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Ulam Stabilities via Pachpatte's Inequality for Volterra–Fredholm Delay Integrodifferential Equations in Banach Spaces

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Abstract. In this paper, we will investigate Ulam–Hyers and Ulam–Hyers–Rassias stabilities of nonlinear Volterra–Fredholm delay integrodifferential equations in Banach spaces. Pachpatte's inequality and its extended version are utilized to obtain these stabilities. Examples are given in support of the results we obtained.

Keywords: Volterra–Fredholm Integrodifferential Equations, Ulam–Hyers stability; Ulam–Hyers–Rassias stability; Integral inequality.

MSC 2000 classification: 45N05, 45M10, 34G20, 35A23.

1 Introduction

The Ulam stability problem of functional equation [1] have been extended to different kinds of equations such as ordinary differential equations, integral equations, difference equations, fractional differential equations and Partial differential equations. In the past recent years, several authors proved the Ulam–Hyers and Ulam–Hyers–Rassias stabilities of various forms of differential and integrodifferential equations by utilizing different techniques [2, 3, 4, 5, 6, 7, 10, 11, 12].

Rus [8] and Akkouchi et al. [9] by utilizing the tools of Gronwall inequality and fixed point technique, investigated the Ulam–Hyers and Ulam–Hyers–Rassias stabilities of ordinary semilinear differential equations in Banach space

$$x'(t) = Ax(t) + f(t, x(t)),\ t \in I = [a, b] \text{ or } [a, +\infty),$$

where $A : X \to X$ is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t\geq 0}$ in a Banach space $(X, \|\cdot\|)$ (see[16], [19]).

Recently, Kucche and Shikhare [13] by employing Pachpatte's inequality extended the study of [8, 9] to semilinear Volterra integrodifferential equations

$$x'(t) = Ax(t) + f\left(t, x(t), \int_0^t g(t, s, x(s))ds\right), \ t \in J = [0, b] \text{ or } [0, +\infty), \tag{1.1}$$

and semilinear Volterra delay integrodifferential equations

$$x'(t) = Ax(t) + f\left(t, x_t, \int_0^t g(t, s, x_s)ds\right), \ t \in J = [0, b] \text{ or } [0, +\infty), \tag{1.2}$$

in a Banach space.

Inspired by the work mentioned above, in this paper, by employing Pachpatte's inequality and its extended version, we investigate the Ulam–Hyers and Ulam–Hyers–Rassias stabilities of semilinear Volterra–Fredholm delay integrodifferential equations (VFDIDE)

$$x'(t) = Ax(t) + f\left(t, x_t, \int_0^t g_1(t, s, x_s)ds, \int_0^b g_2(t, s, x_s)ds\right), \ t \in J = [0, b], \ 0 < b < \infty, \tag{1.3}$$

in a Banach space $(X, \|\cdot\|)$, where $A : X \to X$ is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t\geq 0}$, $f : J \times C \times X \times X \to X$ and $g_i : J \times J \times C \to X$ $(i = 1, 2)$ are given continuous nonlinear functions, $C = C([-r, 0], X)$ is the Banach space of continuous functions endowed with supremum norm $\|\cdot\|_C$, $B = C([-r, b], X)$ is the Banach space of all continuous functions with supremum norm $\|\cdot\|_B$ and for any $x \in B$, $t \in [0, b]$ we denote by x_t the element of C defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$.

The novelty of this paper is that by applying the Pachpatte inequality, we have obtained Ulam–Hyers and Ulam–Hyers–Rassias stability results for more general equation (1.3) with the only Lipschitz type conditions on the functions f and g_i $(i = 1, 2)$ involved in the equation. Further, the results obtained in this paper includes the study of [8] and [9] (when $r = 0$, $g_i = 0\,(i = 1, 2)$), [2] (when $A = 0$, $r = 0$, $g_i = 0\,(i = 1, 2)$), [13] (when $g_2 = 0$) and also may be regarded as generalization of some of the results obtained in [3], [5] and [6].

We remark that the existence, uniqueness and other qualitative properties of the variants of equation (1.3) with initail condition $x(t) = \phi(t)$, $t \in [-r, 0]$ have been studied by Kucche et al. [14, 15].

The paper is organized as follows. In section 2, we give definitions and the statements of the theorem that are utilized in this paper. In Section 3, variant of Pachpatte's inequality is derived. Section 4 deals with Ulam–Hyers and

Ulam–Hyers–Rassias stabilities of nonlinear Volterra–Fredholm delay integrodifferential equations. Finally in Section 5, examples are given to illustrate our main results.

2 Preliminaries

Definition 2.1. Let $\{T(t)\}_{t\geq 0}$ is a C_0- semigroup of bounded linear operators in X with infinitesimal generator A. Then a continuous function which satisfies the integral equations

$$x(t) = T(t)\phi(0) + \int_0^t T(t-s)f\left(s, x_s, \int_0^s g_1(s,\tau,x_\tau)d\tau, \int_0^b g_2(s,\tau,x_\tau)d\tau\right)ds,\ t\in J,$$
$$x(t) = \phi(t),\ t\in[-r,0].$$

is called a mild solution of initial value problem

$$x'(t) = Ax(t) + f\left(t, x_t, \int_0^t g_1(t,s,x_s)ds, \int_0^b g_2(t,s,x_s)ds\right),\ t\in J=[0,b],\ 0<b<\infty,$$
$$x(t) = \phi(t),\ t\in[-r,0].$$

Theorem 2.1 ([16])**.** Let $\{T(t)\}_{t\geq 0}$ is a C_0- semigroup. There exists constant $\omega\geq 0$ and $M\geq 1$ such that $\|T(t)\|\leq Me^{\omega t},\ 0<t<\infty$.

For the details on C_0- semigroup theory, we refer to the monographs of Pazy [16] and Engel and Nagel [19].

To establish Ulam–Hyers stabilities for VFDIDE (1.3) we need the following integral inequality investigated by B. G. Pachpatte.

Theorem 2.2 ([17], p-47)**.** Let $z(t),\ u(t),\ v(t),\ w(t)\in C([\alpha,\beta],R_+)$ and $k\geq 0$ be a real constant and

$$z(t)\leq k + \int_\alpha^t u(s)\left[z(s) + \int_\alpha^s v(\sigma)z(\sigma)d\sigma + \int_\alpha^\beta w(\sigma)z(\sigma)d\sigma\right]ds, \quad \text{for} \quad t\in[\alpha,\beta].$$

If

$$r^* = \int_\alpha^\beta w(\sigma)\exp\left(\int_\alpha^\sigma [u(\tau)+v(\tau)]d\tau\right)d\sigma < 1,$$

then

$$z(t)\leq \frac{k}{1-r^*}\exp\left(\int_\alpha^t [u(s)+v(s)]ds\right), \quad \text{for} \ \ t\in[\alpha,\beta].$$

3 A Variant of Pachpatte's inequality

The following corollary is the variant of the Pachpatte's inequality given in Theorem 2.2. It's proof is very close to the proof of Theorem 1.7.4 ([18], page 39) and can be completed on similar line. The variant of the Pachpatte's inequality established below will be utilized to obtain Ulam–Hyers–Rassias stabilities for VFDIDE (1.3).

Corollary 24. Let $z(t),\ u(t),\ v(t),\ w(t) \in C([\alpha,\beta], R_+)$ and $n(t)$ be a positive and nondecreasing continuous function defined on $[\alpha,\beta]$ for which inequality

$$z(t) \le n(t) + \int_\alpha^t u(s)\left[z(s) + \int_\alpha^s v(\sigma)z(\sigma)d\sigma + \int_\alpha^\beta w(\sigma)z(\sigma)d\sigma\right]ds, \text{ for } t \in [\alpha,\beta]. \tag{3.1}$$

If

$$r^* = \int_\alpha^\beta w(\sigma)\exp\left(\int_\alpha^\sigma [u(\tau)+v(\tau)]d\tau\right)d\sigma < 1,$$

then

$$z(t) \le \frac{n(t)}{1-r^*}\exp\left(\int_\alpha^t [u(s)+v(s)]ds\right), \text{ for } t \in [\alpha,\beta].$$

Proof. Noting that $n(t)$ be a positive and nondecreasing continuous function defined on $[\alpha,\beta]$ and $\alpha \le \sigma \le s \le t \le \beta$, from inequality (3.1) we have

$$\begin{aligned}
&z(t)\\
&\le n(t) + \int_\alpha^t u(s)z(s)ds + \int_\alpha^t u(s)\left(\int_\alpha^s v(\sigma)z(\sigma)d\sigma\right)ds + \int_\alpha^t u(s)\left(\int_\alpha^\beta w(\sigma)z(\sigma)d\sigma\right)ds\\
&= n(t) + \int_\alpha^t u(s)\frac{z(s)}{n(s)}n(s)ds + \int_\alpha^t u(s)\left(\int_\alpha^s v(\sigma)\frac{z(\sigma)}{n(\sigma)}n(\sigma)d\sigma\right)ds\\
&\quad + \int_\alpha^t u(s)\left(\int_\alpha^\beta w(\sigma)\frac{z(\sigma)}{n(\sigma)}n(\sigma)d\sigma\right)ds\\
&\le n(t) + \int_\alpha^t u(s)\frac{z(s)}{n(s)}n(t)ds + \int_\alpha^t u(s)\left(\int_\alpha^s v(\sigma)\frac{z(\sigma)}{n(\sigma)}n(t)d\sigma\right)ds\\
&\quad + \int_\alpha^t u(s)\left(\int_\alpha^\beta w(\sigma)\frac{z(\sigma)}{n(\sigma)}n(t)d\sigma\right)ds\\
&= n(t)\left[1 + \int_\alpha^t u(s)\frac{z(s)}{n(s)}ds + \int_\alpha^t u(s)\left(\int_\alpha^s v(\sigma)\frac{z(\sigma)}{n(\sigma)}d\sigma\right)ds\right.\\
&\quad \left.+ \int_\alpha^t u(s)\left(\int_\alpha^\beta w(\sigma)\frac{z(\sigma)}{n(\sigma)}d\sigma\right)ds\right].
\end{aligned}$$

Therefore

$$\frac{z(t)}{n(t)} \leq 1 + \int_{\alpha}^{t} u(s) \left[\frac{z(s)}{n(s)} + \int_{\alpha}^{s} v(\sigma) \frac{z(\sigma)}{n(\sigma)} d\sigma + \int_{\alpha}^{\beta} w(\sigma) \frac{z(\sigma)}{n(\sigma)} d\sigma \right] ds.$$

Apply the inequality given in the Theorem 2.2 to above inequality with $\frac{z(t)}{n(t)}$ in place of $z(t)$ and 1 in place of k to obtain

$$\frac{z(t)}{n(t)} \leq \frac{1}{1-r^*} \exp\left(\int_{\alpha}^{t} [u(s) + v(s)] ds \right), \text{ for } t \in [\alpha, \beta],$$

which gives the desired inequality. QED

4 Ulam type stabilities for VFDIDE

The definitions of Ulam type stabilities for VFDIDE are based on the papers by Rus [2, 8].

Definition 4.1. We say that equation (1.3) has the Ulam–Hyers stability if there exists a non negative constant C such that for each $\varepsilon \geq 0$, if $y : [-r, b] \to X$ in B satisfies

$$\left\| y'(t) - Ay(t) - f\left(t, y_t, \int_0^t g_1(t, s, y_s) ds, \int_0^b g_2(t, s, y_s) ds \right) \right\| \leq \varepsilon, \ t \in J, \quad (4.1)$$

then there exists a solution $x : [-r, b] \to X$ in B of the equation (1.3) with

$$\|y - x\|_B \leq C\, \varepsilon.$$

Definition 4.2. We say that equation (1.3) has the generalised Ulam–Hyers stability if there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$ such that for each solution $y : [-r, b] \to X$ in B of (4.1) there exists a solution $x : [-r, b] \to X$ in B of the equation (1.3) with

$$\|y - x\|_B \leq \theta_f(\varepsilon).$$

Definition 4.3. We say that equation (1.3) has the Ulam–Hyers–Rassias stability with respect to a positive nondecreasing continuous function $\psi : [-r, b] \to \mathbb{R}_+$, if there exists $C_{f,\psi} \geq 0$ (depending on f and ψ) such that for each $\varepsilon \geq 0$, if $y : [-r, b] \to X$ in B satisfies

$$\left\| y'(t) - Ay(t) - f\left(t, y_t, \int_0^t g_1(t, s, y_s) ds, \int_0^b g_2(t, s, y_s) ds \right) \right\| \leq \varepsilon \psi(t), \ t \in J, \quad (4.2)$$

then there exists a solution $x : [-r, b] \to X$ in B of the equation (1.3) with

$$\|y(t) - x(t)\| \leq C_{f,\psi}\, \varepsilon\, \psi(t), \forall\ t \in [-r, b].$$

Definition 4.4. We say that equation (1.3) has the generalised Ulam–Hyers–Rassias stability with respect to a positive nondecreasing continuous function $\psi : [-r, b] \to \mathbb{R}_+$, if there exists $C_{f,\psi} \geq 0$ (depending on f and ψ) such that if $y : [-r, b] \to X$ in B satisfies

$$\left\| y^{'}(t) - Ay(t) - f\left(t, y_t, \int_0^t g_1(t,s,y_s)ds, \int_0^b g_2(t,s,y_s)ds\right) \right\| \leq \psi(t),\ t \in J, \tag{4.3}$$

then there exists a solution $x : [-r, b] \to X$ in B of the equation (1.3) with

$$\|y(t) - x(t)\| \leq C_{f,\psi}\, \psi(t), \forall\ t \in [-r, b].$$

Remark 25. A function $y \in B$ is a solution of inequation (4.1) if there exists a function $a_y \in C(J, X)$ (which depend on y) such that

(i) $\|a_y(t)\| \leq \varepsilon\ \ t \in J.$

(ii) $y^{'}(t) = Ay(t) + f\left(t, y_t, \int_0^t g_1(t,s,y_s)ds, \int_0^b g_2(t,s,y_s)ds\right) + a_y(t)\ t \in J.$

Remark 26. If $y \in B$ satisfies inequation (4.1) then y is a solution of the following integral inequation

$$\left\| y(t) - T(t)y(0) - \int_0^t T(t-s) f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) ds \right\|$$
$$\leq \varepsilon \int_0^t \|T(t-s)\| ds,\ t \in J. \tag{4.4}$$

Indeed, if $y \in B$ satisfies inequation (4.1), by Remark 25 we have

$$y^{'}(t) = Ay(t) + f\left(t, y_t, \int_0^t g_1(t,s,y_s)ds, \int_0^b g_2(t,s,y_s)ds\right) + a_y(t), \in J.$$

This implies that

$$\begin{aligned} y(t) &= T(t)y(0) + \int_0^t T(t-s)\left[f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) + a_y(s)\right] ds \\ &= T(t)y(0) + \int_0^t T(t-s) f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) ds \\ &\quad + \int_0^t T(t-s)a_y(s)ds\, t \in J. \end{aligned}$$

Therefore

$$\left\| y(t) - T(t)y(0) - \int_0^t T(t-s) f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) ds \right\|$$

$$\leq \int_0^t \|T(t-s)\|\|a_y(s)\|ds$$
$$\leq \varepsilon \int_0^t \|T(t-s)\|ds.$$

One can obtain similar type of estimations for the inequations (4.2) and (4.3).

4.1 Ulam–Hyers Stability

Theorem 4.1. We suppose that

(i) $f \in C(J\times C\times X\times X;X)$ and $g_i \in C(J\times J\times C;X)\,(i=1,2)$;

(ii) there exists $L(\cdot)\in C(J,R_+)$ such that

$$\|f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)\| \leq L(t)\left(\|x_1-y_1\|_C+\|x_2-y_2\|+\|x_3-y_3\|\right),$$

for all $t,s\in J,\ x_1,y_1\in C$ and $x_2,x_3,y_2,y_3\in X$;

(iii) there exists $G_i(\cdot)\in C(J,R_+)$ for $i=1,2$ such that

$$\|g_i(t,s,x_1)-g_i(t,s,y_1)\| \leq G_i(t)\left(\|x_1-y_1\|_C\right),\ \forall\ t,s\in J,\ x_1,y_1\in C.$$

Then, the VFDIDE (1.3) is Ulam–Hyers stable, provided

$$q^* = \int_0^b G_2(\sigma)\exp\left(\int_0^\sigma [ML(\tau)e^{w(b-\tau)}+G_1(\tau)]d\tau\right)d\sigma < 1. \tag{4.5}$$

Proof. Let $y\in B$ satisfies the inequation (4.1). Let $x\in B$ be the mild solution of the following problem

$$x'(t) = Ax(t) + f\left(t,x_t,\int_0^t g_1(t,s,x_s)ds,\int_0^b g_2(t,s,x_s)ds\right),\ t\in J,$$
$$x(t)=y(t),\ t\in[-r,0].$$

Then

$$x(t) = T(t)y(0) + \int_0^t T(t-s)f\left(s,x_s,\int_0^s g_1(s,\tau,x_\tau)d\tau,\int_0^b g_2(s,\tau,x_\tau)d\tau\right)ds. \tag{4.6}$$

Using the Theorem 2.1 and the inequation (4.4), we obtain

$$\left\|y(t)-T(t)y(0)-\int_0^t T(t-s)f\left(s,y_s,\int_0^s g_1(s,\tau,y_\tau)d\tau,\int_0^b g_2(s,\tau,y_\tau)d\tau\right)ds\right\|$$

$$\leq \varepsilon \int_0^t \|T(t-s)\| ds \leq \varepsilon \int_0^t M e^{\omega(t-s)} ds = \frac{\varepsilon M}{\omega}\left(e^{\omega t}-1\right)$$
$$\leq \frac{\varepsilon M}{\omega}\left(e^{\omega b}-1\right) \leq \frac{\varepsilon M}{\omega} e^{\omega b},\ t \in J.$$

From above inequation and the equation (4.6), for any $t \in J$, we have

$$\begin{aligned}
&\|y(t)-x(t)\| \\
&= \left\| y(t) - T(t)y(0) - \int_0^t T(t-s) f\left(s, x_s, \int_0^s g_1(s,\tau,x_\tau)d\tau, \int_0^b g_2(s,\tau,x_\tau)d\tau\right) ds \right\| \\
&\leq \left\| y(t) - T(t)y(0) - \int_0^t T(t-s) f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) ds \right\| \\
&\quad + \left\| \int_0^t T(t-s) \left[f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) \right.\right. \\
&\qquad \left.\left. - f\left(s, x_s, \int_0^s g_1(s,\tau,x_\tau)d\tau, \int_0^b g_2(s,\tau,x_\tau)d\tau\right)\right] ds \right\| \\
&\leq \frac{\varepsilon M}{\omega} e^{\omega b} + \int_0^t \|T(t-s)\| \left\| f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) \right. \\
&\qquad \left. - f\left(s, x_s, \int_0^s g_1(s,\tau,x_\tau)d\tau, \int_0^b g_2(s,\tau,x_\tau)d\tau\right) \right\| ds \\
&\leq \frac{\varepsilon M}{\omega} e^{\omega b} + \int_0^t M e^{\omega(t-s)} L(s) \left[\|y_s - x_s\|_C + \int_0^s G_1(\tau) \|y_\tau - x_\tau\|_C \, d\tau \right. \\
&\qquad \left. + \int_0^b G_2(\tau) \|y_\tau - x_\tau\|_C \, d\tau \right] ds.
\end{aligned}$$

Consider the function defined by $\mu(t) = \sup\{\|(y-x)(s)\| : s \in [-r,t]\},\ t \in J$, then $\|(y-x)_t\|_C \leq \mu(t)$ for all $t \in J$ and there is $t^* \in [-r,t]$ such that $\mu(t) = \|(y-x)(t^*)\|$. Hence for $t^* \in [0,t]$ we have

$$\begin{aligned}
\mu(t) &\leq \frac{\varepsilon M}{\omega} e^{\omega b} + \int_0^{t^*} M L(s) e^{\omega(b-s)} \left[\|(y_s - x_s)\|_C + \int_0^s G_1(\tau) \|y_\tau - x_\tau\|_C \, d\tau \right. \\
&\qquad \left. + \int_0^b G_2(\tau) \|y_\tau - x_\tau\|_C \, d\tau \right] ds \\
&\leq \frac{\varepsilon M}{\omega} e^{\omega b} + \int_0^t M L(s) e^{\omega(b-s)} \left[\mu(s) + \int_0^s G_1(\tau)\mu(\tau)d\tau + \int_0^b G_2(\tau)\mu(\tau)d\tau \right] ds.
\end{aligned} \tag{4.7}$$

If $t^* \in [-r,0]$ then $\mu(t) = 0$ and the inequality (4.7) hold obviously, since $M \geq 1$. Applying the Pachpatte inequality given in the Theorem 2.2 to the inequation (4.7) with

$$z(t) = \mu(t),\ k = \frac{\varepsilon M}{\omega} e^{\omega b},\ u(t) = ML(t)e^{\omega(b-t)},\ v(t) = G_1(t) \text{ and } w(t) = G_2(t),$$

we obtain

$$\begin{aligned} \mu(t) &\le \varepsilon \frac{Me^{\omega b}}{\omega(1-q^*)} \exp\left(\int_0^t \left[L(s)Me^{\omega(b-s)} + G_1(s)\right] ds\right) \\ &\le \varepsilon \frac{Me^{\omega b}}{\omega(1-q^*)} \exp\left(\int_0^b \left[L(s)Me^{\omega(b-s)} + G_1(s)\right] ds\right). \end{aligned}$$

Therefore

$$\|y-x\|_B \le \varepsilon \frac{Me^{\omega b}}{\omega(1-q^*)} \exp\left(\int_0^b \left[L(s)Me^{\omega(b-s)} + G_1(s)\right] ds\right). \tag{4.8}$$

Putting $C = \frac{Me^{\omega b}}{\omega(1-q^*)} \exp\left(\int_0^b \left[L(s)Me^{\omega(b-s)} + G_1(s)\right] ds\right)$ in (4.8), we obtain

$$\|y-x\|_B \le \varepsilon\, C.$$

This completes the proof. QED

Corollary 27. Under the assumptions of Theorem 4.1 the VFDIDE (1.3) is generalized Ulam–Hyers stable, provided that the condition (4.5) is satisfied.

Proof. Define $\theta_f(\varepsilon) = \varepsilon \frac{Me^{\omega b}}{\omega(1-q^*)} \exp\left(\int_0^b \left[L(s)Me^{\omega(b-s)} + G_1(s)\right] ds\right)$, where q^* is given in condition (4.5). Then we have $\theta_f \in C(R_+, R_+)$, $\theta_f(0) = 0$ and the inequation (4.8) takes the form

$$\|y-x\|_B \le \theta_f(\varepsilon).$$

This proves (1.3) is generalized Ulam–Hyers stable. QED

4.2 Ulam–Hyers–Rassias Stability

Theorem 4.2. Assume that f and $g_i\,(i = 1,2)$ satisfy the conditions of Theorem 4.1. Moreover assume that $\psi : [-r,b] \to \mathbb{R}_+$ is positive, nondecreasing, continuous function and there exists $\lambda > 0$ such that

$$\int_0^t \|T(t-s)\|\psi(s)ds \le \lambda\psi(t),\ t \in [-r,b].$$

Then, the equation (1.3) is Ulam–Hyers–Rassias stable with respect to ψ, provided that the condition (4.5) is satisfied.

Proof. Let $y \in B$ be solution of inequation (4.2) then proceeding as in Remark 26 and using the hypothesis, we have

$$\left\| y(t) - T(t)y(0) + \int_0^t T(t-s) f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) ds \right\| \\ \leq \int_0^t \|T(t-s)\| \, \|a_y(s)\| \, ds \leq \int_0^t \|T(t-s)\| \, \varepsilon \, \psi(s) ds \leq \varepsilon\lambda\psi(t), \ t \in J. \tag{4.9}$$

Let us denote by $x \in B$ the mild solution of the following problem

$$x'(t) = Ax(t) + f\left(t, x_t, \int_0^t g_1(t,s,x_s)ds, \int_0^b g_2(t,s,x_s)ds\right), \ t \in J = [0,b], \\ x(t) = y(t), \ t \in [-r, 0].$$

Then

$$x(t) = T(t)y(0) + \int_0^t T(t-s) f\left(s, x_s, \int_0^s g_1(s,\tau,x_\tau)d\tau, \int_0^b g_2(s,\tau,x_\tau)d\tau\right) ds. \tag{4.10}$$

Using the equation (4.10) and the inequation (4.9), we have

$$\begin{aligned} &\|y(t) - x(t)\| \\ &= \left\| y(t) - T(t)y(0) - \int_0^t T(t-s) f\left(s, x_s, \int_0^s g_1(s,\tau,x_\tau)d\tau, \int_0^b g_2(s,\tau,x_\tau)d\tau\right) ds \right\| \\ &\leq \left\| y(t) - T(t)y(0) - \int_0^t T(t-s) f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) ds \right\| \\ &+ \left\| \int_0^t T(t-s) \left[f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) \right.\right. \\ &\qquad \left.\left. - f\left(s, x_s, \int_0^s g_1(s,\tau,x_\tau)d\tau, \int_0^b g_2(s,\tau,x_\tau)d\tau\right)\right] ds \right\| \\ &\leq \varepsilon\lambda\psi(t) + \int_0^t \|T(t-s)\| \left\| f\left(s, y_s, \int_0^s g_1(s,\tau,y_\tau)d\tau, \int_0^b g_2(s,\tau,y_\tau)d\tau\right) \right. \\ &\qquad \left. - f\left(s, x_s, \int_0^s g_1(s,\tau,x_\tau)d\tau, \int_0^b g_2(s,\tau,x_\tau)d\tau\right) \right\| ds \\ &\leq \varepsilon\lambda\psi(t) + \int_0^t M e^{\omega(b-s)} L(s) \left[\|y_s - x_s\|_C + \int_0^s G_1(\tau) \|y_\tau - x_\tau\|_C \, d\tau \right. \\ &\qquad \left. + \int_0^b G_2(\tau) \|y_\tau - x_\tau\|_C \, d\tau \right] ds. \end{aligned}$$

Considering the function μ defined in the proof of Theorem 4.1 and proceeding in similar manner we obtain

$$\mu(t) \leq \varepsilon\, \lambda\psi(t) + \int_0^t ML(s)e^{\omega(b-s)} \left[\mu(s) + \int_0^s G_1(\tau)\mu(\tau)d\tau + \int_0^b G_2(\tau)\mu(\tau)d\tau\right] ds. \tag{4.11}$$

Applying the inequality given in the Corollary 24 to inequation (4.11) with

$z(t) = \mu(t),\ n(t) = \varepsilon\, \lambda\psi(t),\ u(t) = ML(t)e^{\omega(b-t)},\ v(t) = G_1(t)$ and $w(t) = G_2(t)$,

we obtain

$$\begin{aligned}\mu(t) &\leq \frac{\varepsilon\lambda\psi(t)}{1-q^*} \exp\left(\int_0^t \left[L(s)Me^{\omega(b-s)} + G_1(s)\right] ds\right)\\ &\leq \frac{\varepsilon\lambda\psi(t)}{1-q^*} \exp\left(\int_0^b \left[L(s)Me^{\omega(b-s)} + G_1(s)\right] ds\right).\end{aligned}$$

Taking

$$C_{f,\psi} = \frac{\lambda}{(1-q^*)} \exp\left(\int_0^b \left[L(s)Me^{\omega(b-s)} + G_1(s)\right] ds\right),$$

we get

$$\mu(t) \leq \varepsilon\, C_{f,\psi}, \psi(t),\ t \in J.$$

Therefore

$$\|y(t) - x(t)\| \leq \varepsilon\, C_{f,\psi}, \psi(t),\ t \in [-r, b].$$

This proves (1.3) is Ulam–Hyers–Rassias stable with respect to ψ . QED

Corollary 28. Under the assumptions of Theorem 4.1 the VFDIDE (1.3) is generalized Ulam–Hyers–Rassias stable with respect to ψ, provided that the condition (4.5) is satisfied.

Proof. Proof follows by taking $\varepsilon = 1$ in the proof of Theorem 4.2. QED

5 Application

We know the following initial value problem for difference equation

$$x'(t) = Ax(t) + f\left(t, x(t-r), \int_0^t g_1(t,s,x(s-r))ds, \int_0^b g_2(t,s,x(s-r))ds\right),\ t \in [0,b], \tag{5.1}$$

$$x(t) = \phi(t),\ t \in [-r, 0], \tag{5.2}$$

is the particular case of VFDIDE (1.3) with initial condition $x(t) = \phi(t),\ t \in [-r, 0]$. Thus all the results we obtained in this paper for VFDIDE are also applicable to the difference equations (5.1)–(5.2).

6 Examples

In this section we present a illustrative example.

Example 6.1. Consider the nonlinear Volterra–Fredholm delay integrodifferential equations in the Banach space $(\mathbb{R}, |\cdot|)$:

$$\begin{aligned} x^{'}(t) &= -\frac{1}{80} + \frac{\sin(8)}{640} + \frac{\sin(4)}{80} - \frac{x(t-4)}{320} + \frac{\sin(2x(t-4))}{640} + \frac{1}{10}\int_0^t \frac{\sin^2(x(s-4))}{16} ds \\ &\quad - \frac{1}{16}\int_0^\pi \frac{\cos(x(s-4))}{10} ds,\ t \in [0, \pi], \qquad (6.1) \\ x(t) &= t,\ t \in [-4, 0]. \qquad (6.2) \end{aligned}$$

Consider the functions $g_i : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ are defined by

$$g_1(t, s, x(s-4)) = \frac{\sin^2(x(s-4))}{16},$$

$$g_2(t, s, x(s-4)) = \frac{\cos(x(s-4))}{10},$$

and the function $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{aligned} & f\left(t, x(t-4), \int_0^t g_1(t, s, x(s-4))ds, \int_0^b g_2(t, s, x(s-4))ds\right) \\ &= -\frac{1}{80} + \frac{\sin(8)}{640} + \frac{\sin(4)}{80} - \frac{x(t-4)}{320} + \frac{\sin(2x(t-4))}{640} + \frac{1}{10}\int_0^t \frac{\sin^2(x(s-4))}{16} ds \\ &\quad - \frac{1}{16}\int_0^\pi \frac{\cos(x(s-4))}{10} ds. \end{aligned}$$

Then the initial value problem (6.1)–(6.2) can be written in the form of (5.1)–(5.2) with infinitsimal generator $A = 0$.

Note that:

(i) For any $t, s \in [0, \pi]$ and $x_1, y_1 \in \mathbb{R}$, we have

$$|g_1(t, s, x_1) - g_1(t, s, y_1)| \leq \frac{1}{16}\left|\sin^2 x_1 - \sin^2 y_1\right| \leq \frac{1}{8}|x_1 - y_1|.$$

(ii) For any $t, s \in [0,1]$ and $x_1, y_1 \in \mathbb{R}$, we have

$$|g_2(t,s,x_1) - g_2(t,s,y_1)| \leq \frac{1}{10} |\cos x_1 - \cos y_1| \leq \frac{1}{10} |x_1 - y_1| .$$

(iii) For any $t, s \in [0,1]$ and $x_1, y_1, x_2, y_2, x_3, y_3 \in \mathbb{R}$, we have

$$|f(t,s,x_1,x_2,x_3) - f(t,s,y_1,y_2,y_3)| \leq \frac{1}{10} \{|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|\} .$$

The functions $f,\ g_1$ and g_2 in the equation (6.1) verifies the assumption with $L(t) = \frac{1}{10},\ G_1(t) = \frac{1}{8},\ G_2(t) = \frac{1}{10}$. Note that for the infinitesimal generator $A = 0$ the C_0 semigroup is $T(t) = 1,\ t \geq 0$ and the corresponding constants in the Theorem 2.1 are $M = 1,\ \omega = 0$. Thus we have,

$$\begin{aligned} q^* &= \int_0^b G_2(\sigma) \exp\left(\int_0^\sigma [ML(\tau)e^{\omega(b-\tau)} + G_1(\tau)]d\tau \right) d\sigma \\ &= \int_0^b G_2(\sigma) \exp\left(\int_0^\sigma [L(\tau) + G_1(\tau)]d\tau \right) d\sigma \\ &= \int_0^\pi \frac{1}{10} \exp\left(\int_0^\sigma \left[\frac{1}{10} + \frac{1}{8} \right] d\tau \right) d\sigma = 0.456716 < 1. \end{aligned}$$

Therefore by the Theorem 4.1 the equation (6.1) is Ulam–Hyres stable on $[0,\pi]$.

We now discuss the Ulam–Hyres stability of the equation (6.1) by showing that given any values of $\varepsilon > 0$ and given solutions $y(t)$ of the inequations

$$\left| y^{'}(t) - f\left(t, y(t-4), \int_0^t g_1(t,s,y(s-4))ds, \int_0^b g_2(t,s,y(s-4))ds \right) \right| < \varepsilon \qquad (6.3)$$

there is a solution $x(t)$ of equation (6.1) satisfying the inequation

$$|y(t) - x(t)| < C\varepsilon,\ t \in [-4,\pi].$$

One can veryify that $x(t) = t,\ t \in [-4,\pi]$ is solution of the intial value problem (6.1)–(6.2).

(i) Choose $\varepsilon = 0.55$ and $y_1(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0,\pi], \\ t & \text{if } t \in [-4,0]. \end{cases}$ Then for $t \in [0,\pi]$, we have

$$\begin{aligned} &\left| y_1^{'}(t) - f\left(t, y_1(t-4), \int_0^t g_1(t,s,y_1(s-4))ds, \int_0^b g_2(t,s,y_1(s-4))ds \right) \right| \\ &= \left| y_1^{'}(t) + \frac{1}{80} - \frac{\sin(8)}{640} - \frac{\sin(4)}{80} + \frac{y_1(t-4)}{320} - \frac{\sin(2y_1(t-4))}{640} \right. \\ &\qquad \left. - \frac{1}{10} \int_0^t \frac{\sin^2(y_1(s-4))}{16} ds + \frac{1}{16} \int_0^\pi \frac{\cos(y_1(s-4))}{10} ds \right| \end{aligned}$$

$$= \left| \frac{1}{2} + \frac{1}{80} - \frac{\sin(8)}{640} - \frac{\sin(4)}{80} + \frac{t-4}{640} - \frac{\sin(t-4)}{640} - \frac{1}{10}\int_0^t \frac{\sin^2\left(\frac{(s-4)}{2}\right)}{16} ds \right.$$
$$\left. + \frac{1}{16}\int_0^\pi \frac{\cos\left(\frac{(s-4)}{2}\right)}{10} ds \right| \leq 0.511872 < \varepsilon.$$

For the solution $x(t) = t$ of (6.1)–(6.2) and constant $C = 3$ we have

$$|y_1(t) - x(t)| = \left| t - \frac{t}{2} \right| \leq \frac{\pi}{2} < C\varepsilon, \; t \in [0, \pi],$$

and

$$|y_1(t) - x(t)| = 0, \; t \in [-4, 0].$$

Therefore

$$|y_1(t) - x(t)| < C\varepsilon, \; t \in [-4, \pi].$$

(ii) Choose $\varepsilon = 2.3$ and $y_2(t) = \begin{cases} 2t & \text{if } t \in [0, \pi], \\ t & \text{if } t \in [-4, 0]. \end{cases}$ Then for $t \in [0, \pi]$, we have

$$\left| y_2'(t) - f\left(t, y_2(t-4), \int_0^t g_1(t, s, y_2(s-4))ds, \int_0^b g_2(t, s, y_2(s-4))ds\right) \right|$$
$$= \left| y_2'(t) + \frac{1}{80} - \frac{\sin(8)}{640} - \frac{\sin(4)}{80} + \frac{y_2(t-4)}{320} - \frac{\sin(2y_2(t-4))}{640} \right.$$
$$\left. - \frac{1}{10}\int_0^t \frac{\sin^2(y_2(s-4))}{16} ds + \frac{1}{16}\int_0^\pi \frac{\cos(y_2(s-4))}{10} ds \right|$$
$$= \left| 2 + \frac{1}{80} - \frac{\sin(8)}{640} - \frac{\sin(4)}{80} + \frac{t-4}{640} - \frac{\sin(4(t-4))}{640} - \frac{1}{10}\int_0^t \frac{\sin^2(2(s-4))}{16} ds \right.$$
$$\left. + \frac{1}{16}\int_0^\pi \frac{\cos(2(s-4))}{10} ds \right| \leq 2.0478 < \varepsilon.$$

For the solution $x(t) = t$ of (6.1)–(6.2) and constant $C = 1.4$ we have

$$|y_2(t) - x(t)| = |2t - t| \leq \pi < C\varepsilon, \; t \in [0, \pi],$$

and

$$|y_2(t) - x(t)| = 0, \; t \in [-4, 0].$$

Therefore

$$|y_2(t) - x(t)| < C\varepsilon, \; t \in [-4, \pi].$$

(iii) Choose $\varepsilon = 6.5$ and $y_2(t) = \begin{cases} t^2 & \text{if } t \in [0,\pi], \\ t & \text{if } t \in [-4,0]. \end{cases}$ Then for $t \in [0,\pi]$, we have

$$\begin{aligned}
&\left| y_3'(t) - f\left(t, y_3(t-4), \int_0^t g_1(t,s,y_3(s-4))ds, \int_0^b g_2(t,s,y_3(s-4))ds\right)\right| \\
&= \left| y_3'(t) + \frac{1}{80} - \frac{\sin(8)}{640} - \frac{\sin(4)}{80} + \frac{y_3(t-4)}{320} - \frac{\sin(2y_2(t-4))}{640} \right. \\
&\qquad \left. - \frac{1}{10}\int_0^t \frac{\sin^2(y_3(s-4))}{16} ds + \frac{1}{16}\int_0^\pi \frac{\cos(y_3(s-4))}{10} ds \right| \\
&= \left| 2t + \frac{1}{80} - \frac{\sin(8)}{640} - \frac{\sin(4)}{80} + \frac{(t-4)^2}{320} - \frac{\sin(2(t-4)^2)}{640} - \frac{1}{10}\int_0^t \frac{\sin^2((s-4)^2)}{16} ds \right. \\
&\qquad \left. + \frac{1}{16}\int_0^\pi \frac{\cos((s-4)^2)}{10} ds \right| \\
&\le 6.29242 < \varepsilon.
\end{aligned}$$

For the solution $x(t) = t$ of (6.1)–(6.2) and constant $C = 2.2$ we have

$$|y_3(t) - x(t)| = \left| t - t^2 \right| < C\varepsilon, \ t \in [0,\pi],$$

and

$$|y_1(t) - x(t)| = 0, \ t \in [-4,0].$$

Therefore

$$|y_1(t) - x(t)| < C\varepsilon, \ t \in [-4,\pi].$$

We conclude that corresponding to given values of ε and given solutions $y(t)$ of the inequation (6.3), we are able to find the the exact solution $x(t)$ of equation (6.1) satisfying $|y(t) - x(t)| \le C\,\varepsilon, \ t \in [-4,\pi]$. Hence equation (6.1) is Ulam–Hyres stable on $[-4,\pi]$.

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