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A general optimal inequality for warped product submanifolds in paracosymplectic manifolds

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Abstract. The aim of this paper is to study the pseudo-Riemannian warped product submanifolds of a paracosymplectic manifold $\bar{M}$. We first, prove some fundamental lemmas and then derive some important results with parallel canonical structures on $\mathcal{PR}$-semi-invariant submanifolds M of $\bar{M}$. Finally, we describe the warped product submanifold M of $\bar{M}$ by developing the general optimal inequality in terms of warping function and squared norm of second fundamental form. We also consider the totally geodesic, mixed geodesic and equality case of the inequality.

Keywords: Paracontact manifold, Submanifold, Warped product

MSC 2000 classification: primary 53B25, 53B30, secondary 53C25, 53D15

1 Introduction

Bishop and O'Neill [1] introduced the notion of warped product, as one of the most effective generalization of Riemannian product manifold. Thereafter, O'Neill, B. [18] generalized the study of warped products for semi-Riemannian manifolds. However, the theory gain recognition after Chen came with a new class of CR-submanifolds [3] called CR-warped products in Kaehlerian manifold and established the geometric inequalities for the second fundamental form in term of warping function [4]. On the other hand, Hasegawa-Mihai [10] and

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Munteanu [17] continued the study (in the sense of Chen) for Sasakian ambient which earlier studied by Bejancu and Papaghuic [2] and can be observed as an odd-dimensional counterpart of Kähler manifold. Since then several geometers have studied the geometric aspects of warped product submanifolds in almost Hermitian and contact manifolds (c.f., [6, 14, 22, 21]). Here, it is important to mention that the Riemannian geometric configuration may not found appropriate in mathematical physics particularly, in the theory of space-time and black hole, where the metric is not necessarily positive definite. Thus, the geometry of warped product submanifolds with negative definite metric became an area of investigation. Beside this the premises of warped product has perceived several interesting contributions in semi-Riemannian geometries, and has been effectively utilized in the study of general theory of relativity and black holes (c.f., [7, 12, 13, 18]). Recently, B.Y. Chen and M.I. Munteanuo initiated the geometry of pseudo-Riemannian warped products submanifolds (called them PR-warped product) in para-Kähler manifolds and established a optimal (or sharp) geometric inequality: as the equality case holds for certain cases it follows that the inequality cannot be improved, involving intrinsic and extrinsic invariants [5]. Motivated to [5], we developed an optimal geometric inequality for $\mathcal{PR}$-semi-invariant warped product submanifold M of $\bar{M}$ that can be viewed as an odd-dimensional counterpart of para-Kähler manifolds.

The brief description of the present paper is as follows: In Sect. 2, we recall some definitions, basic facts and properties of paracosymplectic manifold and its submanifolds in the context of paracontact structure. Sect. 3 deals with the definition of a $\mathcal{PR}$-semi-invariant submanifold M of a paracosymplectic manifold $\bar{M}$, some constructive lemmas, integrablity and totally geodesic conditions for the distributions associated with the definition of M, and important results concerning parallel canonical structure. A general optimal inequality (in the sense of [5]) for a $\mathcal{PR}$-semi-invariant warped product submanifold M considering its totally geodesic, mixed geodesic and equality cases in the setting of paracosymplectic manifold are derived in Sect. 4.

2 Preliminaries

Let $\bar{M}$ be an odd dimensional smooth manifold. An almost paracontact structure on $\bar{M}$ is a triplet (ϕ, ξ, η) [11], consisting of a tensor field ϕ of type-$(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions:

$$\phi^2 = I - \eta \otimes \xi,\ \eta(\xi) = 1 \tag{2.1}$$

where I is the identity transformation and the tensor field ϕ induces on $2m$-dimensional horizontal distribution $D := \ker(\eta)$ an almost paracomplex structure J; that is $J^2 = I$ and the eigen subbundles $D^{\pm}$ corresponding to the eigenvalues ± 1 of J respectively, have equal dimension m; hence $D = D^{+} \oplus D^{-}$. From the equation (2.1), it can be directly obtained that

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \quad \text{and} \quad \operatorname{rank}(\phi) = 2m. \tag{2.2}$$

The manifold $\bar{M}$ is said to be an almost paracontact manifold if it is endowed with an almost paracontact structure (ϕ, ξ, η) (c.f., [19, 23]). If an almost paracontact manifold $\bar{M}$ admits a pseudo-Riemannian metric g satisfying:

$$g(X, Y) = -g(\phi X, \phi Y) + \eta(X)\, \eta(Y), \tag{2.3}$$

for any vector fields X and Y, where signature of g is certainly $(m+1,\, m)$; then the quadruple (ϕ, ξ, η, g) is called an almost paracontact metric structure and the manifold $\bar{M}$ endowed with paracontact metric structure is called an almost paracontact metric manifold denoted by $\bar{M}$. Furthermore with respect to g, η is metrically dual to ξ, that is

$$g(X, \xi) = \eta(X). \tag{2.4}$$

With the consequences of Eqs. (2.1), (2.2) and (2.3), we deduce that

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.5}$$

for any $X, Y \in \Gamma(T\bar{M})$ where $\Gamma(T\bar{M})$ denote the sections of tangent bundle $T\bar{M}$ of $\bar{M}$. The fundamental 2-form Φ on $\bar{M}$ is given by

$$g(X, \phi Y) = \Phi(X, Y). \tag{2.6}$$

Definition 2.1. An almost paracontact metric manifold $\bar{M}$ is said to be *paracosymplectic* if the forms η and Φ are parallel with respect to the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$, i.e.,

$$\bar{\nabla}\eta = 0 \quad \text{and} \quad \bar{\nabla}\Phi = 0. \tag{2.7}$$

(see [8, 15]).

Let M be a N-dimensional submanifold immersed in a $(2m+1)$-dimensional almost paracontact manifold $\bar{M}$; we denote by the same symbol g the induced metric on M. If $\Gamma(TM)$ denotes the tangent bundle of submanifold M and $\Gamma(T^{\perp}M)$ the set of vector fields normal to M then Gauss and Weingarten formulas are given by respectively,

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.8}$$

$$\bar{\nabla}_X \zeta = -A_{\zeta} X + \nabla^{\perp}_X \zeta. \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$ and $\zeta \in \Gamma(T^{\perp}M)$, where ∇ is the induced connection, $\nabla^{\perp}$ is the normal connection on the normal bundle $\Gamma(T^{\perp}M)$, h is the second fundamental form, and the shape operator A_{ζ} associated with the normal section ζ is given by

$$g\left(A_{\zeta}X, Y\right) = g\left(h(X, Y), \zeta\right). \tag{2.10}$$

If we write, for all $X \in \Gamma(TM)$ and $\zeta \in \Gamma(T^{\perp}M)$ that

$$\varphi X = tX + nX, \tag{2.11}$$
$$\varphi\zeta = t'\zeta + n'\zeta, \tag{2.12}$$

where tX (resp., nX) is tangential (resp., normal) part of φX and $t'\zeta$ (resp., $n'\zeta$) is tangential (resp., normal) part of $\varphi\zeta$. Then the submanifold M is said to be *invariant* if n is identically zero and *anti-invariant* if t is identically zero. From Eqs.(2.5) and (2.11), we obtain that

$$g(X, tY) = -g(tX, Y). \tag{2.13}$$

Since M is a submanifold of a paracosymplectic manifold $\bar{M}$ therefore for any $X, Y \in \Gamma(TM)$, we deduce by virtue of Eqs. (2.7), (2.8), (2.9) and (2.10) that

$$\left(\nabla_X t\right) Y = A_{nY}X + t'h(X, Y), \tag{2.14}$$
$$\left(\nabla_X n\right) Y = n'h(X, Y) - h(X, tY), \tag{2.15}$$

where the covariant derivatives of the tensor fields t and n are, respectively, defined by

$$(\nabla_X t)Y = \nabla_X tY - t\nabla_X Y, \tag{2.16}$$
$$(\nabla_X n)Y = \nabla^{\perp}_X nY - n\nabla_X Y. \tag{2.17}$$

The canonical structure t and n on a submanifold M are said to be *parallel* if $\nabla t = 0$ and $\nabla n = 0$, respectively. Let $p \in M$ and $\{e_1, \cdots, e_N, \cdots, e_{2m+1}\}$ be an orthonormal basis of the tangent space $T_p\bar{M}$ such that $e_1, \cdots, e_N$ are tangent to M at p. The mean curvature vector H of M is given by

$$H(p) = \frac{1}{N}\operatorname{trace} h. \tag{2.18}$$

Also, we set

$$h^k_{ij} = g(h(e_i, e_j), e_k), \quad i, j \in \{e_1, \cdots, e_N\}, \quad k \in \{e_{N+1}, \cdots, e_{2m+1}\}. \tag{2.19}$$

Moreover, the squared norm of the second fundamental form h is defined as

$$||h||^2 = \sum_{i,j=1}^{N} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.20}$$

A submanifold M is said to be [5]

- *totally geodesic* if its second fundamental form vanishes identically.
- *umbilical* in the direction of a normal vector field ζ on M, if $A_\zeta = \delta Id$, for certain function δ on M, here such ζ is called a umbilical section.
- *totally umbilical* if M is umbilical with respect to every (local) normal vector field.

Let (B, g_B) and (F, g_F) be two pseudo-Riemannian manifolds and f be a positive smooth function on B. Consider the product manifold $B \times F$ with canonical projections

$$\pi : B \times F \to B \quad \text{and} \quad \sigma : B \times F \to F. \tag{2.21}$$

Then the manifold $M = B \times_f F$ is said to be *warped product* if it is equipped with the following warped metric

$$g(X, Y) = g_B(\pi_*(X), \pi_*(Y)) + (f \circ \pi)^2 g_F(\sigma_*(X), \sigma_*(Y)) \tag{2.22}$$

for all $X, Y \in \Gamma(TM)$ and '$*$' stands for derivation map, or equivalently,

$$g = g_B + f^2 g_F. \tag{2.23}$$

The function f is called *the warping function* and a warped product manifold M is said to be *trivial* if f is constant. In view of simplicity, we will identify a vector field X on B with its lift $\bar{X}$ and a vector field Z on F with its lift $\bar{Z}$ on $M = B \times_f F$ (see also [1, 5]). Now, we recall the following proposition for the warped product manifolds [1]:

Proposition 2.2. For $X, Y \in \Gamma(TB)$ and $U, V \in \Gamma(TF)$, we obtain on warped product manifold $M = B \times_f F$ that

(1) $\nabla_X Y \in \Gamma(TB)$,

(2) $\nabla_X U = \nabla_U X = X(\ln f)U$,

(3) $\nabla_U V = \nabla'_U V - g(U, V)\text{grad}(\ln f)$,

where, ∇ and ∇' denotes the Levi-Civita connections on M and F respectively. Furthermore, *grad*$(\ln f)$ is the gradient of $\ln f$ with respect to the ambient metric g.

3 $\mathcal{PR}$-semi-invariant submanifolds

This section includes the definition of $\mathcal{PR}$-semi-invariant submanifolds M of $\bar{M}$ and conditions for distributions equipped with the definition of M and an important results allied to the characterization of M with parallel canonical structure.

Definition 3.1. Let M is an isometrically immersed pseudo-Riemannian submanifold of an almost paracontact manifold $\bar{M}$. Then M is said to be a $\mathcal{PR}$-*semi-invariant submanifold* [20], if it is furnished with a pair of orthogonal distribution $(\mathfrak{D}, \mathfrak{D}^{\perp})$ satisfying the conditions:

(i) $TM = \mathfrak{D} \oplus \mathfrak{D}^{\perp} \oplus \{\xi\}$,

(ii) the distribution $\mathfrak{D}$ is invariant under ϕ, i.e., $\phi(\mathfrak{D}_p) \subseteq \mathfrak{D}_p$, for each $p \in M$ and

(iii) the distribution $\mathfrak{D}^{\perp}$ is anti-invariant under ϕ, i.e., $\phi({\mathfrak{D}_p}^{\perp}) \subseteq T_p^{\perp}M$, for each $p \in M$.

Here, $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$ are the distributions on invariant B and anti-invariant F submanifold of $\bar{M}$, respectively. Now, we can say that a $\mathcal{PR}$-*semi-invariant submanifold* M is *invariant*, if $\dim {\mathfrak{D}_p}^{\perp} = 0$; *anti-invariant*, if $\dim \mathfrak{D}_p = 0$; *proper* if $\dim \mathfrak{D}_p \neq 0$ and $\dim {\mathfrak{D}_p}^{\perp} \neq 0$; *mixed totally geodesic*, if $h(\mathfrak{D}_p, {\mathfrak{D}_p}^{\perp}) = 0$ for each $p \in M$.

Let M be a $\mathcal{PR}$-semi-invariant submanifolds of a paracosymplectic manifold $\bar{M}$ such that ξ is tangent to M. Then we can write

$$T_p\bar{M} = T_pM \oplus T_p^{\perp}M, \quad p \in M,$$

and the normal bundle $T^{\perp}M$ of M of a paracosymplectic manifold $\bar{M}$ can be decomposed as

$$T^{\perp}M = \phi(\mathfrak{D}^{\perp}) \oplus \nu,$$

where ν is normal subbundle orthogonal to $\phi(\mathfrak{D}^{\perp})$ and invariant under ϕ.
Now, we give an important lemmas for later use:

Lemma 3.2. If M be a $\mathcal{PR}$-semi-invariant submanifold M of a paracosymplectic manifold $\bar{M}$ such that $\xi \in \Gamma(TM)$. Then for any $X \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^{\perp})$, we have

$$\nabla_X\xi = 0, \quad h(X, \xi) = 0, \tag{3.1}$$

$$\nabla_Z\xi = 0, \quad h(Z, \xi) = 0, \tag{3.2}$$

$$\nabla_\xi\xi = 0, \quad h(\xi, \xi) = 0. \tag{3.3}$$

Proof. The proof of the lemma can be easily derive by using the fact that the manifold is paracosymplectic and Gauss-Weingertan formulas. $\boxed{QED}$

Lemma 3.3. If M be a $\mathcal{PR}$-semi-invariant submanifold M of a paracosymplectic manifold $\bar{M}$ such that $\xi \in \Gamma(TM)$. Then we have

$$(\nabla_X t)\xi = (\nabla_X n)\xi = (\nabla_\xi t)X = (\nabla_\xi n)X = 0; \quad (\nabla_\xi t')Z = (\nabla_\xi n')Z = 0; \quad (3.4)$$

for any $X \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^\perp)$.

Proof. By virtue of lemma 3.2 and Eqs. (2.14)-(2.17), we complete the proof of the lemma. $\boxed{QED}$

Proposition 3.4. Let M be a $\mathcal{PR}$-semi-invariant submanifold M of a paracosymplectic manifold $\bar{M}$ such that $\xi \in \Gamma(TM)$. Then for any $Y, Z \in \Gamma(TM)$, canonical structure t is parallel if and only if $A_{nY}Z = A_{nZ}Y$.

Proof. We can write by the virtue of Eq. (2.14), that $g((\widetilde{\nabla}_X t)Y, Z) = g(A_{nY}X + t'h(X,Y), Z)$ for all $X \in \Gamma(TM)$. Thus, by using Eqs. (2.5), (2.10) and (2.12) in above expression, we complete the proof of the proposition. $\boxed{QED}$

Proposition 3.5. Let M be a $\mathcal{PR}$-semi-invariant submanifold M of a paracosymplectic manifold $\bar{M}$. Then the invariant distribution $(\mathfrak{D} \oplus \xi)$ is

(a) integrable if and only if $h(X, \phi Y) = h(\phi X, Y)$;

(b) totally geodesic if and only if $g(h(X, \phi Y), \phi Z) = 0$;

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \xi)$ and $Z \in \Gamma(\mathfrak{D}^\perp)$.

Proof. By virtue of lemma 3.2, Eqs.(2.14)-(2.17) and the fact that h is symmetric, we obtain for any $X, Y \in \Gamma(\mathfrak{D} \oplus \xi))$, that

$$\begin{aligned} n([X,Y]) &= n(\nabla_X Y) - n(\nabla_Y X) \\ &= \nabla_X^\perp nY - (\nabla_X n)Y - \nabla_Y^\perp nX + (\nabla_Y n)X \\ &= (\nabla_Y n)X - (\nabla_X n)Y \\ &= n'h(Y,X) - h(Y,tX) - n'h(Y,X) + h(X,tY). \end{aligned}$$

Thus from above last expression we achieve the proof of the formula-(a). For formula-(b), we have from Eqs. (2.3), lemma 3.2 and the definition of paracosymplectic manifold that $g(\nabla_X Y, Z) = g(\nabla_X \phi Y, \phi Z)$. Now, using Eqs. (2.9), (2.10) and property of Levi-Civita connection ∇ in previous expression, we derive the required formula. This completes the proof of the proposition. $\boxed{QED}$

Proposition 3.6. Let M be a $\mathcal{PR}$-semi-invariant submanifold M of a paracosymplectic manifold $\bar{M}$. Then the anti- invariant distribution $(\mathfrak{D}^{\perp})$ is

- integrable if and only if $A_{nZ}W = A_{nW}Z$;
- totally geodesic if and only if $g(h(Z, \phi X), \phi W) = 0$;

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \xi)$ and $Z, W \in \Gamma(\mathfrak{D}^{\perp})$.

Proof. The proof of the proposition can be achieved by following same steps as the proof of proposition 3.5. $\boxed{QED}$

Lemma 3.7. If M be a $\mathcal{PR}$-semi-invariant submanifold M of a paracosymplectic manifold $\bar{M}$. Then we have

$$g(\nabla_S Z, X) = g(\phi A_{\phi Z} S, X), \forall\ X \in \Gamma(\mathfrak{D} \oplus \xi),\ Z \in \Gamma(\mathfrak{D}^{\perp}),\ S \in \Gamma(TM). \quad (3.5)$$

$$A_{\phi Z} W = A_{\phi W} Z, \quad \forall\ Z, W \in \Gamma(\mathfrak{D}^{\perp}). \quad (3.6)$$

$$A_{\zeta} X = \phi A_{\phi \zeta} X, \quad \forall\ X \in \Gamma(\mathfrak{D} \oplus \xi),\ \zeta \in \Gamma(\nu). \quad (3.7)$$

$$g(A_{\zeta} X, Z) = -g(A_{\phi\zeta} \phi X, Z), \forall\ X \in \Gamma(\mathfrak{D} \oplus \xi),\ \zeta \in \Gamma(\nu) \text{ and } Z \in \Gamma(\mathfrak{D}^{\perp}). \quad (3.8)$$

Proof. We have from Eqs. (2.5), (2.7), (2.9) and lemmas 3.2, 3.3, that

$$g(\phi A_{\phi Z} S, X) = g(\phi \bar{\nabla}_S Z, \phi X).$$

Thus, formula-(3.5) follows from Eqs. (2.3), (2.8), lemma 3.2 and property of Levi-Civita connection $\bar{\nabla}$. To prove formula-(3.6), we have from the definition of paracosymplectic and Eq. (2.9) that $g(A_{\phi Z} W, S) = g(\bar{\nabla}_W Z, \phi S)$ which implies $g(A_{\phi Z} W - A_{\phi W} Z, S) = g([Z, W], \phi S)$. Therefore from the fact of integrablity of $(\mathfrak{D}^{\perp})$, $[Z, W] \in \Gamma(\mathfrak{D}^{\perp})$ while $\phi S \in \phi(\mathfrak{D} \oplus \mathfrak{D}^{\perp})$ conclude the formula-(3.6). Finally, the formulas-(3.7) and (3.8) follows from expressions $g(A_{\phi\zeta} X, S) = g(\zeta, h(X, S))$, $g(A_{\zeta} \phi X, S) = -g(\zeta, h(S, X))$ and the fact that distributions $\mathfrak{D}$ and ν are invariant under ϕ. The expressions above can be obtained in light of Eqs. (2.5), (2.8) ,(2.9), (2.10) and the fact symmetry of h. This completes the proof of the lemma. $\boxed{QED}$

Theorem 3.8. For a parallel canonical structure n, the $\mathcal{PR}$-semi-invariant submanifolds M of a paracosymplectic manifold $\bar{M}$ satisfies
(i) $A_{\nu} \mathfrak{D}^{\perp} = 0$, (ii) $A_{n\mathfrak{D}^{\perp}}(\mathfrak{D} \oplus \xi) = 0$ and (iii) M is $(\mathfrak{D} \oplus \xi, \mathfrak{D}^{\perp})$-mixed totally geodesic.

Proof. For any $X \in \Gamma(\mathfrak{D} \oplus \xi)$, $Z \in \Gamma(\mathfrak{D}^{\perp})$ the theorem is obvious in light of lemmas 3.2 and 3.3. Let $S, T \in \Gamma(TM)$, then from Eq. (2.15) and the fact that canonical structure n is parallel we find that

$$n' h(S, T) = h(S, tT). \quad (3.9)$$

In particular if we replace T by $Z \in \Gamma(\mathfrak{D}^{\perp})$ in previous expression we deduce that

$$n'h(S,Z) = 0. \tag{3.10}$$

Furthermore, by the use of Eqs. (2.5), (2.10) and (2.12), we obtain that

$$g(n'h(S,Z),\zeta) = -g(A_{\phi\zeta}Z,S)$$

for any $\zeta \in \nu$. Therefore previous expression using Eq. (3.10) and the fact that ν is invariant reduced to $g(A_{\zeta}Z,S) = 0$. Hence proves the formula-(i) since, S is a non null tangent vector field on M. For formula -(ii), we have from the certainty that n is parallel and Eqs. (2.3), (2.11), (2.12) that $g(h(S,tX),nZ) = g(n'h(S,X),\phi Z) = -g(h(S,X),Z) = 0$ for any $X \in \Gamma(\mathfrak{D})$. Now, using Eq. (2.10) and the fact that X belongs to the invariant distribution in equation $g(h(S,tX),nZ) = 0$, we achieve the required formula. Hence we can conclude using Eq. (3.9) that $n'^2h(X,Z) = n'h(X,tZ) = 0$, now again on the other side $n'^2h(X,Z) = n'^2h(Z,X) = n'h(Z,tX) = h(Z,X)$ for all $X \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}^{\perp})$. Finally, formula -$(iii)$ can be derived using previous expressions and the fact of symmetry of h. This completes the proof of the theorem. $\boxed{QED}$

4 An optimal geometric inequality

In this section, we first recall the definitions and results from [20] and then establish general sharp inequality (in the sense of B.Y.Chen [5]) for $\mathcal{PR}$-*semi-invariant warped product* submanifolds of the form $B \times_f F$ with the characteristic vector field ξ tangent to B in paracosymplectic manifolds.

Definition 4.1. A $\mathcal{PR}$-semi-invariant submanifold is called a $\mathcal{PR}$-*semi-invariant warped product* if it is a warped product of the form: $B \times_f F$ or $F \times_f B$, where B is an invariant submanifold, F is an anti-invariant submanifold of an almost paracontact manifold $\bar{M}$ and f is a non-constant positive smooth function on the first factor. If the warping function f is constant then a $\mathcal{PR}$-semi-invariant warped product submanifold is said to be a $\mathcal{PR}$-*semi-invariant product* [20].

For further systematic analysis of $\mathcal{PR}$-semi-invariant warped product submanifolds in a paracosymplectic manifold we recall some findings which implies the existence and nonexistence of the warped products of different structure.

Proposition 4.2. [20] There doesn't exist a $\mathcal{PR}$-semi-invariant warped product submanifold $M = B \times_f F$ of a paracosymplectic manifold $\bar{M}$ such that the characteristic vector field ξ is tangent to F.

Proposition 4.3. [20] There doesn't exist a $\mathcal{PR}$-semi-invariant warped product submanifold $M = B \times_f F$ of a paracosymplectic manifold $\bar{M}(\phi, \xi, \eta, g)$ such that the characteristic vector field ξ is normal to M.

Proposition 4.4. [20] Let $M \to \bar{M}$ be an isometric immersion of a pseudo-Riemannian manifold M into a paracosymplectic manifold $\bar{M}$. Then a necessary and sufficient condition for M to be a $\mathcal{PR}$-semi-invariant warped product $B \times_f F$ submanifold is that the shape operator of M satisfies

$$A_{\phi Z}X = -\phi X(\mu)Z,\ X \in \Gamma(\mathfrak{D} \oplus < \xi >), Z \in \Gamma(\mathfrak{D}^\perp), \tag{4.1}$$

for some function μ on M such that $V(\mu) = 0,\ V \in \Gamma(\mathfrak{D}^\perp)$.
Now here we establish some geometric inequalities for the warped structure of the form $B \times_f F$ such that the characteristic vector field ξ is tangent to B. We first gave an important lemma for later use;

Lemma 4.5. If $M = B \times_f F$ be a $\mathcal{PR}$-semi-invariant warped product submanifold M of a paracosymplectic manifold $\bar{M}$ such that ξ is tangent to B. Then we have

$$g(A_{\phi W}Z, tX) = -g(\nabla_X Z, W); \tag{4.2}$$

$$g(A_{\phi\zeta}Z, X) = -g(\nabla^\perp_X \phi Z, \zeta); \tag{4.3}$$

$$g(A_{\phi\zeta}Z, W) = -g(\nabla^\perp_Z \phi W, \zeta); \tag{4.4}$$

for any $X \in \Gamma(\mathfrak{D})$, $Z, W \in \Gamma(\mathfrak{D}^\perp)$ and $\zeta \in \Gamma(T^\perp M)$.

Proof. The proof of the lemma is obvious for $X = \xi$ by virtue of lemma 3.2, moreover for all $\xi \neq X \in \Gamma(\mathfrak{D})$, $Z, W \in \Gamma(\mathfrak{D}^\perp)$ and $\zeta \in \Gamma(T^\perp M)$ the lemma can be achieved in light of equations (2.3), (2.7)-(2.12), and proposition 2.2. This completes the proof of the lemma. $\boxed{QED}$

Theorem 4.6. Let $\bar{M}$ be an odd-dimensional paracosymplectic manifold and $M = B \times_f F$ be a warped product submanifold of $\bar{M}$ such that B is a $(2\alpha + 1)$-dimensional invariant submanifold tangent to ξ and F a β-dimensional anti-invariant submanifold of M. If we suppose that the submanifold F is timelike and the normal sub-bundle $\phi(F)$ in $\bar{M}$ is invariant under the action of $\nabla^\perp$ *i.e.*, $\nabla^\perp \phi(F) \subseteq \phi(F)$. Then the squared norm of the second fundamental form $||h||^2$ of M satisfies

$$||h||^2 \geq 2\beta||\nabla(\ln f)||^2 + ||h^{\mathfrak{D}}_\nu||^2 \tag{4.5}$$

where $\nabla(\ln f)$ is the gradient of $\ln f$ and $||h^{\mathfrak{D}}_\nu||^2 = g(h_\nu(\mathfrak{D}, \mathfrak{D}), h_\nu(\mathfrak{D}, \mathfrak{D}))$ with its ν component and invariant distribution $\mathfrak{D}$.

Proof. Let us consider the local orthonormal frame on B by $\{e_0 = \xi, e_i, e_{i^\star} = \phi(e_i)\}$, for any $i, j = \{1, \cdots, \alpha\}$ such that $\epsilon_0 = g(e_0, e_0) = 1$, $\epsilon_i = g(e_i, e_i) = 1$ accordingly $\epsilon_{i^\star} = g(e_{i^\star}, e_{i^\star}) = -1$ for all i, and on F by $\{\bar{e}_a\}$ for any $a, b, ... = \{1, \cdots, \beta\}$ such that $\bar{\epsilon}_a = g(\bar{e}_a, \bar{e}_a) = -1\ \forall\ a$. Furthermore, on ν the orthonormal frame can be assumed as $\{\zeta_k, \zeta_{k^\star} = \phi(\zeta_k)\}$, for any $k = \{1, \cdots, q\}$ such that $\epsilon_k = g(\zeta_k, \zeta_k) = 1$ accordingly $\epsilon_{k^\star} = g(\zeta_{k^\star}, \zeta_{k^\star}) = -1$ for all k. Now, we can write by definition of squared norm of the second fundamental form that

$$||h||^2 = g(h, h) = \\ g\big(h(\mathfrak{D}, \mathfrak{D}), h(\mathfrak{D}, \mathfrak{D})\big) + 2\, g\big(h(\mathfrak{D}, \mathfrak{D}^\perp), h(\mathfrak{D}, \mathfrak{D}^\perp)\big) + g\big(h(\mathfrak{D}^\perp, \mathfrak{D}^\perp), h(\mathfrak{D}^\perp, \mathfrak{D}^\perp)\big). \tag{4.6}$$

The first factor of the right hand side of Eq. (4.6) with the definition of $||h||^2$, can be written as;

$$\begin{aligned} g\big(h(\mathfrak{D}, \mathfrak{D}),&h(\mathfrak{D}, \mathfrak{D})\big) \\ = \sum_{i,j=1}^{\alpha} &\big[\epsilon_i\epsilon_j g\big(h(e_i, e_j), h(e_i, e_j)\big) + \epsilon_{i^\star}\epsilon_j g\big(h(e_{i^\star}, e_j), h(e_{i^\star}, e_j)\big) \\ &+\epsilon_i\epsilon_{j^\star} g\big(h(e_i, e_{j^\star}), (h(e_i, e_{j^\star})\big) + \epsilon_{i^\star}\epsilon_{j^\star} g\big(h(e_{i^\star}, e_{j^\star}), h(e_{i^\star}, e_{j^\star})\big)\big] \\ &+\sum_{i=1}^{\alpha} \big[\epsilon_0\epsilon_i g\big(h(e_0, e_i), h(e_0, e_i)\big) + \epsilon_0\epsilon_{i^\star} g\big(h(e_0, e_{i^\star}), h(e_0, e_{i^\star})\big)\big] \\ &+\epsilon_0\epsilon_0 g\big(h(e_0, e_0), h(e_0, e_0)\big). \end{aligned} \tag{4.7}$$

Furthermore, using the fact that $\mathfrak{D}$ is totally geodesic, as a consequence from proposition 3.5–(b), we conclude that $h(\mathfrak{D}, \mathfrak{D}) \in \nu$. So we can state the expressions as;

$$\begin{aligned} &h(e_i, e_j) = h^k_{ij}\zeta_k + h^{k^\star}_{ij}\zeta_{k^\star}, \quad h(e_i, e_{j^\star}) = h^k_{ij^\star}\zeta_k + h^{k^\star}_{ij^\star}\zeta_{k^\star}, \\ &h(e_{i^\star}, e_j) = h^k_{i^\star j}\zeta_k + h^{k^\star}_{i^\star j}\zeta_{k^\star}, \quad h(e_{i^\star}, e_{j^\star}) = h^k_{i^\star j^\star}\zeta_k + h^{k^\star}_{i^\star j^\star}\zeta_{k^\star}, \\ &h(e_0, e_i) = h^k_{0i}\zeta_k + h^{k^\star}_{0i}\zeta_{k^\star}, \quad h(e_0, e_{i^\star}) = h^k_{0i^\star}\zeta_k + h^{k^\star}_{0i^\star}\zeta_{k^\star}, \\ &h(e_0, e_0) = h^k_{00}\zeta_k + h^{k^\star}_{00}\zeta_{k^\star}. \end{aligned}$$

Above expressions in light of lemma 3.2 and Eq. (2.19), reduced to

$$\begin{aligned} &h(e_i, e_j) = h^k_{ij}\zeta_k + h^{k^\star}_{ij}\zeta_{k^\star}, \quad h(e_i, e_{j^\star}) = h^k_{ij^\star}\zeta_k + h^{k^\star}_{ij^\star}\zeta_{k^\star}, \\ &h(e_{i^\star}, e_j) = h^k_{i^\star j}\zeta_k + h^{k^\star}_{i^\star j}\zeta_{k^\star}, \quad h(e_{i^\star}, e_{j^\star}) = h^k_{i^\star j^\star}\zeta_k + h^{k^\star}_{i^\star j^\star}\zeta_{k^\star}, \\ &h(e_0, e_i) = h(e_0, e_{i^\star}) = h(e_0, e_0) = 0. \end{aligned} \tag{4.8}$$

Employing Eq. (4.8) in Eq. (4.7), we derive

$$g\big(h(\mathfrak{D},\mathfrak{D}),h(\mathfrak{D},\mathfrak{D})\big)$$
$$= \sum_{i,j=1}^{\alpha}\sum_{k=1}^{q}\big[\big\{(h_{ij}^{k})^2-(h_{ij}^{k^\star})^2\big\}-\big\{(h_{i^\star j}^{k})^2-(h_{i^\star j}^{k^\star})^2\big\}$$
$$-\big\{(h_{ij^\star}^{k})^2-(h_{ij^\star}^{k^\star})^2\big\}+\big\{(h_{i^\star j^\star}^{k})^2-(h_{i^\star j^\star}^{k^\star})^2\big\}\big] \tag{4.9}$$

Since $\mathfrak{D}$ is integrable, therefore by the use of proposition 3.5-(a) and formula (3.7) of lemma 3.7 in Eq. (4.9), we deduce that

$$g\big(h(\mathfrak{D},\mathfrak{D}),h(\mathfrak{D},\mathfrak{D})\big)=\sum_{i,j=1}^{\alpha}\sum_{k=1}^{q}\big[(h_{ij}^{k})^2-(h_{ij}^{k^\star})^2\big] \tag{4.10}$$

Next, we evaluate the second factor of right hand side of Eq. (4.6);

$$g\big(h(\mathfrak{D},\mathfrak{D}^{\perp}),h(\mathfrak{D},\mathfrak{D}^{\perp})\big)$$
$$=\sum_{i=1}^{\alpha}\sum_{a=1}^{\beta}\big[\epsilon_i\bar{\epsilon}_a g\big(h(e_i,\bar{e}_a),h(e_i,\bar{e}_a)\big)+\epsilon_{i^\star}\bar{\epsilon}_a g\big(h(e_{i^\star},\bar{e}_a),h(e_{i^\star},\bar{e}_a)\big)\big]$$
$$+\sum_{a=1}^{\beta}\epsilon_0\bar{\epsilon}_a g\big(h(e_0,\bar{e}_a),h(e_0,\bar{e}_a)\big). \tag{4.11}$$

Analogous to above, we can express the following as:

$$h(e_i,\bar{e}_a)=h_{ia}^{b}n\bar{e}_b+h_{ia}^{k}\zeta_k+h_{ia}^{k^\star}\zeta_{k^\star},\quad h(e_{i^\star},\bar{e}_a)=h_{i^\star a}^{b}n\bar{e}_b+h_{i^\star a}^{k}\zeta_k+h_{i^\star a}^{k^\star}\zeta_{k^\star},$$
$$h(e_0,\bar{e}_a)=h_{0a}^{b}n\bar{e}_b+h_{0a}^{k}\zeta_k+h_{0a}^{k^\star}\zeta_{k^\star}.$$

From lemma 3.2, above expressions reduced to

$$h(e_i,\bar{e}_a)=h_{ia}^{b}n\bar{e}_b+h_{ia}^{k}\zeta_k+h_{ia}^{k^\star}\zeta_{k^\star},\quad h(e_{i^\star},\bar{e}_a)=h_{i^\star a}^{b}n\bar{e}_b+h_{i^\star a}^{k}\zeta_k+h_{i^\star a}^{k^\star}\zeta_{k^\star},$$
$$h(e_0,\bar{e}_a)=0. \tag{4.12}$$

Equation (4.11), by the use of Eq. (4.12), becomes

$$g\big(h(\mathfrak{D},\mathfrak{D}^{\perp}),h(\mathfrak{D},\mathfrak{D}^{\perp})\big)$$
$$=\sum_{i=1}^{\alpha}\sum_{a,b=1}^{\beta}\sum_{k=1}^{q}\big[\big\{(h_{i^\star a}^{b})^2+(h_{i^\star a}^{k})^2-(h_{i^\star a}^{k^\star})^2\big\}-\big\{(h_{ia}^{b})^2+(h_{ia}^{k})^2-(h_{ia}^{k^\star})^2\big\}\big] \tag{4.13}$$

Formula (3.8) of lemma 3.7 in Eq. (4.13), yields that

$$g\big(h(\mathfrak{D},\mathfrak{D}^\perp),h(\mathfrak{D},\mathfrak{D}^\perp)\big) = \sum_{i=1}^{\alpha}\sum_{a,b=1}^{\beta}\big[(h^b_{i^\star a})^2-(h^b_{ia})^2\big]+2\sum_{i=1}^{\alpha}\sum_{a=1}^{\beta}\sum_{k=1}^{q}\big[(h^{k^\star}_{ia})^2-(h^k_{ia})^2\big]. \tag{4.14}$$

Using formula (4.2) of lemma (4.5) and Eq. (2.10) in Eq. (4.14), we obtain that

$$g\big(h(\mathfrak{D},\mathfrak{D}^\perp),h(\mathfrak{D},\mathfrak{D}^\perp)\big) = \beta\sum_{i=1}^{\alpha}\big[(e_i(\ln f))^2-(e_{i^\star}(\ln f))^2\big]+2\sum_{i=1}^{\alpha}\sum_{a=1}^{\beta}\sum_{k=1}^{q}\big[(h^{k^\star}_{ia})^2-(h^k_{ia})^2\big]. \tag{4.15}$$

Now, by employing the hypothesis $\nabla^\perp\phi(F)\subseteq\phi(F)$ and formula (4.3) of lemma (4.5), we conclude that $h(\mathfrak{D},\mathfrak{D}^\perp)\subseteq\phi(\mathfrak{D}^\perp)$. Thus Eq. (4.15), simplified to

$$g\big(h(\mathfrak{D},\mathfrak{D}^\perp),h(\mathfrak{D},\mathfrak{D}^\perp)\big) = \beta\sum_{i=1}^{\alpha}[(e_i(\ln f))^2-(e_{i^\star}(\ln f))^2]=\beta g\big(\nabla(\ln f),\nabla(\ln f)\big) \tag{4.16}$$

Finally, we have

$$g\big(h(\mathfrak{D}^\perp,\mathfrak{D}^\perp),h(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\big)=\sum_{a,b=1}^{\beta}\big[\bar\epsilon_a\bar\epsilon_b g\big(h(\bar e_a,\bar e_b),h(\bar e_a,\bar e_b)\big)\big]. \tag{4.17}$$

To compute the above equation we need to analyze $h(\bar e_a,\bar e_b)$ to do so, we express it in the form $h(\bar e_a,\bar e_b)=h^c_{ab}n\bar e_c+h^k_{ab}\zeta_k+h^{k^\star}_{ab}\zeta_{k^\star}$. Using this expression in Eq. (4.17), we arrive at

$$g\big(h(\mathfrak{D}^\perp,\mathfrak{D}^\perp),h(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\big)=\sum_{a,b,c=1}^{\beta}\sum_{k=1}^{q}\big[\bar\epsilon_a\bar\epsilon_b\big\{(h^c_{ab})^2+(h^k_{ab})^2-(h^{k^\star}_{ab})^2\big\}\big]. \tag{4.18}$$

From formula (4.4) of lemma (4.5) and using the assumption $\nabla^\perp\phi(F)\subseteq\phi(F)$, we get $h(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\subseteq\phi(\mathfrak{D}^\perp)$ as a consequence of which $h^k_{ab}=h^{k^\star}_{ab}$. Eq. (4.18) accordingly give

$$g\big(h(\mathfrak{D}^\perp,\mathfrak{D}^\perp),h(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\big)=\sum_{a,b,c=1}^{\beta}\big\{(h^c_{ab})^2\big\}. \tag{4.19}$$

By substituting, Eqs. (4.10), (4.16) and (4.19) in Eq. (4.6) we can achieve the inequality (4.5). This completes the proof of the theorem. $\boxed{QED}$

Now, we can state the following as a direct consequences of Theorem 4.6;

Remark 4.7. If M is mixed totally geodesic warped product submanifold in $\bar{M}$ and $\nabla^{\perp}\phi(F) \subseteq \phi(F)$ then the inequality (4.5) looks like

$$||h||^2 \geq ||h_{\nu}^{\mathfrak{D}}||^2.$$

Remark 4.8. If $h(\mathfrak{D}^{\perp}, \mathfrak{D}^{\perp}) = 0$ and $\nabla^{\perp}\phi(F) \subseteq \phi(F)$ for M in $\bar{M}$ then equality sign for inequality (4.5) holds identically.

Theorem 4.9. Let $\bar{M}$ be a $(2m+1)$-dimensional paracosymplectic manifold and $M = B \times_f F$ a warped product submanifold of $\bar{M}$ such that B is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and F a β-dimensional anti-invariant submanifold of M. If we assume that submanifold F is spacelike and $\nabla^{\perp}\phi(F) \subseteq \phi(F)$. Then the squared norm of the second fundamental form $||h||^2$ of M satisfies

$$||h||^2 \leq 2\beta||\nabla(\ln f)||^2 + ||h_{\nu}^{\mathfrak{D}}||^2; \tag{4.20}$$

- If $h(\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$ then inequality (4.20) shall be replaced by,

$$||h||^2 \leq ||h_{\nu}^{\mathfrak{D}}||^2,$$

- Equality sign for inequality (4.20) holds identically when,

$$h(\mathfrak{D}^{\perp}, \mathfrak{D}^{\perp}) = 0$$

where $\nabla(\ln f)$ is the gradient of $\ln f$ and $||h_{\nu}^{\mathfrak{D}}||^2 = g(h_{\nu}(\mathfrak{D}, \mathfrak{D}), h_{\nu}(\mathfrak{D}, \mathfrak{D}))$ with its ν component and the invariant distribution $\mathfrak{D}$.

Proof. The proof of this Theorem can be achieved by following the proof of Theorem 4.6 or the Theorem 4.1 (in the case of para-Kähler ambient) of [5]. $\boxed{QED}$

Theorem 4.10. Let $M = B \times_f F$ a warped product submanifold of a paracosymplectic manifold $\bar{M}$ such that B is a $(2\alpha + 1)$-dimensional invariant submanifold tangent to ξ and F a β-dimensional anti-invariant submanifold of M. If we suppose that the submanifold F is timelike (respecvtively spaclike), $\nabla^{\perp}\phi(F) \subseteq \phi(F)$ and B is totally geodesic in $\bar{M}$. Then the squared norm of the second fundamental form $||h||^2$ on M satisfies

$$||h||^2 \geq 2\beta||\nabla(\ln f)||^2 \quad \big(\text{respectively, } ||h||^2 \leq 2\beta||\nabla(\ln f)||^2\big); \tag{4.21}$$

- Equality sign for inequality (4.21) holds identically when,

$$h(\mathfrak{D}^{\perp}, \mathfrak{D}^{\perp}) = 0$$

where, $\nabla(\ln f)$ is the gradient of $\ln f$ and $\mathfrak{D}^{\perp}$ is the anti-invariant distribution on F.

Proof. The inequality (4.21) can be drawn from the proof of the Theorem 4.6 and Theorem 4.9 respectively, whereas the case of equality can be obtained from the proof of the inequality (4.21). $\boxed{QED}$

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