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On May Modules of Finite Rank and the Jacobson Radicals of Their Endomorphism Rings

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Abstract. In [5], W. May studied the question of when isomorphisms of the endomorphism rings of mixed modules are necessarily induced by isomorphisms of the underlying modules. In so doing he introduced a class of mixed modules over a complete discrete valuation domain; in [4] these modules were renamed after their inventor. The class of May modules contains the class of Warfield modules. In this work, an intermediate class of finite rank modules is considered, called the Butler-May modules, that parallels the idea of a Butler torsion-free abelian group. Results of M. Flagg from [2] on the Jacobson radicals of the endomorphism rings of finitely generated Warfield modules are generalized to May modules. Finally, a negative example is given to an interesting and unresolved question from [2].

Keywords: May module, complete discrete valuation ring, endomorphism ring, Jacobson Radical, totally projective

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1 Introduction and Review

Throughout, $\mathbf{R}$ will denote a fixed complete discrete valuation ring with $p \in \mathbf{R}$ a prime. We denote the quotient field of $\mathbf{R}$ by $\mathbf{Q}$ and let $\mathbf{K} = \mathbf{Q}/\mathbf{R}$. Except where explicitly noted, all modules will be over $\mathbf{R}$. Most of our terminology follows that of [3]; in particular, all totally projective modules will be assumed to be reduced.

We will also use the language of *valuated modules* (see, for example, [6]), which we briefly review: A valuation on a module V is a function $|\ |$ from V to the ordinals (with ∞ adjoined) such that for all ordinals γ, $V(\gamma) := \{\, v \in V \mid |v| \geq \gamma \,\}$ is a submodule of V with $pV(\gamma) \subseteq V(\gamma+1)$. If we wish to emphasize the context, we may use the notation $|v|_V$. When we are discussing valuated modules, but we are really not worried about the valuation, we may use the adjective *algebraically*, as in "V is algebraically torsion-free".

If $v \in V$, then the sequence of ordinals $\|v\| := (|v|, |pv|, |p^2v|, \dots)$ is called the *value sequence* of v. If $S \subseteq V$ is some arbitrary subset of V, then $|S| = \{\, |s| \mid s \in S\,\}$ is the *value spectrum* of S.

A submodule $N \subseteq V$ is *nice* if every coset $v + N$ has an element of maximal value (i.e., it is *proper* with respect to N). If v_p is this proper element, then setting $|v+N|_{V/N} = |v_p|_V$ makes V/N into a valuated module.

If V is a valuated module that is algebraically the (internal or external) direct sum $\oplus_{i\in I} V_i$, then we will say that the decomposition is *valuated* if $|v_{i_1} + \dots + v_{i_k}| = \min\{|v_1|, \dots, |v_{i_k}|\}$ whenever $v_{i_1} \in V_{i_1}, \dots, v_{i_k} \in V_{i_k}$. In this case the value spectra will satisfy $|V| = \cup_{i\in I}|V_i|$.

If V and W are valuated modules, then a homomorphism $V \to W$ is *valuated* if it does not decrease values, an *embedding* if it is injective and preserves all values and an *isometry* if it is a bijective embedding.

There is a generic way to construct valuated modules. Suppose M is any module and $V \subseteq M$ is a submodule. If we just restrict the height function on M to V, the result is trivially a valuation on V. We will call M an *NT-realization of* V if, in addition, V is nice in M and M/V is totally projective. It follows from ([6], Theorem 1) that any valuated module has an NT-realization.

Recall that a submodule $F \subseteq M$ is *full-rank* if M/F is torsion. Borrowing from [5], in [4] a class of modules was defined as follows:

Definition 1. Suppose M is a module.

(a) A submodule $B \subseteq M$ is an **NFT-submodule** if B is algebraically free and M is an NT-realization of B.

(b) M is a **May** module if every full-rank submodule contains an NFT-submodule.

The class of May modules is somewhat reminiscent of the class of torsion-free abelian groups. For example, a torsion-free abelian group is *completely decomposable* if it is isomorphic to a direct sum of groups of rank one. Suppose B is a valuated module that is algebraically torsion-free. We will say B is a *completely decomposable valuated module* if it is the valuated direct sum of algebraically cyclic modules. A basis corresponding to this representation is called a *decomposition basis* for B. The module M is *Warfield* if it is an NT-realization of a completely decomposable valuated submodule. In some respects, then, the class of Warfield modules is analogous to the class of completely decomposable torsion-free abelian groups. It is elementary that any Warfield module will be a May module.

The purpose of this work is to investigate some properties of May modules of finite rank (this term will always refer to the *torsion-free* rank). A particular class of finite rank torsion-free abelian groups that has drawn considerable attention is known as the *Butler groups.* In Section 2 we present a parallel class of May modules, which we refer to as *Butler-May modules.* In particular, the classical dual characterizations of Butler groups using completely decomposable groups is shown to translate into dual characterizations of the valuated modules whose NT-realizations are these Butler-May modules (Theorem 1).

We will observe that any finite rank Warfield module is a Butler-May module, and that any Butler-May module is a May module. In Section 3 we provide examples showing that the converses of these statements do not hold. We first construct a Butler-May module that is not a Warfield module (Example 1), and then we exhibit a finite rank May module that is not a Butler-May module (Example 2).

May modules were originally defined in order to study the endomorphism rings of mixed modules (i.e., neither torsion nor torsion-free). Suppose M is a mixed May module and N is any other module for which there is an isomorphism of the endomorphism rings $E(M) \to E(N)$. When can we conclude that this will necessarily be induced by a module isomorphism $M \to N$? Clarifying a result from [5], in [4] it was shown that this almost always is true, and the exceptions, called *E-torsion* May modules, were completely described. In particular, E-torsion May modules must have finite rank.

If M is a module, we denote the Jacobson radical of the endomorphism ring $E(M)$ by $\mathcal{J}(M)$. In ([2], Theorem 5.2) it was shown that if M and N are finite rank Warfield modules such that the torsion submodule $T \subseteq M$ is unbounded and there is an isomorphism (of non-unitary $\mathbf{R}$-algebras) $\mathcal{J}(M) \to \mathcal{J}(N)$, then M is isomorphic to N as modules. Further, in that proof it was shown that if M/T is divisible, then the isomorphism of these Jacobson radicals is in fact induced by a module isomorphism $M \to N$.

We extend these two results in a couple of ways (Corollary 7, Theorem 2). First, we show that in both we can replace the condition that M and N be Warfield modules by the more general requirement that they be May modules. Second, if M/T divisible and M is a non-E-torsion May module, then in order to guarantee that our isomorphism is induced all we need to assume is that N is reduced.

Finally, if M and N are finite rank Warfield modules for which T is unbounded, there is an isomorphism $\mathcal{J}(M) \to \mathcal{J}(N)$ and M/T *fails* to be divisible, though it was shown in [2] that there must be an isomorphism $M \cong N$, it was not determined whether such an isomorphism can always be found that will induce the given isomorphism on their Jacobson radicals. We show, perhaps surprisingly, that this actually fails by producing an automorphism of the Jacobson radical of a rank one Warfield module that is not induced by an automorphism of the module itself (Example 3). This example also contradicts ([2], Theorem 4.1).

2 Butler-May Modules

We begin with a simple observation that we will use frequently. The completeness of $\mathbf{R}$ easily implies that if M is a module and N is a finitely generated submodule of M, then N will necessarily be nice in M. And if $N \subseteq M$ is finitely generated and torsion, then M is totally projective iff M/N is totally projective.

Proposition 1. *Suppose M is a module of finite rank. If one full-rank free submodule $B \subseteq M$ is an NFT-submodule, then every full-rank free submodule $C \subseteq M$ is an NFT-submodule. In particular, M must be a May module.*

Proof. We know that C is nice in M, so we need only show that M/C is totally projective. Since $B/(B \cap C)$ is finitely generated torsion, M/B is totally projective and $M/B \cong (M/(B \cap C))/(B/(B \cap C))$, it follows that $M/(B \cap C)$ is totally projective. Similarly, since $C/(B \cap C)$ is finitely generated torsion, $M/(B \cap C)$ is totally projective and $M/C \cong (M/(B \cap C))/(C/(B \cap C))$, it again follows that M/C is totally projective. QED

It is not hard to see that any direct sum of a (possibly infinite) collection of May modules is also a May module. Regarding closure under summands, we have the following:

Corollary 1. *Suppose M is a finite rank May module. If N is a summand of M, then N is also a finite rank May module.*

Proof. Suppose $M = N \oplus L$, B_N is free and full-rank in N and B_L is free and full-rank in L. Since B_N has finite rank, it is nice in N. By Proposition 1, $B := B_N \oplus B_L$ is an NFT-submodule of M. It follows that M/B is totally projective. And since N/B_N is isomorphic to a summand of M/B, it too is totally projective. Therefore, B_N is an NFT-submodule of N. Again by Proposition 1, N must be a May module. QED

The following seems to be harder than it might appear:

Conjecture. *Suppose M is a May module. If N is a summand of M, then N is also a May module.*

In a later work it will be shown that if M has countably infinite rank, then any summand of M is a May module, but the authors do not know the answer for modules of uncountable rank. It is perhaps worth noting that one characterization of Warfield modules is that they are precisely the summands of simply presented modules. From this it immediately follows that the Warfield modules are closed under summands.

The following observation is well known.

Proposition 2. *Suppose M is a finite rank reduced module whose torsion submodule, T, is totally projective. Then M is a May module.*

Proof. Let $B \subseteq M$ be free and full-rank. Since B is nice in M, M/B must be a reduced torsion module. The composition $T \subseteq M \to M/B$ will be injective and its cokernel, which is isomorphic to $M/(T+B)$, will be countably generated. So by ([7], Theorem 1), M/B is totally projective, completing the argument. QED

On the other hand, if M is a (finite rank or not) May module such that $p^{\omega_1}M \neq 0$ is an NFT-submodule of M, then it can be seen that T will not be totally projective. (It will, though, be an S-module-see [8].)

We want to consider the following idea:

Definition 2. A valuated module is **Butler** if it can be embedded as a pure submodule of a finite rank completely decomposable valuated module.

We need to review some simple, but important, ideas. Suppose n is a positive integer, $\{s_j\}_{j\in\mathcal{N}}$, where $\mathcal{N} = \{1, 2, \ldots, n\}$, is a basis for an algebraically free module F. If B is a pure submodule of F, let $\mathcal{S}_B = \mathrm{supp}(B\backslash\{0\})$ be the collection of non-empty subsets of $\mathcal{N}$ which are the supports of non-zero elements of B. And if $J \subseteq \mathcal{N}$, let $B_J = B \cap \langle s_j : j \in J\rangle = \{ x \in B \mid \mathrm{supp}(x) \subseteq J \}$.

The following are easily checked:

(A) If $J \in \mathcal{S}_B$, then J is minimal in $\mathcal{S}_B$ under inclusion iff B_J has rank 1.

(B) If $\{J_i\}_{i\in\mathcal{M}}$, where $\mathcal{M} = \{1, 2, \ldots, m\}$, are the minimal elements of $\mathcal{S}_B$ under inclusion, and for $i \in \mathcal{M}$, $B_{J_i} = \langle a_i\rangle$, then $B = \sum_{i\in\mathcal{M}}\langle a_i\rangle$.

The following result and its proof parallel a classical result of Butler for finite rank torsion-free abelian groups ([3], Theorem 14.1.4).

Theorem 1. *A valuated module B is Butler iff there is a completely decomposable valuated module E of finite rank and a pure submodule $K \subseteq E$ such that the quotient valuated module E/K is isometric to B.*

Proof. Suppose first that B is Butler, F is a completely decomposable valuated module with decomposition basis $\{s_j\}_{j\in\mathcal{N}}$ (where $\mathcal{N} = \{1, 2, \ldots, n\}$) and B is a pure submodule of F. Suppose $\{J_i\}_{i\in\mathcal{M}}$ (where $\mathcal{M} = \{1, 2, \ldots, m\}$) are the elements of $\mathcal{S}_B$ that are minimal under inclusion. For $i \in \mathcal{M}$, let $A_i = \langle a_i\rangle = B_{J_i}$. If E is the (external) valuated direct sum $\bigoplus_{i\in\mathcal{M}} A_i$, then E is a completely decomposable valuated module. Let $\phi : E \to B$ be the sum map; so by (B), ϕ

is a surjection. If K is the kernel of ϕ, then we claim that the resulting short exact sequence

$$0 \to K \to E \xrightarrow{\phi} B \to 0$$

is valuated; that is, for every ordinal γ,

$$0 \to K(\gamma) \to E(\gamma) \to B(\gamma) \to 0$$

is exact.

To verify this, we need only show that for each γ, $E(\gamma) \to B(\gamma)$ is surjective. First, it is easy to see that $F(\gamma) = \bigoplus_{j \in \mathcal{N}}[\langle s_j \rangle(\gamma)]$ is completely decomposable and $B(\gamma)$ is pure in $F(\gamma)$.

Next, suppose $\mathcal{N}^\gamma \subseteq \mathcal{N}$ is the set of all $j \in \mathcal{N}$ such that $\langle s_j \rangle(\gamma) \neq 0$. It is easily seen that

$$\mathcal{S}_{B(\gamma)} = \{\, S \in \mathcal{S}_B \mid S \subseteq \mathcal{N}^\gamma \,\}.$$

So the minimal sets in $\mathcal{S}_{B(\gamma)}$ will be $\{J_i\}_{i \in \mathcal{M}^\gamma}$, where $\mathcal{M}^\gamma = \{\, i \in \mathcal{M} \mid J_i \subseteq \mathcal{N}^\gamma \,\}$. In addition, it is clear that for each $i \in \mathcal{M}^\gamma$ that $(B(\gamma))_{J_i} = B_{J_i} \cap B(\gamma) = A_i(\gamma)$. Therefore, by (B) above (using $B = B(\gamma)$ and $F = F(\gamma)$), we have

$$B(\gamma) = \sum_{i \in \mathcal{M}^\gamma} A_i(\gamma) = \phi[E(\gamma)],$$

which proves this implication.

We prove the converse by induction on the rank of B. Certainly, if B has rank 1, then it is already completely decomposable and the result is trivial. So suppose B has rank exceeding 1, there is a valuated direct sum $E = \bigoplus_{i \in \mathcal{M}} A_i$, where again $A_i = \langle a_i \rangle$, and $\phi : E \to B$ has kernel K, so that B is isometric to the quotient valuated module E/K. There is no loss of generality in assuming that each $a_i \in B$ and ϕ is simply the sum map. In addition, replacing each A_i by its purification in B, we may clearly assume that each A_i is pure in B.

Now, for each $i \in \mathcal{M}$ we have a valuated quotient module $\widehat{B}_i := B/A_i$. Define a homomorphism as follows:

$$\mu : B \to \widehat{B} := \bigoplus_{i \in \mathcal{M}} \widehat{B}_i, \quad \text{where} \quad \mu(x) = \sum_{i \in \mathcal{M}} (x + A_i).$$

We claim that μ is a valuated embedding; that is, it does not increase values. Supposing otherwise, there is an $x \in B$ such $\gamma := |x|_B < |\mu(x)|_{\widehat{B}}$. This mean that for each $i \in \mathcal{M}$, we can find a $y_i \in A_i$ such that $|x|_B < |x + y_i|_B$. In particular, this means that $\gamma = |x|_B = |y_i|_B$. And since A_i is cyclic, this implies that $A_i(\gamma)/A_i(\gamma + 1)$ has p-rank 1, so that $E(\gamma)/E(\gamma + 1)$ has p-rank m.

On the other hand, in B, $y_i = -x + (x + y_i) \in \langle -x \rangle + B(\gamma + 1)$ for each $i \in \mathcal{M}$. So we have

$$B(\gamma) = \sum_{i \in \mathcal{M}} A_i(\gamma) = \sum_{i \in \mathcal{M}} \langle y_i \rangle \subseteq \langle -x \rangle + B(\gamma + 1) \subseteq B(\gamma).$$

Therefore, $B(\gamma)/B(\gamma + 1)$ has p-rank 1. Now there is a short exact sequence

$$0 \to K(\gamma)/K(\gamma + 1) \to E(\gamma)/E(\gamma + 1) \to B(\gamma)/B(\gamma + 1) \to 0.$$

Since $B(\gamma)/B(\gamma + 1)$ has p-rank 1 and $E(\gamma)/E(\gamma + 1)$ has p-rank m, we can conclude that $K(\gamma)/K(\gamma + 1)$ has p-rank $m - 1$. This means that the (torsion-free) rank of K is at least $m - 1$. However, since the rank of B is at least 2, the rank of K can be at most $m - 2$. This contradiction implies that μ is a valuated embedding.

Observe that if the above argument is applied not to $|\ |$, but to the height valuation on B and $\widehat{B}$, it would imply that μ does not increase heights either, i.e., $\mu(B)$ is pure in $\widehat{B}$.

Now, each $\widehat{B}_i$ is also the valuated image $E \to B \to \widehat{B}_i$. And since B_i will have rank $m - 1 < m$, by induction each $\widehat{B}_i$ can be embedded as a pure submodule in a completely decomposable valuated module. Therefore, $\widehat{B} = \bigoplus_{i \in \mathcal{M}} \widehat{B}_i$ can also be so embedded, which implies that our original $B \cong \mu(B) \subseteq \widehat{B}$ is Butler, as required.

QED

Proposition 3. *Suppose C is a full-rank valuated submodule of the finite rank algebraically free valuated module B. Then B is Butler iff C is Butler.*

Proof. Suppose first that C is Butler. By induction, it suffices to assume that $B/C \cong \mathbf{R}/p\mathbf{R}$. By Theorem 1, there is a completely decomposable valuated module E and a valuated surjection $\phi : E \to C$. Let $b \in B \setminus C$ be proper with respect to C. If we consider the valuated external direct sum $E \oplus \langle b \rangle$, it is straightforward to check that the map $E \oplus \langle b \rangle \to B$ given by $(e, \alpha b) \mapsto \phi(e) + \alpha b$ is a valuated surjection, showing that B is Butler as a valuated module.

Conversely, suppose B is Butler. If F is a finite rank completely decomposable valuated module containing B as a pure valuated submodule and $k < \omega$, then $p^k F$ will also be a finite rank completely decomposable valuated module containing $p^k B$ as a pure valuated submodule. Therefore $p^k B$ will always be Butler.

Clearly, for some $k < \omega$, $p^k B \subseteq C$ is full-rank. And since $p^k B$ is Butler, it follows from the first part of the proof that C will also be Butler. QED

Corollary 2. *If B is a Butler valuated module and $C \subseteq B$ is a valuated submodule, then C is also Butler.*

Proof. Suppose B is a pure valuated submodule of the completely decomposable valuated module F. Then the purification, C_*, of C in B will also be a pure valuated submodule of F, so that C_* is also Butler. And since C is full-rank in C_*, it follows from Proposition 3 that C is Butler, as required. QED

We will say a module M is **Butler-May** if it is an NT-realization of a (finite rank) Butler valuated module. It is clear that any finite rank Warfield module is a Butler-May module. And since any Butler valuated module has finite rank, by Proposition 1 any Butler-May module is a May module.

Corollary 3. *Suppose M is a Butler-May module. Then any free submodule $B \subseteq M$ is a Butler valuated module.*

Proof. Suppose M is an NT-realization of the Butler module C. It follows from Corollary 2 that $B \cap C \subseteq C$ is Butler. And since $B \cap C$ is full-rank in B, it follows from Proposition 3 that B is Butler. QED

Corollary 4. *Suppose M is a Butler-May module and $N \subseteq M$ is an isotype submodule. If N is a May module, then N is a Butler-May module.*

Proof. Let B be an NFT-submodule of N. By Corollary 3, B will be a Butler valuated module. So N is a Butler-May module, as required. QED

Corollary 5. *Suppose M is a countably generated Butler-May module. If $N \subseteq M$ is an isotype submodule, then N is a Butler-May module.*

Proof. If T_N is the torsion submodule of N, then since T_N is countably generated, by Proposition 2, N is a May module. So the result follows from Corollary 4. QED

Corollary 6. *Suppose M is a Butler-May module. If N is a summand of M, then N is also a Butler-May module.*

Proof. By Corollaries 1 and 4. QED

Recall that if $\overline{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \dots)$ is a strictly increasing sequence of ordinals and $k < \omega$, then $p^k\overline{\alpha} = (\alpha_k, \alpha_{k+1}, \alpha_{k+2}, \dots)$. We will say a valuated module B is *homogeneous* if for every pair of non-torsion elements $x, y \in B$ there are $j, k \in \omega$ such that $\|p^j x\| = p^j\|x\| = p^k\|y\| = \|p^k y\|$. In particular, this definition applies to ordinary modules with the height valuation.

The following result again parallels a classical result on Butler finite rank torsion-free groups ([3], Corollary 14.1.5).

Proposition 4. *Suppose M is a homogeneous finite rank module. Then M is a Butler-May module iff it is a Warfield module.*

Proof. Since any finite rank Warfield module is a Butler-May module, assume that M is a homogeneous Butler-May module with NFT-submodule B. So there are elements $b_1, \ldots, b_k \in B$ such that the sum map $E := \langle b_1 \rangle \oplus \cdots \oplus \langle b_k \rangle \to B$ is a valuated surjection (where, of course, E is a valuated direct sum).

For each $j \in \mathcal{K} := \{ 1, 2, \ldots, k \}$ we can find an $n_j < \omega$ such that whenever $j, j' \in \mathcal{K}$, we have $p^{n_j} \|b_j\| = p^{n_{j'}} \|b_{j'}\|$. Let $\overline{\alpha} = \|p^{n_1} b_1\|$ be this common value sequence. Note that $E(\alpha_0) = \langle b_1 \rangle(\alpha_0) \oplus \cdots \oplus \langle b_k \rangle(\alpha_0) \to B(\alpha_0)$ is a valuated surjection. In particular, this also means that $B(\alpha_0)$ is full-rank in M, and hence an NFT-submodule.

Replacing B with $B(\alpha_0)$ and each b_j with $p^{n_j} b_j$, we may assume that $\|b_j\| = \overline{\alpha}$ for all $j \in \mathcal{K}$. Observe that this means that we can compute the valuation on E simply in terms of $\overline{\alpha}$ and the height valuation on E. In other words, if $x \in E$ has p-height $m < \omega$, then $|x|_E = \alpha_m = |p^m b_1|$. But this can be seen to mean that any algebraic basis for E will also be a decomposition basis.

Let C be the kernel of the valuated surjection $E \to B$. If $c_1, \ldots, c_i$ is a (decomposition) basis for C which we extend to a (decomposition) basis $c_1, \ldots, c_i, d_1, \ldots, d_\ell$ for E, then it follows that there is an isometry $B \cong \langle d_1 \rangle \oplus \cdots \oplus \langle d_\ell \rangle$. Therefore, M is an NT-realization of a totally decomposable valuated module, so that it is Warfield, as required. QED

3 Examples

We want to construct a Butler-May module that is not a Warfield module. We start with a completely decomposable valuated module $F := \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$, where

$$\|x\| = (1, 3, 4, 7, 9, 10, 13, 15, 16, ...);$$
$$\|y\| = (1, 2, 5, 7, 8, 11, 13, 14, 17, ...);$$
$$\|z\| = (0, 3, 5, 6, 9, 11, 12, 15, 17, ...).$$

In other words, for all $m < \omega$, one of $|p^m x|$, $|p^m y|$ and $|p^m z|$ is $2m$ and the other two are $2m + 1$.

Let B be the collection of $v = \delta x + \epsilon y + \rho z \in F$ for which $\delta, \epsilon, \rho \in \mathbf{R}$ satisfy $\delta + \epsilon + \rho = 0$. Of course, B inherits a valuation from F. Since B is clearly pure in F, it follows that it is a Butler valuated module of rank 2.

It is easily seen that

(1) $\overline{\alpha} := \|x - y\| = (1, 2, 4, 7, 8, 10, 13, \ldots, 6n + 1, 6n + 2, 6n + 4, 6(n + 1) + 1, \ldots);$

(2) $\overline{\beta} := \|y - z\| = (0, 2, 5, 6, 8, 11, 12, \dots, 6n, 6n + 2, 6n + 5, 6(n + 1), \dots)$;
(3) $\overline{\gamma} := \|x - z\| = (0, 3, 4, 6, 9, 10, 12, \dots, 6n, 6n + 3, 6n + 4, 6(n + 1), \dots)$.

We also let

(4) $\overline{\mu} := \quad (0, 2, 4, 6, 8, 10, 12, \dots, 6k, 6k + 2, 6k + 4, 6(k + 1), \dots)$.

Now, if $v = \delta x + \epsilon y + \rho z \in B$, then $\delta + \epsilon + \rho = 0$ implies that at least two of $i := |\delta|_{\mathbf{R}}$, $j := |\epsilon|_{\mathbf{R}}$ and $k := |\rho|_{\mathbf{R}}$ are equal, and this common value is less than or equal to the third. This implies the following:

$$\|v\| = \begin{cases} p^i\overline{\alpha} & \text{if } i = j < k; \\ p^j\overline{\beta} & \text{if } j = k < i; \\ p^k\overline{\gamma} & \text{if } k = i < j; \\ p^i\overline{\mu} & \text{if } i = j = k. \end{cases} \tag{$\dagger$}$$

Example 1. There is a Butler-May module of rank 2 that is not a Warfield module.

Proof. Let B be as above and M be an NT-realization of B. Clearly, M is a Butler-May module, so we need only show that M is not Warfield. Assume otherwise, and let $C \subseteq M$ be an NFT-submodule that is a completely decomposable valuated module. Replacing C by p^nC for some $n < \omega$, we may assume that $C \subseteq B$. Suppose C is the valuated direct sum $\langle c_1\rangle \oplus \langle c_2\rangle$. It follows from ($\dagger$) that $|c_1|$ and $|c_2|$ must be multiplies of either $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ or $\overline{\mu}$. Suppose, for example, that $|c_1| = p^j\overline{\beta}$ and $|c_2| = p^k\overline{\gamma}$ (all of the other cases are totally analogous).

It follows from computations (1)-(4) that every *odd* element of the value spectrum $|C \setminus \{0\}| = p^j\overline{\beta} \cup p^k\overline{\gamma}$ is congruent to either 3 or 5 modulo 6. However, for a sufficiently large value of m, $p^{3m}(x-y) \in C$, and $|p^{3m}(x-y)|$ is congruent to 1 modulo 6. This contradiction shows that M is not Warfield. QED

The following shows that there are finite rank May modules that are not Butler-May modules.

Example 2. There is a May module of rank 2 that fails to be a Butler-May module.

Proof. For each $n \in \mathcal{N} := \{2, 3, \dots\}$, if we let $j_n = n^2 - n$, and $k_n = j_n/2 = n(n-1)/2$, then $j_n = 2, 6, 12, 20, \dots$, and $k_n = 1, 3, 6, 10, \dots$. We have relations:

$$n^2 = j_n + n, \quad j_{n+1} = n^2 + n = j_n + 2n, \quad k_{n+1} = k_n + n.$$

For $n \in \mathcal{N}$, let $u_n = 1 + p^{n-1}$; so, for example, $u_3 = 1 + p^2$. Clearly, each u_n is a unit of $\mathbf{R}$, and if $n, t \in \mathcal{N}$ with $n < t$, then $u_n \not\equiv u_t \pmod{p^n}$.

Let B be an algebraically free module with basis e_1, e_2. For each $n \in \mathcal{N}$, let $v_n = e_1 + u_n e_2 \in B$. We define a valuation on B as follows: Let $B(0) = B$, $B(1) = pB$. If $\alpha \in \mathcal{N}$, then we can find a unique $n \in \mathcal{N}$ such that $j_n \leq \alpha < j_{n+1}$. In turn, there is a unique $i \in \{0, 1, \ldots, n-1\}$ such that exactly one of $\alpha = j_n + i$ or $\alpha = n^2 + i$ holds. If $j_n \leq \alpha = j_n + i < n^2$, we let

$$B(\alpha) = p^{k_n}(\langle e_1 + e_2 \rangle + p^i B),$$

so for all $n \in \mathcal{N}$ we have $B(j_n) = p^{k_n} B$. If $n^2 \leq \alpha = n^2 + i < j_{n+1}$, we let

$$B(\alpha) = p^{k_n}(p^i \langle v_n \rangle + p^n B).$$

Observe that

$$B(n^2) = p^{k_n}(\langle e_1 + u_n e_2 \rangle + p^n B),$$

and if $n^2 < \alpha = n^2 + i < j_{n+1}$, then

$$B(\alpha) = p^{k_n}(p^i \langle e_1 + (1 + p^{n-1}) e_2 \rangle + p^n B) = p^{k_n}(p^i \langle e_1 + e_2 \rangle + p^n B).$$

It is easily seen that whenever $\alpha < \omega$, $B(\alpha + 1) \subseteq B(\alpha)$ and $pB(\alpha) \subseteq B(\alpha + 1)$. Therefore, this descending sequence of submodules is determined by a valuation on B.

Let $\mathcal{S} = \{ n^2 \mid n \in \mathcal{N} \} = \{4, 9, 16, \ldots\}$. For each $n \in \mathcal{N}$, $|p^{k_n} v_n| = n^2$, so we can conclude that $\mathcal{S} \subseteq |B|$. In addition, it is straightforward to verify that for all $m < \omega$, $|p^m e_1| = |p^m e_2|$ (i.e., e_1, e_2 have the same value sequence) and that this common value is never an element of $\mathcal{S}$.

We claim that for any non-zero $b \in B$, the value sequence $\|b\|$ contains at most one element of $\mathcal{S}$. Let $b = \sigma e_1 + \tau e_2$. If $|\sigma|_{\mathbf{R}} \neq |\tau|_{\mathbf{R}}$, then without loss of generality, assume $m := |\sigma|_{\mathbf{R}} < |\tau|_{\mathbf{R}}$. So

$$|\sigma e_1| = |p^m e_1| = |p^m e_2| < |\tau e_2|.$$

It follows that $|b| = |p^m e_1|$ which we know is not in $\mathcal{S}$.

On the other hand, suppose $m := |\sigma|_{\mathbf{R}} = |\tau|_{\mathbf{R}}$; let $b' = p^{-m} b$. It follows that $\|b\| = p^m \|b'\| \subseteq \|b'\|$; so after possibly replacing b by b', there is no loss of generality in assuming that $m = 0$. This implies that σ and τ will be units in $\mathbf{R}$. Now if $b'' = \sigma^{-1} b$, then it trivially follows that $\|b''\| = \|b\|$; so after possibly replacing b by b'', there is no loss of generality in assuming that $b = e_1 + \tau e_2$.

Assume $\ell \in \omega$ and $|p^\ell b| = n^2$ for some $n \in \mathcal{N}$. It follows that $p^\ell b \in B(n^2) \setminus B(n^2 + 1)$, so that

$$p^\ell (e_1 + \tau e_2) = p^\ell b = \gamma p^{k_n} v_n + c = \gamma p^{k_n}(e_1 + u_n e_2) + c,$$

where $c \in p^{k_n+n}B$ and $\gamma \in \mathbf{R}$ is a unit. By considering heights in B, we must have $\ell = k_n$. Considering the e_1 coordinates of the two sides of this equation, we can conclude that $p^{k_n} \equiv \gamma p^{k_n} \pmod{p^{k_n+n}}$, i.e., $\gamma \equiv 1 \pmod{p^n}$. And considering the e_2 coordinates, we can conclude that $p^{k_n}\tau \equiv \gamma p^{k_n} u_n \pmod{p^{k_n+n}}$, i.e., $\tau \equiv \gamma u_n \equiv u_n \pmod{p^n}$.

Suppose there is a second $t > n$ such that t^2 also appears in $\|b\|$. The above computation will imply that $\tau \equiv u_t \pmod{p^t}$, which implies that $u_n \equiv \tau \equiv u_t \pmod{p^n}$. This contradicts the construction of the us, and completes the proof of the claim.

Now let M be any NT-realization of B, so that M is a May module. If M were Butler-May, it would follow from Corollary 3 that B is Butler as a valuated module. This would mean that we could find a finite collection of elements $b_1, \dots, b_k \in B$ such that the sum map, $\langle b_1 \rangle \oplus \cdots \oplus \langle b_k \rangle \to B$, is a valuated surjection.

Since the kernel of this map is nice, this would imply that

$$|B| \subseteq |\langle b_1 \rangle \oplus \cdots \oplus \langle b_k \rangle| = |\langle b_1 \rangle| \cup \cdots \cup |\langle b_k \rangle|.$$

However $\mathcal{S} \subseteq |B|$, whereas each $|\langle b_j \rangle|$ has at most one element of $\mathcal{S}$. This contradiction shows that B cannot be Butler. QED

4 Jacobson Radicals of Endomorphism Rings

To review, for any module M we denote the endomorphism ring of M by $E(M)$ and the Jacobson radical of $E(M)$ by $\mathcal{J}(M)$. As in [4], we will write $E(M)$ and $\mathcal{J}(M)$ as operating on M on the right, so M is a left $\mathbf{R}$, right $\mathcal{J}(M)$-module. We let $p_M \in E(M)$ be defined by $xp_M = px$ for all $x \in M$.

In this section we strengthen a result from [2].

For an arbitrary module M with torsion submodule T, let $M_a \subseteq M$ be defined by the condition that M_a/T is the maximal divisible submodule of M/T. Clearly, T and M_a are fully invariant in M (i.e., for all $\phi \in E(M)$ we have $T\phi \subseteq T$ and $M_a\phi \subseteq M_a$). And trivially, $M_f := M/M_a$ is a reduced torsion-free module.

We begin with a generalization of ([2], Proposition 11). In the proof we do not actually need that our ring is complete, only that it is a discrete valuation ring. In addition, we do not require that we are looking at Warfield modules, or even May modules.

Proposition 5. *Suppose R is a discrete valuation ring with completion $\mathbf{R}$, and $p \in R$. If M is any finite rank reduced R-module, then $p_M \in \mathcal{J}(M)$.*

Proof. Suppose $\phi \in E(M)$; we need to show that $1_M + p_M\phi$ is an automorphism of M.

First, observe that $1_M + p_M\phi$ is necessarily injective. So see this, suppose $x \neq 0$ is in its kernel. We have

$$x = x + px\phi - px\phi = x(1_M + p_M\phi) - px\phi = -px\phi.$$

Using the height valuation, this implies that

$$|x|_M = |px\phi|_M > |x\phi|_M \geq |x|_M,$$

which clearly cannot happen when M is reduced.

Next, since T is fully invariant, ϕ restricts to an endomorphism $\phi_T \in E(T)$. The homomorphism $1_T + p_T\phi_T$ agrees with the identity on the socle $T[p]$, so that it is an automorphism of T.

Since $1_M + p_M\phi$ restricts to an automorphism on T and is injective on the fully invariant subgroup M_a, it readily follows that it induces an injective endomorphism on M_a/T. However, this quotient is a finite rank torsion-free divisible module. This means that this induced endomorphism on M_a/T is also surjective, and hence an automorphism. This easily implies that $1_M + p_M\phi$ restricts to an automorphism of M_a.

Finally, $1_M + p_M\phi$ will induce an endomorphism $1_{M_f} + p_{M_f}\phi_{M_f}$ on M_f. If we knew $R = \mathbf{R}$ was complete, we could conclude that M_f is a finite rank free module, and it would follow that $p_{M_f} \in \mathcal{J}_{M_f}$. Therefore, $1_{M_f} + p_{M_f}\phi_{M_f}$ would be an automorphism of M. This would imply that $1_M + p_M\phi_M$ is an automorphism, completing the proof.

So consider the case where $R \neq \mathbf{R}$ is not complete. Note that $N := (M_f)^\bullet = \mathrm{Ext}(\mathbf{K}, M_f)$ is a finite rank free $\mathbf{R}$-module containing M_f as a pure R-submodule with R-torsion-free divisible quotient. By the last paragraph, $\gamma := 1_{M_f} + p_{M_f}\phi_{M_f}$ will induce an automorphism of N such that $(M_f)\gamma \subseteq M_f$. It will suffice to prove that γ is surjective when restricted to M_f, so suppose $y \in M_f$. We know that $y = x\gamma$ for some $x \in N$. Since M_f has finite R-rank and γ is injective on M_f it follows that for some $k < \omega$ we have $p^k y \in (M_f)\gamma$. So if $p^k y = z\gamma$ where $z \in M_f$, then $p^k x\gamma = p^k y = z\gamma$, so that $p^k x = z \in M_f$. But since N/M_f is torsion-free, we can conclude that $x \in M_f$. Therefore, $y = x\gamma \in (M_f)\gamma$, as required. QED

Proposition 5 often fails if M is either of infinite rank or not reduced. To see the first, suppose $M = \bigoplus_{i<\omega} \langle b_i \rangle$ is a free module of countable rank. If $\phi \in E(M)$ is defined by $b_i\phi = b_{i+1}$ for all $i < \omega$, then it is easily seen that $1_M + p_M\phi \in E(M)$ is not surjective (b_0 fails to be in its image); so $p_M \notin \mathcal{J}(M)$. To see the second, suppose M is any non-zero torsion-free divisible module of finite rank. If $\phi \in E(M)$ is multiplication by $-p^{-1}$, then $1_M + p_M\phi = 0$; and again $p_M \notin \mathcal{J}(M)$

Suppose again that $\mathbf{R}$ is complete. So if M_f has finite rank, then it is free and there is a splitting $M \cong M_f \oplus M_a$. In this splitting the submodule M_f is uniquely determined up to isomorphism, but there may be many different submodules that are complementary summands of M_a. For our purposes, it will not matter which complementary summand is chosen.

The next result is a restatement of ideas from [2].

Lemma 1. *Suppose M is a reduced module with unbounded torsion T. Let $\mathbf{t}\mathcal{J}(M)$ be the torsion subalgebra of $\mathcal{J}(M)$.*

(a) $L_M := \{\phi \in \mathcal{J}(M) \mid \mathbf{t}\mathcal{J}(M)\phi = 0\} = \{\phi \in \mathcal{J}(M) \mid M_a\phi = 0\}$.

(b) $A_M := \{\gamma \in L_M \mid \gamma L_M = 0\} = \{\gamma \in L_M \mid M\gamma \subseteq M_a\}$.

(c) If M_f has rank at least $n < \omega$, then L_M/A_M has rank at least n^2.

Proof. Regarding (a), suppose $M_a\phi = 0$. If $\alpha \in \mathbf{t}\mathcal{J}(M)$, then $M\alpha\phi \subseteq T\phi \subseteq M_a\phi = 0$, from which it follows that $\phi \in L_M$.

Conversely, suppose $\phi \in L_M$. For every $t \in T$, it is straightforward to construct an $\alpha \in \mathbf{t}\mathcal{J}(M)$ such that $t \in M\alpha$. It follows that $t\phi \in M\alpha\phi = M0 = 0$. Therefore, $T\phi = 0$. And since M_a/T is divisible and M is reduced, we must have $M_a\phi = 0$.

Turning to (b), if $\gamma \in L_M$ satisfies $M\gamma \subseteq M_a$, then by (a), for all $\phi \in L_M$ we have $M\gamma\phi \subseteq M_a\phi = 0$. Therefore, $\gamma L_M = 0$, i.e., $\gamma \in A_M$.

Conversely, suppose $\gamma \in A_M$. If $M\gamma$ is not contained in M_a, then find an $m \in M\gamma$ such that $m \notin M_a$. It follows that we can find a decomposition $M = C \oplus Z$, where C is a cyclic summand, $M_a \subseteq Z$ and $m = c + z$, with $z \in Z$ and $0 \neq c \in C$. If ϕ is multiplication by p on C and $C\phi = 0$, then clearly $\phi \in \mathcal{J}(M)$. By (a), $\phi \in L_M$. And since $0 \neq pc = m\phi \in M\gamma\phi = M0 = 0$, this contradiction shows that we must have $M\gamma \subseteq M_a$.

Regarding (c), our hypotheses guarantee that there is a decomposition $C \oplus Z$, where C is a free module of rank n and $M_a \subseteq Z$. If λ is any element of $\mathcal{J}(C) = pE(C)$, then extend it to $\widehat{\lambda} \in E(M)$ by setting $\widehat{\lambda}(Z) = 0$. If $\lambda \neq 0$, then it is easy to check that $\widehat{\lambda} \in L_M \setminus A_M$. Therefore, $\lambda \mapsto \widehat{\lambda}$ gives an injective homomorphism $pE(C) \to L_M/A_M$, completing the argument. QED

Suppose once again that M_f has finite rank, so that $M \cong M_f \oplus M_a$. Since M_a is fully-invariant, it is well-known that we can identify $\mathcal{J}(M)$ with $\mathcal{J}(M_f) \oplus \mathrm{Hom}(M_f, M_a) \oplus \mathcal{J}(M_a)$. Observe that by (a) we can identify L_M with $\mathcal{J}(M_f) \oplus \mathrm{Hom}(M_f, M_a)$. From this it follows that there is a natural isomorphism $\mathcal{J}(M_a) \cong \mathcal{J}(M)/L_M$. Similarly, by (b) we can identify A_M with $\mathrm{Hom}(M_f, M_a)$, so that there is a natural isomorphism $L_M/A_M \cong \mathcal{J}(M_f) = pE(M_f)$; in particular, L_M/A_M is a finite rank free module.

This leads to the following result, which is implicit in the discussions in [2].

Proposition 6. *Suppose M is a reduced module of finite rank with unbounded torsion T and N is a second reduced module such that there is an isomorphism $\mathcal{J}(M) \to \mathcal{J}(N)$.*

(a) N_f has finite rank and the isomorphism

$$\mathcal{J}(M_f) \to L_M/A_M \to L_N/A_N \to \mathcal{J}(N_f)$$

is induced by an isomorphism $M_f \to N_f$.

(b) There is an isomorphism

$$\mathcal{J}(M_a) \cong \mathcal{J}(M)/L_M \cong \mathcal{J}(N)/L_N \cong \mathcal{J}(N_a).$$

Proof. If $\phi \in \mathcal{J}(M)$, let ϕ' denote the corresponding element of $\mathcal{J}(N)$. By ([2], Theorem 3.4), if T' is the torsion submodule of N, then there is an isomorphism $\gamma : T \to T'$ such that the assignment $\phi|_T \mapsto \phi'|_{T'}$ is induced by γ. In particular, T' is also unbounded.

Regarding (a), we know that $L_M/A_M \cong L_N/A_N$ has finite rank. Therefore, Lemma 1(c) implies that N_f also has finite rank. In particular, there is a splitting $N \cong N_f \oplus N_a$ and the stated isomorphism follows. By ([1], Theorem 3.3), this isomorphism is induced by an isomorphism $M_f \to N_f$, as required.

Finally, (b) follows immediately from the above discussion. QED

Next, we consider when the isomorphism in Proposition 6(b) is also induced. In [4] a May module M was said to be *E-torsion* if the restriction map $E(M) \to E(T)$ is an isomorphism. It was shown ([4], Theorem 3.2) that when M is not torsion, then it is E-torsion iff (1) the length of T equals $\mu + n$, where $n < \omega$ and μ is a limit ordinal of uncountable cofinality and (2) $p^{\mu+n}M$ is a finite-rank NFT-submodule of M.

Theorem 2. *Suppose M is a May module of finite rank with unbounded torsion T and M/T is divisible (i.e., $M = M_a$). Let N be a second reduced module. If either N is also a May module or M is not E-torsion, then any isomorphism $\mathcal{J}(M) \to \mathcal{J}(N)$ is induced by a module isomorphism $M \to N$.*

Proof. Again using ([2], Theorem 3.4), we can identify the torsion submodules of M and N in such a way that if $\phi \in \mathcal{J}(M)$ corresponds to $\phi' \in \mathcal{J}(N)$, then ϕ and ϕ' restrict to the same endomorphism of T.

Let $T^\bullet = \mathrm{Ext}(\mathbf{K}, T)$ be the cotorsion-hull of T. It follows that we can think of T as the torsion submodule of $T^\bullet$ and the quotient $T^\bullet/T$ is divisible. It is straightforward to see that this means that we can also view M as a submodule of $T^\bullet$. In addition, we can identify $E(M)$ with the collection of all $\phi \in E(T^\bullet)$ such that $M\phi \subseteq M$ (see, for example, [4]. Lemma 2.6).

By Proposition 6(a), we know $N_f = 0$, so that N/T is divisible as well. So as in the last paragraph, we may assume $N \subseteq T^\bullet$ and $E(N) = \{ \phi \in E(T^\bullet) \mid N\phi \subseteq N \}$.

This means that $E(M)$ and $E(N)$ are subrings of $E(T^\bullet)$ and this identification can be set up so that $\mathcal{J} := \mathcal{J}(M) = \mathcal{J}(N)$. Since M/T is divisible and torsion-free, so is $T^\bullet/M$. Now, if $\phi \in E(T^\bullet)$, $\alpha \in \mathbf{R}$ and $\alpha\phi \in E(M)$, then $\alpha\phi(M) \subseteq M$ implies $\phi(M) \subseteq M$, i.e., $E(T^\bullet)/E(M)$ is torsion-free. The same reasoning shows $E(T^\bullet)/E(N)$ is torsion-free.

Since M has finite rank, by Proposition 5, $p_M = p_N$ is in this common Jacobson radical, $\mathcal{J}$. Since $E(T^\bullet)/E(M)$ and $E(T^\bullet)/E(N)$ are torsion-free, we can conclude that

$$E(M) = \{ \phi \in E(T^\bullet) \mid p\phi \in \mathcal{J} \} = E(N).$$

Suppose first that M is not E-torsion. Then it follows from ([4], Theorem 3.5) that $M = N$, so that our isomorphism is induced. Similarly, suppose N is also assumed to be a May module. Then it follows from ([4], Proposition 3.7) that $M = N$, and again, our isomorphism is induced. QED

Putting together Proposition 6 and Theorem 2 we have the following result:

Corollary 7. *Suppose M is a May module of finite rank with unbounded torsion T. Let N be a reduced module such that there is an isomorphism $\mathcal{J}(M) \to \mathcal{J}(N)$. If either N is also a May module or M_a is not E-torsion, then $M \cong N$.*

Observe that if M is a mixed E-torsion May module, then we will have $E(M) \cong E(T)$, and so $\mathcal{J}(M) \cong \mathcal{J}(T)$. However, since M is mixed, it is not isomorphic to T.

It is worth emphasizing that Corollary 7 generalizes ([2], Theorem 5.2) in two ways. In both results we had M of finite rank, T unbounded and N a second reduced module for which there is an isomorphism $\mathcal{J}(M) \to \mathcal{J}(N)$ and we wanted to conclude $M \cong N$. In the earlier result both M and N were assumed to be Warfield modules. First, our result holds if both M and N are in the more general class of May modules. Second, if M is a non-E-torsion May module, then we actually do not need to assume anything additional about N.

5 An Example

Suppose, as in Corollary 7, M is a finite rank May module with unbounded torsion and N is a reduced module such that there is an isomorphism $\mathcal{J}(M) \to \mathcal{J}(N)$. If either N is a May module or M_a is not E-torsion, then we concluded that there is an isomorphism $M \to N$, but we did not claim that this isomorphism induces the given isomorphism of their Jacobson radicals. In this section we show that it may not, in fact, be possible to "glue together" the isomorphism $M_f \to N_f$ from Proposition 6(a) and the isomorphism $M_a \to N_a$ from Theorem 2 in such a way as to induce our given $\mathcal{J}(M) \to \mathcal{J}(N)$. In so doing we answer in the negative a question from [2].

In this section, T will denote some reduced unbounded torsion module. We will use the following elementary observation.

Lemma 2. *If $t \in T \setminus p^\omega T$, then there is a $\nu \in \mathcal{J}(T)$ such that $t\nu \notin p^\omega T$.*

Proof. It can be seen that there is a decomposition $T = B \oplus C$, where B is a maximal p^k-bounded summand of T and $t = b + c$, where $0 \neq b \in B$ and $c \in C$. There is clearly an endomorphism $\nu \in E(T)$ such that $B\nu \subseteq C$, $b\nu \notin p^\omega T$ and $C\nu = 0$. It is elementary to show that $\nu \in \mathcal{J}(T)$. QED

We mention in passing the following easy and familiar factoid.

Lemma 3. *Suppose α is an ordinal such that $E(T)$ acts transitively on the α-th Ulm factor $U := (p^\alpha T)[p]/(p^{\alpha+1}T)[p]$; in other words, if $x, y \in U$ and $x \neq 0$, then there is an endomorphism $\phi \in E(T)$ such that the induced endomorphism $\phi_U \in E(U)$ maps x to y. If $\nu \in \mathcal{J}(T)$, then $((p^\alpha T)[p])\nu \subseteq (p^{\alpha+1}T)[p]$*

Proof. If this failed for some $\nu \in \mathcal{J}(T)$, then the induced endomorphism $\nu_U \in E(U)$ would be non-zero. This means we could find a non-zero $x \in U$ such that $y := x\nu_U \neq 0$. By hypothesis, we could find a $\phi \in E(T)$ such that $y\phi_U = -x$. This would imply that $x(1_U + \nu_U\phi_U) = x + y\phi_U = x - x = 0$. Therefore, $1_U + \nu_U\phi_U$ is not an automorphism of U, so that $1_T + \nu\phi$ is not an automorphism of T. This contradicts that $\nu \in \mathcal{J}(T)$ and completes the argument. QED

The last result applies, for example, whenever $\alpha < \omega$ is finite, T is totally projective or the α-th Ulm factor is isomorphic to $\mathbf{R}/p\mathbf{R}$. The following apparently technical observation is a key step in our construction.

Lemma 4. *If $\mathcal{J}(T)^2 \subseteq \mathcal{J}(T)$ is the submodule generated by all products $\nu\mu$ for $\nu, \mu \in \mathcal{J}(T)$, then the quotient $\mathcal{J}(T)/\mathcal{J}(T)^2$ is p-bounded and non-zero.*

Proof. Since $p \in \mathcal{J}(T)$, $\mathcal{J}(T)/\mathcal{J}(T)^2$ is clearly p-bounded.

We will assume that f_T, the Ulm function of T, satisfies $f_T(0) \neq 0$, $f_T(1) \neq 0$ (if the first two non-zero Ulm invariants are larger than 0,1, an obvious translation of our argument works). So $T = \langle x \rangle \oplus \langle y \rangle \oplus T'$, where $\langle x \rangle \cong \mathbf{R}/p\mathbf{R}$ and $\langle y \rangle \cong \mathbf{R}/p^2\mathbf{R}$. Consider $\gamma \in E(T)$ defined as follows: $x\gamma = py$ and $(\langle y \rangle \oplus T')\gamma = 0$; clearly $\gamma \in \mathcal{J}(T)$. Note that by Lemma 3, if ν is any element of $\mathcal{J}(T)^2$, then $x\nu \in (p^2T)[p]$, so that $x\nu \neq py$. This shows that $\gamma \notin \mathcal{J}(T)^2$, so that $\gamma + \mathcal{J}(T)^2$ is a non-zero element of our quotient. QED

We put the last result to work in the next.

Lemma 5. *Suppose $p^\omega T \neq 0$ is p-bounded and $E(T)$ acts transitively on $p^\omega T$. There is a (module) homomorphism $\phi : \mathcal{J}(T) \to T$, which we will write on the left, such that*

(a) $\phi(\nu) \in p^\omega T = (p^\omega T)[p]$ for every $\nu \in \mathcal{J}(T)$;

(b) $\phi(\nu\mu) = 0 = \phi(\nu)\mu$ for every $\nu, \mu \in \mathcal{J}(T)$;

(c) it is not the case that there is a fixed $t \in T$ such that $\phi(\nu) = t\nu$ for every $\nu \in \mathcal{J}(T)$.

Proof. Using Lemma 4, we can construct a non-zero (module) homomorphism $\mathcal{J}(T)/\mathcal{J}(T)^2 \to p^\omega T$. Define ϕ to be the composite $\mathcal{J}(T) \to \mathcal{J}(T)/\mathcal{J}(T)^2 \to p^\omega T \subseteq T$. Now, (a) is immediate. As to (b), since $\nu\mu \in \mathcal{J}(T)^2$, $\phi(\nu\mu) = 0$; and since $\phi(\nu) \in (p^\omega T)[p]$, by Lemma 3, $\phi(\nu)\mu \in p^{\omega+1}T = 0$.

Finally, regarding (c), we assume such a t exists and derive a contradiction. Suppose first that $t \in p^\omega T$. So by Lemma 3, $\phi(\nu) = t\nu \in p^{\omega+1}T = 0$ for all $\nu \in \mathcal{J}(T)$, contradicting that ϕ is non-zero. Next, if $t \notin p^\omega T$, then by Lemma 2 we can find a $\nu \in \mathcal{J}(T)$ such that $t\nu \notin p^\omega T$. This would then imply that $\phi(\nu) = t\nu \notin p^\omega T$, which again contradicts that $\phi(\mathcal{J}(T)) \subseteq p^\omega T$, completing the proof. QED

This brings us to the main step in our construction. It clearly works whenever **R** is a discrete valuation ring, and in particular, when it is complete.

Theorem 3. *Suppose $p^\omega T \neq 0$ is p-bounded and $E(T)$ acts transitively on $p^\omega T$. If $M := \mathbf{R} \oplus T$, then there is an automorphism $\Phi : \mathcal{J}(M) \to \mathcal{J}(M)$ that is not induced by an automorphism $M \to M$. That is, not every automorphism of $\mathcal{J}(M)$ is inner.*

Proof. We will express the elements of M as row vectors, and endomorphisms on M will be represented as right multiplication by matrices.

We identify $E(\mathbf{R})$ with $\mathbf{R}$ and $\mathrm{Hom}(\mathbf{R}, T)$ with T. With this, we can identify

$$E(M) = \begin{bmatrix} \mathrm{Hom}(\mathbf{R},\mathbf{R}) & \mathrm{Hom}(\mathbf{R},T) \\ \mathrm{Hom}(T,\mathbf{R}) & \mathrm{Hom}(T,T) \end{bmatrix} = \begin{bmatrix} \mathbf{R} & T \\ 0 & E(T) \end{bmatrix}.$$

In this identification we have

$$\mathcal{J}(M) = \begin{bmatrix} p\mathbf{R} & T \\ 0 & \mathcal{J}(T) \end{bmatrix}.$$

Let $\phi : \mathcal{J}(T) \to T$ be as in Lemma 5. Now, let $\Phi : \mathcal{J}(M) \to \mathcal{J}(M)$ (which we will also write on the left) be defined as follows:

$$\Phi\left(\begin{bmatrix} \alpha & x \\ 0 & \nu \end{bmatrix}\right) = \begin{bmatrix} \alpha & x + \phi(\nu) \\ 0 & \nu \end{bmatrix},$$

where $\nu \in \mathcal{J}(T)$, $x \in T$ and $\alpha \in p\mathbf{R}$. Clearly, Φ is an automorphism of modules, where

$$\Phi^{-1}\left(\begin{bmatrix}\alpha & x\\ 0 & \nu\end{bmatrix}\right) = \begin{bmatrix}\alpha & x-\phi(\nu)\\ 0 & \nu\end{bmatrix}.$$

We need to show that Φ preserves products. Suppose $A, B \in \mathcal{J}(M)$, where

$$A = \begin{bmatrix}\alpha & x\\ 0 & \nu\end{bmatrix}, \quad B = \begin{bmatrix}\beta & y\\ 0 & \mu\end{bmatrix}.$$

We have

$$\Phi(AB) = \Phi\left(\begin{bmatrix}\alpha\beta & \alpha y + x\mu\\ 0 & \nu\mu\end{bmatrix}\right) = \begin{bmatrix}\alpha\beta & \alpha y + x\mu + \phi(\nu\mu)\\ 0 & \nu\mu\end{bmatrix}$$

and

$$\Phi(A)\Phi(B) = \begin{bmatrix}\alpha & x+\phi(\nu)\\ 0 & \nu\end{bmatrix}\begin{bmatrix}\beta & y+\phi(\mu)\\ 0 & \mu\end{bmatrix} = \begin{bmatrix}\alpha\beta & \alpha y + \alpha\phi(\mu) + x\mu + \phi(\nu)\mu\\ 0 & \nu\mu\end{bmatrix}$$

By Lemma 5(b) we have $\phi(\nu\mu) = 0 = \phi(\nu)\mu$. And since $\alpha \in p\mathbf{R}$, by Lemma 5(a) we have $\alpha\phi(\mu) = 0$, so the two sides are equal. (In fact, $\Phi(AB) = AB = \Phi(A)\Phi(B)$.)

We want to show that Φ is not induced by an automorphism $P : M \to M$; so we assume it is and derive a contradiction. Using our matrix representation, P must be right multiplication by a matrix of the form

$$P = \begin{bmatrix}\gamma & c\\ 0 & \delta\end{bmatrix} \in E(M),$$

where $\delta \in E(T)$, $c \in T$ and $\gamma \in \mathbf{R}$. Since P is an automorphism of M, δ must be an automorphism of T and γ must be a unit in $\mathbf{R}$. It follows that for any $\nu \in \mathcal{J}(T)$ we have

$$\begin{aligned}\begin{bmatrix}0 & \phi(\nu)\\ 0 & \nu\end{bmatrix} &= \Phi\left(\begin{bmatrix}0 & 0\\ 0 & \nu\end{bmatrix}\right)\\ &= P^{-1}\begin{bmatrix}0 & 0\\ 0 & \nu\end{bmatrix}P\\ &= \begin{bmatrix}\gamma^{-1} & -\gamma^{-1}c\delta^{-1}\\ 0 & \delta^{-1}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \nu\end{bmatrix}\begin{bmatrix}\gamma & c\\ 0 & \delta\end{bmatrix}\\ &= \begin{bmatrix}\gamma^{-1} & -\gamma^{-1}c\delta^{-1}\\ 0 & \delta^{-1}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \nu\delta\end{bmatrix}\\ &= \begin{bmatrix}0 & -\gamma^{-1}c\delta^{-1}\nu\delta\\ 0 & \delta^{-1}\nu\delta\end{bmatrix}\end{aligned}$$

Let $t := -\gamma^{-1}c \in T$. For all $\nu \in \mathcal{J}(T)$, looking at the lower right corner, we have $\nu = \delta\nu\delta^{-1}$. And then looking at the upper right corner we have

$$\phi(\nu) = -\gamma^{-1}c\delta^{-1}\nu\delta = (-\gamma^{-1}c)(\delta^{-1}\nu\delta) = t\nu.$$

Since this contradicts Lemma 5(c), such a P cannot exist, and the result is established. QED

Example 3. There is a Warfield module M of torsion-free rank 1 and unbounded torsion such that there is an automorphism $\Phi : \mathcal{J}(M) \to \mathcal{J}(M)$ that is not induced by an automorphism $M \to M$.

Proof. In Theorem 3, just let T be, for example, a reduced countable group of length $\omega + 1$. QED

Again, Example 3 provides a counter-example to ([2], Theorem 4.1).

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