $[x, y] = xyx^{-1}y^{-1}$

Example 2.1.

(1) Any group G is a multiplicative Lie ring under $\{x,y\} = xyx^{-1}y^{-1}$ for all $x, y \in G$, which is denoted by $G_{[,]}$.

(2) Any group G is also a multiplicative Lie ring under $\{x,y\} = 1$ for all $x, y \in G$. It is called the *abelian multiplicative Lie ring* of G and is sometimes denoted by $G_{\bullet}$.

(3) Any ordinary Lie ring L is a multiplicative Lie ring under the Lie product. Moreover, if L is a multiplicative Lie ring whose underlying group is abelian, then L is an ordinary Lie ring.

- A *morphism* $\phi : L \to L'$ of multiplicative Lie rings is a group homomorphism such that $\phi\{x,y\} = \{\phi(x), \phi(y)\}$ for all $x, y \in L$.
- A subgroup A of a multiplicative Lie ring L will be a subring of L if $\{x,y\} \in A$ for all $x, y \in A$. It will be an *ideal* of L, if it is a normal subgroup and if $\{x,y\} \in A$ for all $x \in A$ and $y \in L$.
- Let A and B be subgroups of L. The subgroup of L generated by all elements $\{x,y\}, x \in A, y \in B$, is denoted by $\{A, B\}$. The subgroup $\{L, L\}$ is an ideal and is called the *Lie commutator* of the multiplicative Lie ring L.

Definition 2.1. Let L and A be two multiplicative Lie rings. By *an action of L on A*, we mean an underlying group action of L on A, given by a group homomorphism $h : L \to Aut(A)$, together with a map $\rho : L \times A \to A$, satisfying the following conditions:

$$\rho(x, yy') = \rho(x, y)\rho({}^{y}x, {}^{y}y') \tag{2.8}$$

$$\rho(xx', y) = \rho({}^{x}x', {}^{x}y)\rho(x, y) \tag{2.9}$$

$$\rho(\{x, x'\}, {}^{x'}y)\rho({}^{y}x, \rho(x', y))^{-1}\rho({}^{x}x', \rho(x, y)^{-1})^{-1} = 1 \tag{2.10}$$

$$\rho({}^{y'}x, \{y, y'\})\{{}^{y}y', \rho(x, y)\}\{{}^{x}y, \rho(x, y')^{-1}\} = 1, \tag{2.11}$$

where $x, x' \in L,\ y, y' \in A, {}^{x}y = h_x(y), {}^{x}x' = xx'x^{-1}, {}^{y}y' = yy'y^{-1}$.

3 Multiplicative Lie ring extensions and factor systems

Given multiplicative Lie rings L and A, we try to find different multiplicative Lie rings, such as E, that contain the ideal A with $\frac{E}{A} \simeq L$. Such multiplicative Lie rings are called extensions of A by L. In this section, we reduce the classification of equivalence classes of such multiplicative Lie ring extensions to so called factor systems. Schreier's theorem yields a bijection between equivalence classes of multiplicative Lie ring extensions and the equivalence classes of the factor systems. Through this section we fix multiplicative Lie rings L and A.

Definition 3.1 (Multiplicative Lie ring extension)**.** An *extension of A by L* is a multiplicative Lie ring E along with a monomorphism $i : A \to E$ and an epimorphism $\pi : E \to L$ such that $\ker \pi = Imi$. We usually refer to an extension (E, i, π) simply by the multiplicative Lie ring E. Let $\varepsilon(L, A)$ denote the set of all extensions of A by L.

Definition 3.2. Let (E_1, i_1, π_1) and (E_2, i_2, π_2) be extensions. We say that *they are equivalent* if there exists a homomorphism $\zeta : E_1 \to E_2$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{i_1} & E_1 & \xrightarrow{\pi_1} & L & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \zeta} & & \| & & \\
1 & \longrightarrow & A & \xrightarrow{i_2} & E_2 & \xrightarrow{\pi_2} & L & \longrightarrow & 1
\end{array}$$

This relation is symmetric, reflexive, and transitive, so we may speak of the equivalence classes of extensions of A by L. We use $[E]$ to denote the equivalence class of E.

The following definition of a factor system is analogous to the definition of a factor triple in Lie rings (see [6]).

Definition 3.3. A factor system of (L, A) is (f, g, ρ, h) of maps $f, g : L \times L \to A$,

$$\rho : L \times A \to A \quad and \quad h : L \to Aut(A)$$

such that h sends each $\ell \in L$ to h_e, with the following properties:

$$(i) \quad g(\ell, \ell) = 1, \tag{3.1}$$

$$(ii) \quad f(\ell, \ell^{-1}) = f(\ell^{-1}, \ell) = 1, \tag{3.2}$$

$$(iii) \quad f(1, \ell) = f(\ell, 1) = 1, \tag{3.3}$$

$$(iv) \quad f(\ell_1, \ell_2)f(\ell_1\ell_2, \ell_3) = h_{\ell_1}(f(\ell_2, \ell_3))f(\ell_1, \ell_2\ell_3), \tag{3.4}$$

$$(v) \quad \rho(\ell_1, f(\ell_2, \ell_3))f(\ell_2, \ell_3)g(\ell_1, \ell_2\ell_3)h_{\{\ell_1,\ell_2\ell_3\}}(f(\ell_2, \ell_3)^{-1}) \tag{3.5}$$

$$= g(\ell_1, \ell_2)f(\{\ell_1, \ell_2\}, \ell_2)h_{\{\ell_1,\ell_2\}\ell_2}(g(\ell_1, \ell_3)$$

$$f(\{\ell_1, \ell_3\}, \ell_2^{-1}))f(\{\ell_1, \ell_2\}\ell_2, \{\ell_1, \ell_3\}\ell_2^{-1}) \tag{3.6}$$

and

$$f(\ell_1, \ell_2)g(\ell_1\ell_2, \ell_3)h_{\{\ell_1\ell_2,\ell_3\}}\Big(\rho\big(\ell_3, f(\ell_1, \ell_2)\big)f(\ell_1, \ell_2)\Big)^{-1}$$

$$= h_{\ell_1}(g(\ell_2,\ell_3))f(\ell_1,\{\ell_2,\ell_3\})f(\ell_1\{\ell_2,\ell_3\},\ell_1^{-1})h_{\ell_1\{\ell_2,\ell_3\}}(g(\ell_1,\ell_3))$$
$$f({}^{\ell_1}\{\ell_2,\ell_3\},\{\ell_1,\ell_3\}),$$

$$(vi)\quad h_{\ell_3}(g(\ell_1,\ell_2)f(\{\ell_1,\ell_2\},\ell_3^{-1}))f(\ell_3,\{\ell_1,\ell_2\}\ell_3^{-1}) \tag{3.7}$$
$$={}^{f(\ell_3,\ell_1)f(\ell_3\ell_1,\ell_3^{-1})}\,\rho({}^{\ell_3}\ell_1,f(\ell_3,\ell_2)f(\ell_3\ell_2,\ell_3^{-1}))$$
$$\{f(\ell_3,\ell_1)f(\ell_3\ell_1,\ell_3^{-1}),f(\ell_3,\ell_2)f(\ell_3\ell_2,\ell_3^{-1})\}f(\ell_3,\ell_2)f(\ell_3\ell_2,\ell_3^{-1})f(\ell_3,\ell_1)$$
$$f(\ell_3\ell_1,\ell_3^{-1})g({}^{\ell_3}\ell_1,{}^{\ell_3}\ell_2)h_{\{{}^{\ell_3}\ell_1,{}^{\ell_3}\ell_2\}}\Big(f(\ell_3,\ell_2)f(\ell_3\ell_2,\ell_3^{-1})$$
$$\rho({}^{\ell_3}\ell_2,f(\ell_3,\ell_1)f(\ell_3\ell_1,\ell_3^{-1}))f(\ell_3,\ell_1)f(\ell_3\ell_1,\ell_3^{-1})\Big)^{-1},$$

$$(vii)\quad Ah_{\{\{\ell_1,\ell_2\}\},{}^{\ell_2}\ell_3\}}(B)f(\{\{\ell_1,\ell_2\},{}^{\ell_2}\ell_3\},\{\{\ell_2,\ell_3\},{}^{\ell_3}\ell_1\})h_{\{{}^{\ell_1}\ell_2,\{\ell_3,\ell_1\}\}}(C)=1, \tag{3.8}$$

where the following D defines, respectively, A, B, and C by replacing (ℓ_1,ℓ_2,ℓ_3), (ℓ_2,ℓ_3,ℓ_1), and (ℓ_3,ℓ_1,ℓ_2) with (a,b,c):

$$D={}^{g(a,b)}\,\rho\big(\{a,b\},f(b,c)f(bc,b^{-1})\big)\{g(a,b),f(b,c)f(bc,b^{-1})\} \tag{3.9}$$
$$f(b,c)f(bc,b^{-1})g(a,b)g(\{a,b\},{}^{b}c)$$
$$h_{\{\{a,b\},{}^{b}c\}}\big(f(b,c)f(bc,b^{-1})\rho({}^{b}c,g(a,b))\big)^{-1},$$

$$(viii)\quad h_{\ell_1}\big(h_{\ell_2}(a)\big)={}^{f(\ell_1,\ell_2)}\big(h_{\ell_1\ell_2}(a)\big), \tag{3.10}$$

$$(ix)\quad \rho(\ell,a_1a_2)=\rho(\ell,a_1)\rho({}^{a_1}\ell,{}^{a_1}a_2), \tag{3.11}$$

$$(x)\quad \rho(\ell_1\ell_2,a)={}^{f(\ell_1\ell_2,\ell_1^{-1})}\,\rho({}^{\ell_1}\ell_2,{}^{\ell_1}a) \tag{3.12}$$
$${}^{f(\ell_1,\ell_2)^{-1}}\big(\{f(\ell_1,\ell_2)f(\ell_1\ell_2,\ell_1^{-1}),{}^{\ell_1}a\}\rho(\ell_1,a)\big)\{f(\ell_1,\ell_2)^{-1},a\},$$

$$(xi)\quad \rho(\{\ell_1,\ell_2\},{}^{\ell_2}a)\Big(\rho\big({}^{a}\ell_1,\rho(\ell_2,a)\big)\Big)^{-1}\Big(\rho\Big({}^{\ell_1}\ell_2,\big(\rho(\ell_1,a)\big)^{-1}\Big)\Big)^{-1}\neq 1, \tag{3.13}$$

$$(xii)\quad \rho({}^{a_2}\ell,\{a_1,a_2\})\big\{{}^{a_1}a_2,\rho(\ell,a_1)\big\}\big\{{}^{\ell}a_1,\big(\rho(\ell,a_2)\big)^{-1}\big\}=1, \tag{3.14}$$

for all $\ell_1,\ell_2,\ell_3,\ell\in L$ and $a,a_1,a_2\in A$.

- Note that if A is an abelian group and abelian multiplicative Lie ring, then the conditions (vii), (ix), and (x) become

  $(vii)'\ h_{\ell_1\ell_2}(a)=h_{\ell_1}\big(h_{\ell_2}(a)\big)$,

  $(ix)'\ \rho(\ell_1\ell_2,a)=\rho({}^{\ell_1}\ell_2,{}^{\ell_1}a)\rho(\ell_1,a)$,

  $(x)'\ \rho(\{\ell_1,\ell_2\},{}^{\ell_2}a)\Big(\rho\big({}^{a}\ell_1,\rho(\ell_2,a)\big)\Big)^{-1}\rho\Big({}^{\ell_1}\ell_2,\big(\rho(\ell_1,a)\big)^{-1}\Big)^{-1}=1$, for all $\ell_1, \ell_2 \in L$ and all $a \in A$, and hence h and ρ together is an action of L on A.

We denote the set of all factor systems of (L, A) by $F(L, A)$.

Definition 3.4. Let (f, g, ρ, h) and (f', g', ρ', h') be two factor systems of (L, A). We say that these factor systems are *equivalent,* if there exists a map $c : L \to A$ such that $c(1) = 1$ and

$$(i)\ h'_x(y) = {}^{c(x)}h_x(y), \tag{3.15}$$

$$(ii)\ \rho'(x, y) = {}^{c(x)}\rho(x, y)\{c(x), y\}, \tag{3.16}$$

$$\begin{aligned}(iii)\ g'(x, x') = \ & {}^{c(x)}\rho\big(x, c(x')\big)\{c(x), c(x')\}c(x')c(x)g(x, x')h_{\{x,x'\}} \\ & \Big(c(x)^{-1}\rho\big(x', c(x)\big)^{-1}c(x')^{-1}\Big)c\{x, x'\}^{-1},\end{aligned} \tag{3.17}$$

$$(iv)\ f'(x, x') = c(x)h_x(c(x'))f(x, x')c(xx')^{-1}, \tag{3.18}$$

for all $x, x' \in L$, all $y \in A$.

It is easy to check that this is an equivalence relation. We will use $[(f, g, \rho, h)]$ to denote the equivalent class of factor system (f, g, ρ, h) of (L, A).

Given two multiplicative Lie rings L and A, we introduce the following notations:

$$\bar{\varepsilon}(L, A) = \{[E] : \quad E \in \varepsilon(L, A)\},$$
$$\bar{F}(L, A) = \{[f, g, \rho, h] : \quad (f, g, \rho, h) \in F(L, A)\}.$$

Let (E, i, π) be an extension of A by L with i being the inclusion map; then we cannot expect to find a homomorphism $\tau : L \to E$ such that $\tau(L)$ is a transversal to A in E.

However, since $L \cong \frac{E}{A}$, we can always find a map $\tau : L \to E$ whose image is a transversal to A in E. Such a map is said to be *a section of the extension.* We may always choose τ such that $\tau(1) = 1$, in which case, τ is called normalized.

Theorem 3.1. Each multiplicative Lie ring extension $1 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} L \longrightarrow 1$ together with normalized section $\tau : L \to E$ defines a factor system $(f_\tau, g_\tau, \rho_\tau, h_\tau)$.

Proof. Let $x \in L \simeq \frac{E}{A}$ be a coset of A in L and let τ be a fixed transversal function $x \to \tau(x)$. It satisfies $\pi\tau = id|_L$. Since A is an ideal in E, then for all $a \in A$ and all $x \in L$, the elements $\tau(x)a\tau(x)^{-1}$ and $\{\tau(x), a\}$ are in A.
We can define

$$\begin{aligned} h :& L \to Aut(A) \\ & x \mapsto h_x \end{aligned}$$

such that $h_x(a) = \tau(x)a\tau(x)^{-1}$ and

$$\begin{aligned} \rho : L \times A &\to A \\ (x, a) &\mapsto \{\tau(x), a\} \end{aligned}$$

for all $a \in A$ and all $x \in L$.

Since π is a homomorphism, we have

$$\begin{aligned} &\pi\big(\tau(x)\tau(y)\tau(xy)^{-1}\big) = 1, \\ &\pi(\{\tau(x), \tau(y)\}(\tau\{x, y\})^{-1}) = 1; \end{aligned}$$

then $\tau(x)\tau(y)\big(\tau(xy)\big)^{-1}$ and $\{\tau(x), \tau(y)\}(\tau\{x, y\})^{-1}$ are elements of $\ker \pi$.

Now we define

$$\begin{aligned} f : L \times L &\to A \\ (x, y) &\mapsto \tau(x)\tau(y)\big(\tau(xy)\big)^{-1} \end{aligned} \quad \text{and} \quad \begin{aligned} g : L \times L &\to A \\ (x, y) &\mapsto \{\tau(x), \tau(y)\}(\tau\{x, y\})^{-1}. \end{aligned}$$

So we have

$$\tau(x)\tau(y) = f(x, y)\tau(xy), \tag{3.19}$$

$$\{\tau(x), \tau(y)\} = g(x, y)\tau\{x, y\}. \tag{3.20}$$

It is easy to check that $h_{x_1}\big(h_{x_2}(a)\big) = {}^{f(x_1,x_2)}\big(h_{x_1x_2}(a)\big)$ for all $x_1, x_2 \in L$, $a \in A$. We have

$$\begin{aligned} &(i)\ \rho(x, yy') = \{\tau(x), yy'\} = \{\tau(x), y\}^y\{\tau(x), y'\} = \{\tau(x), y\}\{{}^y\tau(x), {}^yy'\} \\ &\qquad = \{\tau(x), y\}\{\tau({}^yx), {}^y y'\} = \rho(x, y)\rho({}^yx, {}^y y'), \\ &(ii)\ \rho(xx', y) = f(xx', x^{-1})\rho({}^xx', {}^xy)f(xx', x^{-1})^{-1}f(x, x')^{-1} \\ &\qquad \{f(x, x')f(xx', x^{-1}), {}^xy\}\rho(x, y)f(x, y)\{f(x, x')^{-1}, y\}, \\ &(iii)\rho(\{x, x'\}, {}^{x'}y)(\rho({}^yx, \rho(x', y)))^{-1}(\rho({}^xx', \rho(x, y))^{-1})^{-1} \neq 1, \\ &(iv)\rho({}^{y'}x, \{y, y'\})\{{}^yy', \rho(x, y)\}\{{}^xy, \rho(x, y'))^{-1}\} = 1. \end{aligned}$$

If A is an abelian group and abelian multiplicative Lie ring, then ρ and h together is an action of L on A. Using associativity in E, we have

$$\begin{aligned} &(\tau(x)\tau(y))\tau(z) = \tau(x)(\tau(y)\tau(z)), \\ &f(x, y)\tau(xy)\tau(z) = \tau(x)(f(y, z)\tau(yz)), \\ &f(x, y)f(xy, z)\tau(xyz) = \tau(x)f(y, z)\tau(x)^{-1}\tau(x)\tau(yz), \\ &f(x, y)f(xy, z)\tau(xyz) = h_x(f(y, z))f(x, yz)\tau(xyz), \\ &f(x, y)f(xy, z) = h_x(f(y, z))f(x, yz), \end{aligned}$$

for all $x, y, z \in L$.

This implies (3.4).

Sine E is a multiplicative Lie ring, for each x, y, y' in L, we have

$$\begin{aligned}\{\tau(x), \tau(y)\tau(y')\} &= \{\tau(x), f(y, y')\tau(yy')\} = \{\tau(x), f(y, y')\}^{f(y,y')}\{\tau(x), \tau(yy')\}\\ &= \rho(x, f(y, y'))f(y, y')g(x, yy')h_{\{x,yy'\}}(f(y, y')^{-1})\tau(\{x, yy'\})\end{aligned}$$

and

$$\begin{aligned}&\{\tau(x), \tau(y)\}^{\tau(y)}\{\tau(x), \tau(y')\}\\ &\qquad = g(x, y)\tau\{x, y\}^{\tau(y)}(g(x, y')\tau(\{x, y'\}))\\ &\qquad = g(x, y)f(\{x, y\}, y)\tau(\{x, y\}y)g(x, y')f(\{x, y'\}, y^{-1})\tau(\{x, y'\}y^{-1})\\ &\qquad = g(x, y)f(\{x, y\}, y)h_{\{x,y\}y}(g(x, y')f(\{x, y'\}, y^{-1}))\\ &\qquad\quad f(\{x, y\}y, \{x, y'\}y^{-1})\tau(\{x, y\}^y\{x, y'\}).\end{aligned}$$

Hence

$$\begin{aligned}&\rho(x, f(y, y'))f(y, y')g(x, yy')h_{\{x,yy'\}}(f(x, y')^{-1})\\ &\quad = g(x, y)f(\{x, y\}, y)h_{\{x,y\}y}(g(x, y')f(\{x, y'\}, y^{-1})f(\{x, y\}y, \{x, y'\}y^{-1}).\end{aligned}$$

Similarly, for all x, x', y in L, we have

$$\{\tau(x)\tau(x'), \tau(y)\} = {}^{\tau(x)}\{\tau(x'), \tau(y)\}\{\tau(x), \tau(y)\};$$

then

$$\begin{aligned}&f(x, x')g(xx', y)h_{\{xx',y\}}(\rho(y, f(x, x'))f(x, x'))^{-1}\\ &\qquad = h_x(g(x', y))f(x, \{x', y\})f(x\{x', y\}, x^{-1})\\ &\qquad\qquad h_{\ {}^x\{x',y\}}(g(x, y))f({}^x\{x', y\}, \{x, y\}),\end{aligned}$$

which implies (3.5).

For all $x, y, z \in L$, we have

$${}^{\tau(z)}\{\tau(x), \tau(y)\} = \{{}^{\tau(z)}\tau(x), {}^{\tau(z)}\tau(y)\}.$$

For left side, we have

$$\begin{aligned}{}^{\tau(z)}\{\tau(x), \tau(y)\} &= {}^{\tau(z)}(g(x, y)\tau\{x, y\})\\ &= h_z(g(x, y)f(\{x, y\}, z^{-1}))f(z, \{x, y\}z^{-1})\tau(\ {}^z\{x, y\}),\end{aligned}$$

and for right side, we have

$$\begin{aligned}\{{}^{\tau(z)}\tau(x), {}^{\tau(z)}\tau(y)\} &= \{f(z,x)f(zx,z^{-1})\tau({}^{z}x), f(z,y)f(zy,z^{-1})\tau({}^{z}y)\}\\ &= {}^{f(z,x)f(zx,z^{-1})}\{\tau({}^{z}x), f(z,y)f(zy,z^{-1})\}\\ &\quad \{f(z,x)f(zx,z^{-1}), f(z,y)f(zy,z^{-1})\}^{f(z,y)f(zy,z^{-1})f(z,x)f(zx,z^{-1})}\\ &\quad (\{\tau({}^{z}x), \tau({}^{z}y)\})\ {}^{f(z,y)f(zy,z^{-1})}(\{f(z,x)f(zx,z^{-1}), \tau({}^{x}y)\})\\ &= {}^{f(z,x)f(zx,z^{-1})}\rho({}^{z}x, f(z,y)f(zy,z^{-1}))\\ &\quad \{f(z,x)f(zx,z^{-1}), f(z,y)f(zy,z^{-1})\}\\ &\quad f(z,y)f(zy,z^{-1})f(z,x)f(zx,z^{-1})g({}^{z}x,{}^{z}y)h_{\{{}^{z}x,{}^{z}y\}}\\ &\quad \Big(f(z,y)f(zy,z^{-1})\rho\big({}^{z}y, f(z,x)f(zx,z^{-1})\big)f(z,x)f(zx,z^{-1})\Big)^{-1}\\ &\quad \tau(\{{}^{z}x,{}^{z}y\}).\end{aligned}$$

So (3.7) satisfies. With a similar argument, by using the Lie bracket condition, we can obtain (3.8). $\boxed{QED}$

Theorem 3.2. Let (f,g,ρ,h) be a factor system of (L,A). Then the set $L \times A$ together with the following operation

$$\begin{aligned}(\ell_1,a_1)(\ell_2,a_2) &= (\ell_1\ell_2, a_1h_{\ell_1}(a_2)f(\ell_1,\ell_2))\\ \{(\ell_1,a_1),(\ell_2,a_2)\} &= (\{\ell_1,\ell_2\}, {}^{a_1}\rho(\ell_1,a_2)\{a_1,a_2\}a_2a_1\\ &\qquad g(\ell_1,\ell_2)h_{\{\ell_1,\ell_2\}}(a_2\rho(\ell_2,a_1)a_1)^{-1})\end{aligned}$$

for all $\ell_1,\ell_2 \in L$ and $a_1,a_2 \in A$, is a multiplicative Lie ring denoted by $E_{(f,g,\rho,h)}$. Moreover the maps $\epsilon : A \to E_{(f,g,\rho,h)}$ defined by $a \mapsto (1,a)$ and $v : E_{(f,g,\rho,h)} \to L$ defined by $(\ell,a) \mapsto \ell$ are multiplicative Lie ring homomorphisms such that $(E_{(f,g,\rho,h)},\epsilon,v)$ is an extension of A by L.

Proof. The proof is purely routine. $\boxed{QED}$

By using the notations and definitions of the above theorem, we have the next proposition.

Proposition 3.1. Let (E,i,π) is an extension of A by L, with i being the inclusion map and let τ be a normalize section of E. Then (E,i,π) is equivalent to $(E_{(f_\tau,g_\tau,\rho_\tau,h_\tau)},\epsilon,v)$, where $(f_\tau,g_\tau,\rho_\tau,h_\tau)$ is a factor system of (L,A) arising from the section τ.

Proof. Let $x \in E$ and let $\ell = \pi(x)$. We know that

$$\pi(x)\Big(\pi\big(\tau(\ell)\big)\Big)^{-1} = 1_L \Rightarrow \pi(x\tau(\ell)^{-1}) = 1_L \Rightarrow x\tau(\ell)^{-1} \in A.$$

So there exists a unique element $a \in A$ such that $x = a\tau(\ell)$.

Now we define $\zeta : E \to E_{(f_\tau, g_\tau, \rho_\tau, h_\tau)}$ by $\zeta(x) = (\ell, a)$. To see ζ is a multiplicative Lie ring homomorphism, we let $y \in E$ and write $y = b\tau(m)$, where $m \in L$ and $b \in A$ with $m = \pi(y)$.

We then have

$$xy = a\tau(\ell)b\tau(m) = ah_\ell(b)\tau(\ell)\tau(m) = ah_\ell(b)f(\ell, m)\tau(\ell m).$$

Hence

$$\zeta(xy) = \zeta(ah_\ell(b)f(\ell, m)\tau(\ell m)) = (\ell m, ah_\ell(b)f(\ell, m)).$$

Also

$$\begin{aligned}\{x, y\} &= \{a\tau(\ell), b\tau(m)\} = a\{\tau(\ell), b\}a^{-1}\{a, b\}ba\{\tau(\ell), \tau(m)\}a^{-1}\{a, \tau(m)\}b^{-1} \\ &= {}^{a}\rho(\ell, b)\{a, b\}bag(\ell, m)h_{\{\ell,m\}}(b\rho(m, a)a)^{-1}\tau\{\ell, m\},\end{aligned}$$

from which we see that

$$\zeta\{x, y\} = \big(\{\ell, m\}, {}^{a}\rho(b, \ell)\{a, b\}bag(\ell, m)h_{\{\ell,m\}}(b\rho(a, m)a)^{-1}\big).$$

Now it is easily checked that $\zeta i = \epsilon$ and $v\zeta = \pi$. QED

Lemma 3.1. Let (f_1, g_1, ρ_1, h_1) and (f_2, g_2, ρ_2, h_2) be two factor systems of (L, A). If these factor systems are equivalent, then the associated multiplicative Lie ring extensions are also equivalent.

Proof. Assume that (f_1, g_1, ρ_1, h_1) and (f_2, g_2, ρ_2, h_2) are equivalent, so that there is a map $c : L \to A$ such that $c(1) = 1$ and satisfying (3.15)–(3.18).

Let $\big(E_{(f_1,g_1,\rho_1,h_1)}, \epsilon_1, v_1\big)$ and $\big(E_{(f_2,g_2,\rho_2,h_2)}\epsilon_2, v_2\big)$ be two multiplicative Lie ring extensions of A by L as the construction theorem 3.2. We need to show that both extensions are equivalent, that is, there is a homomorphism $\zeta : E_{(f_1,g_1,\rho_1,h_1)} \to E_{(f_2,g_2,\rho_2,h_2)}$ such that the diagram commutes.

We define ζ by $\zeta(\ell, a) = (\ell, ac(\ell))$. Clearly this map is bijective. It is also easy to check that ζ is a multiplicative Lie ring homomorphism and so $E_{(f_1,g_1,\rho_1,h_1)}$ and $E_{(f_2,g_2,\rho_2,h_2)}$ are equivalent. QED

Lemma 3.2. Equivalent multiplicative Lie ring extensions define equivalent factor systems.

Proof. Assume that we have two arbitrary equivalent multiplicative Lie ring extensions of A by L.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & L & \longrightarrow & 1\ . \\ & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & A & \xrightarrow{\gamma} & E' & \xrightarrow{\delta} & L & \longrightarrow & 1 \end{array}$$

Choose arbitrary normalized sections τ and τ' to the extensions
$1 \longrightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} L \longrightarrow 1$ and $1 \longrightarrow A \xrightarrow{\gamma} E' \xrightarrow{\delta} L \longrightarrow 1$, respectively. Let $E_{(f_\tau,g_\tau,\rho_\tau,h_\tau)}, E_{(f_{\tau'},g_{\tau'},\rho_{\tau'},h_{\tau'})}$ be multiplicative Lie rings arising from τ and τ'. By Proposition 3.1, we know that

$$(E_{(f_\tau,g_\tau,\rho_\tau,h_\tau)}, \epsilon, v) \quad \text{and} \quad (E_{(f_{\tau'},g_{\tau'},\rho_{\tau'},h_{\tau'})}, \epsilon', v')$$

are equivalent.

Then there exists a multiplicative Lie ring homomorphism

$$\zeta : E_{(f_\tau,g_\tau,\rho_\tau,h_\tau)} \to E_{(f_{\tau'},g_{\tau'},\rho_{\tau'},h_{\tau'})}$$

such that $v_2\zeta = v_1$ and $\zeta\epsilon_1 = \epsilon_2$.

For each arbitrary element ℓ of L, there are unique elements $\ell^* \in L$ and $c(\ell) \in A$ such that

$$\zeta(\ell, 1) = (\ell^*, c(\ell)).$$

So we have

$$\ell = v(\ell, 1) = v'\zeta(\ell, 1) = v'(\ell^*, c(\ell)) = \ell^*.$$

Therefore $\ell = \ell^*$; then $\zeta(\ell, 1) = (\ell, c(\ell))$. Consequently, for each $a \in A$, we have

$$\begin{aligned}
\zeta(\ell, a) &= \zeta(1, a)\zeta(\ell, 1) \\
&= \zeta(1, a)(\ell, c(\ell)) \\
&= \zeta\epsilon_1(a)(\ell, c(\ell)) \\
&= \epsilon_2(a)(\ell, c(\ell)) \\
&= (1, a)(\ell, c(e)) \\
&= (\ell, ac(\ell)).
\end{aligned}$$

Comparing both sides of these relations

$$\begin{aligned}
\zeta((\ell_1, 1)(\ell_2, 1)) &= \zeta(\ell_1, 1)\zeta(\ell_2, 1), \\
\zeta\{(\ell_1, 1), (\ell_2, 1)\} &= \{\zeta(\ell_1, 1), \zeta(\ell_2, 1)\}, \\
\zeta\{(1, a), (\ell, 1)\} &= \{\zeta(1, a), \zeta(\ell, 1)\},
\end{aligned}$$

and

$$\zeta((\ell, 1)(1, a)) = \zeta(\ell, 1)\zeta(1, a)$$

for all $\ell_1, \ell_2, \ell \in L$ and $a \in A$, we see that $(f_\tau, g_\tau, \rho_\tau, h_\tau)$ and $(f_{\tau'}, g_{\tau'}, \rho_{\tau'}, h_{\tau'})$ are equivalent, which completes the proof. QED

All the above mentioned results will be used to conclude that Schreier's theorem holds for multiplicative Lie rings.

Theorem 3.3. Let A and L be two multiplicative Lie rings.There is a one-to-one correspondence between the set of equivalence classes of extensions of A by L and the set of equivalence classes of factor systems of (L, A).

Let E be an extension of A by L and let (f, g, ρ, h) be a factor system arising from a normalized section τ of E.
Define

$$\begin{aligned} h_E : L &\to \tfrac{Aut(A)}{Inn(A)} \\ \ell &\to h_e\big(Inn(A)\big) \end{aligned}$$

and

$$\begin{aligned} \rho_E : L \times A &\to \tfrac{A}{\{A,A\}\cap[A,A]} \\ (\ell, a) &\to \rho(\ell, a)\big(\{A, A\} \cap [A, A]\big). \end{aligned}$$

Suppose that τ' is another normalize section of E and $x \in L$. Then $\tau(x)$ and $\tau'(x)$ differ by an element of A. Hence

$$\tau'(x) = c(x)\tau(x),$$

where c is a map from L to A. We have

$$\begin{aligned} &\{\tau'(x), a\} = \{c(x)\tau(x), a\} =^{c(x)} \{\tau(x), a\}\{c(x), a\}, \\ &{}^{\tau(x)}a =^{c(x)\tau(x)} a =^{c(x)} \big({}^{\tau(x)}a\big). \end{aligned}$$

Consequently, for $a \in A$, $\{\tau(x), a\}$ and $\{\tau'(x), a\}$ differ by elements in $\{A, A\} \cap [A, A]$ and also ${}^{\tau(x)}a$, ${}^{\tau'(x)}a$ differ by an inner automorphism of A. Then ρ_E and h_E are independent of the choice of normalized sections.
It is easy to check that h_E is a group homomorphism and ρ_E satisfies in conditions (2.8), (2.9), (2.10) and (2.11). We shall refer to the ρ_E and h_E together as a *coupling* of E. As we saw in above, equivalent extensions have the same coupling, so we can from $\mathcal{E}_{(\rho_E,h_E)}(L, A)$, a subcategory of multiplicative Lie ring extensions of A by L with coupling ρ_E and h_E.

4 Second cohomology

Cohomology of Lie rings has been studied in [6] also, cohomology of Lie groups and Lie algebras are studied in [7, 5].

But our knowledge of cohomology of multiplicative Lie ring is not enough. In this section, we define the second cohomology group of a multiplicative Lie ring and find the relation between the second cohomology, multiplicative Lie ring extension, and factor system.

Definition 4.1. Let L be a multiplicative Lie ring. An *L-module* is an abelian group and abelian multiplicative ring A such that L acts on A.

Throughout this section, suppose that A is an L-module and that L acts on A by a map ρ and a group homomorphism h such that h sends each $\ell \in L$ to h_ℓ.

Definition 4.2. Let $Z^2(L, A, \rho, h)$ be the set of all pairs of functions (f, g), where $f, g : L \times L \to A$ satisfying the following conditions, for all $\ell, \ell_1, \ell_2, \ell_3 \in L$:

(*i*) $g(\ell, \ell) = 1$,

(*ii*) $f(\ell, \ell^{-1}) = f(\ell^{-1}, \ell) = 1$,

(*iii*) $f(1, \ell) = f(\ell, 1) = 1$,

(*iv*) $f(\ell_1, \ell_2) f(\ell_1\ell_2, \ell_3) = h_{\ell_1}(f(\ell_2, \ell_3)) f(\ell_1, \ell_2\ell_3)$,

(*v*)
$$\begin{aligned}&\rho(\ell_1, f(\ell_2, \ell_3)) f(\ell_2, \ell_3) g(\ell_1, \ell_2\ell_3) h_{\{\ell_1, \ell_2\ell_3\}}\big(f(\ell_2, \ell_3)^{-1}\big)\\ &\quad = g(\ell_1, \ell_2) f(\{\ell_1, \ell_2\}, \ell_2) h_{\{\ell_1, \ell_2\}\ell_2}(g(\ell_1, \ell_3) f(\{\ell_1, \ell_3\}, \ell_2^{-1}))\\ &\qquad f(\{\ell_1, \ell_2\}\ell_2, \{\ell_1, \ell_3\}\ell_2^{-1})\end{aligned}$$

and

$$\begin{aligned}&f(\ell_1, \ell_2) g(\ell_1\ell_2, \ell_3) h_{\{\ell_1\ell_2, \ell_3\}}\Big(\rho\big(\ell_3, f(\ell_1, \ell_2)\big) f(\ell_1, \ell_2)\Big)^{-1}\\ &\quad = h_{\ell_1}(g(\ell_2, \ell_3)) f(\ell_1, \{\ell_2, \ell_3\}) f(\ell_1\{\ell_2, \ell_3\}, \ell_1^{-1}) h_{{\ell_1}\{\ell_2, \ell_3\}}(g(\ell_1, \ell_3))\\ &\qquad f({}^{\ell_1}\{\ell_2, \ell_3\}, \{\ell_1, \ell_3\}),\end{aligned}$$

(*vi*)
$$\begin{aligned}&h_{\ell_3}(g(\ell_1, \ell_2) f(\{\ell_1, \ell_2\}, \ell_3^{-1})) f(\{\ell_3, \{\ell_1, \ell_2\}\ell_3^{-1})\\ &\quad = \rho\big({}^{\ell_3}\ell_1, f(\ell_3, \ell_2) f(\ell_3\ell_2, \ell_3^{-1})\big)\\ &\qquad f(\ell_3, \ell_2) f(\ell_3\ell_2, \ell_3^{-1}) f(\ell_3, \ell_1) f(\ell_3\ell_1, \ell_3^{-1}) g({}^{\ell_3}\ell_1, {}^{\ell_3}\ell_2)\\ &\qquad h_{\{{}^{\ell_3}\ell_1, {}^{\ell_3}\ell_2\}}(f(\ell_3, \ell_2) f(\ell_3\ell_2, \ell_3^{-1}) \rho({}^{\ell_3}\ell_2, f(\ell_3, \ell_1) f(\ell_3\ell_1, \ell_3^{-1})))\\ &\qquad f(\ell_3, \ell_1) f(\ell_3\ell_1, \ell_3^{-1}))^{-1},\end{aligned}$$

(*vii*) $A h_{\{\{\ell_1, \ell_2\}\}, {}^{\ell_2}\ell_3\}}(B) f(\{\{\ell_1, \ell_2\}, {}^{\ell_2}\ell_3\}, \{\{\ell_2, \ell_3\}, {}^{\ell_3}\ell_1\}) h_{\{{}^{\ell_1}\ell_2, \{\ell_3, \ell_1\}\}}(C) = 1$,

where the following D defines, respectively, A, B, and C by replacing (ℓ_1, ℓ_2, ℓ_3), (ℓ_2, ℓ_3, ℓ_1), and (ℓ_3, ℓ_1, ℓ_2) with (a, b, c):

$$\begin{aligned}D &= \rho\big(\{a, b\}, f(b, c) f(bc, b^{-1})\big) f(b, c) f(bc, b^{-1})\\ &\quad g(a, b) g(\{a, b\}, {}^{b}c) h_{\{\{a, b\}, {}^{b}c\}}\big(f(b, c) f(bc, b^{-1}) \rho({}^{b}c, g(a, b))\big)^{-1}.\end{aligned}$$

We shall refer to the elements of $Z^2(L, A, \rho, h)$ as *cocycles* of L in A.

Remark 4.1. The set $Z^2(L, A, \rho, h)$ is an abelian group under the pointwise addition of the two functions

$$(f, g) + (f', g') = (f + f', g + g').$$

Let (f,g) be a cocycle of (L,A,ρ,h); then we observe that (f,g,ρ,h) is a factor system of (L,A), because A is an abelian group and abelian multiplicative Lie ring.

Definition 4.3. Let $c : L \to A$ be a map such that $c(1) = 1$. We define

(i) $g(\ell_1,\ell_2) = \rho(\ell_1, c(\ell_2))c(\ell_2)c(\ell_1)h_{\{\ell_1,\ell_2\}}(c(\ell_1)\rho(\ell_2, c(\ell_1))c(\ell_2))^{-1}\big(c(\{\ell_1,\ell_2\})\big)^{-1}$,

(ii) $f(\ell_1,\ell_2) = c(\ell_1)h_{\ell_1}(c(\ell_2))(c(\ell_1\ell_2))^{-1}$,

and refer to the pair (f,g) as a *coboundary* of L in A and denote the set of all coboundaries by $B^2(L,A,\rho,h)$.

One can check that $B^2(L,A,\rho,h)$ is a subgroup of $Z^2(L,A,\rho,h)$.

Definition 4.4. The factor group

$$H^2(L,A,\rho,h) := Z^2(L,A,\rho,h)/B^2(L,A,\rho,h)$$

is called *the second cohomology group* of L with coefficients in A, and the elements $H^2(L,A,\rho,h)$ are called cohomology classes.

Any two cocycles contained in the same cohomology class are said to be *cohomologous*.

Given $(f,g) \in Z^2(L,A,\rho,h)$, we denote by $(\overline{f,g})$ the cohomology class containing (f,g).

Theorem 4.1. There is a one-to-one correspondence between the following three sets

$$\overline{F}(L,A,\rho,h) = \{[(f,g,\rho,h)] | (f,g) \in Z^2(L,A,\rho,h)\},$$
$$\bar{\varepsilon}(L,A,\rho,h) = \{[E] | E \in \mathcal{E}_{(\rho,h)}(L,A)\},$$
$$H^2(L,A,\rho,h).$$

Proof. In the first step, we show that $\overline{F}(L,A,\rho,h)$ and $\bar{\varepsilon}(L,A,\rho,h)$ have a on-to-one correspondence. Let $[E] \in \bar{\varepsilon}(L,A,\rho,h)$ with $\rho_E = \rho$ and $h_E = h$.

Also suppose that (f,g,ρ',h') is a factor system form a normalized section τ of E. Recall that $\rho' : L \times A \to A$ is defined by $(\ell,a) \to \{\tau(\ell),a\}, \ell \in L$ and $a \in A$ and that $h' : L \to Aut(A)$ is defined by $\ell \to h'_\ell$ with $h'_\ell(a) = \tau(\ell)a\tau(\ell)^{-1}, \ell \in L$ and $a \in A$. So

$$\rho_E(\ell,a) = \rho'(\ell,a)(\{A,A\} \cap [A,A]) = \rho,$$
$$h_E(\ell) = h'_\ell(Inn(A)) = h$$

for $a \in A$, $\ell \in L$.

It follows that $[(f,g,\rho,h)] \in \bar{F}(L,A,\rho,h)$. On the other hand, since equivalent extensions result equivalent factor systems, the map $\mu : \bar{\varepsilon}(L,A,\rho,h) \to \bar{F}(L,A,\rho,h)$ by $[E] \mapsto [(f,g,h,\rho)]$ is well-defined and bijection.

In the second step, we show that $\bar{F}(L, A, \rho, h)$ and $H^2(L, A, \rho, h)$ have a one-to-one correspondence.

We define the map $\lambda : \bar{F}(L, A, \rho, h) \to H^2(L, A, \rho, h)$ by $[(f, g, \rho, h)] \mapsto (\overline{f, g})$.

If $[(f, g, \rho, h)] = [(f', g', \rho, h)]$, then there is a map $c : L \to A$ such that $c(1) = 1$ and

$$g(x, y) = g'(x, y)\rho(x, (c(y))c(x)c(y)h_{\{x,y\}}(c(x)\rho(y, c(x))c(y))^{-1}(c\{x, y\})^{-1},$$
$$f(x, y) = f'(x, y)c(x)h_x(c(y))(c(xy))^{-1}$$

for all $x, y \in L$.

This means $(\overline{f, g}) = (\overline{f', g'})$; hence λ is well-defined.

It is easy to see that λ is a bijection.

Now λ and μ are one-to-one correspondences between the sets $H^2(L, A, \rho, h)$, $\bar{F}(L, A, \rho, h)$ and $\bar{\varepsilon}(L, A, \rho, h)$ QED

References

[1] A. Bak, G. Donadze, N. Inassaridze and M. Ladra, Homology of multiplicative Lie rings, *J. Pure Appl. Algebra* **208** (2007), no. 2, 761–777.

[2] G. Donadze, N. Inassaridze and M. Ladra, Non-abelian tensor and exterior products of multiplicative Lie rings, *Forum Math.* **29** (2016), no. 3, 563–574.

[3] G. Donadze and M. Ladra, More on five commutator identities, *J. Homotopy Relat. Struct.* **2** (2007), no. 1, 45–55.

[4] G. J. Ellis, On five well-known commutator identities, *J. Aust. Math. Soc. Ser. A* **54** (1993), no. 1, 1–19.

[5] B. L. Feigin and D. L. Fuchs, Cohomology of Lie group and Lie algebras, *Itogi Nauki Tekh. Ser. Sovrem. Probl. Mat. Fund. Naprav.* **21** (1988), 121–209.

[6] S. Foladi, A. R, Jamali and R. Orfi, Schreier's extension theory for Lie ring, *J. Algebra* **14** (2015), no. 8, 1550127, 10 pp.

[7] A. W. Knapp, *Lie Groups, Lie Algebras, and Cohomology* (Princeton University Press, 1988).

[8] F. Point and P. Wantiez, Nilpotancy crireria for multiplicative Lie algebras, *J. Pure Appl.* **111** (1996), no. 1-3, 229–243.