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New $W-$weighted concepts for continuous random variables with applications

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Abstract. New concepts on fractional probability theory are introduced and some inequalities for the fractional $w-$weighted expectation and the fractional $w-$weighted variance of continuous random variables are obtained. Other fractional results related to the two orders-fractional $w-$weighted moment are also established. Some recent results on integral inequality theory can be deduced as some special cases. At the end, some applications on the uniform random variable are given.

Keywords: Integral inequalities, Riemann-Liouville integral, random variable, fractional $w-$weighted expectation, fractional $w-$weighted variance, fractional $w-$weighed moment.

MSC 2000 classification: primary 26D15, secondary 26A33.

1 Introduction

In recent years, the integral inequalities have emerged as an important area of research, since this theory has many applications in differential equations and applied sciences. In this sense, a large number of papers have been developed, for details, we refer the reader to [1]-[4], [7, 12, 14], [22]-[27] and the references therein. Moreover, the fractional type inequalities have recently been studied by several researchers. For some earlier work on the topic, we refer to [2, 3, 6], [8]-[12] and [14, 17, 19, 20, 23]. In [5], using Korkine identity, N.S. Barnett et al. established some integral inequalities for the expectation and the variance of a continuous random variable X having a probability density function defined on $[a,b]$. In [16], P. Kumar presented new results involving higher moments for continuous random variables. He also established some estimations for the central moments. Other results based on Gruss inequality and some applications of the truncated exponential distribution have been also discussed by the author. In [18], P. Kumar established other good results for Ostrowski type integral inequalities involving moments of a continuous random variable with $p.d.f.$ defined on a finite interval. He also derived new bounds for the $r-$moments. Further, he discussed some important applications of the proposed bounds to the Euler beta mappings. Recently, G.A. Anastassiou et al. [2] proposed a generalization of the weighted Montgomery identity for fractional integrals with weighted fractional Peano kernel. Then, M. Niezgoda [21] proposed some generalizations for the paper [17], by applying Ostrowski-Grüss type inequalities. In [12], the author established several integral inequalities for the fractional dispersion and the fractional variance functions of continuous random variables with probability density functions $p.d.f.$ that are defined on some finite real intervals. Very recently, A. Akkurt et al. [1] proposed new generalizations of the results in [12]. In a very recent work, Z. Dahmani et al. [13] presented new fractional integral results for the $(r,\alpha)-$fractional moments. In fact, by introducing other concepts on the $(r,\alpha)-$orders fractional moments of continuous random variables (noted by $M_{r,\alpha}$), the authors generalized Theorem 1 in the paper [17]. Other results between the quantities $M_{2r,\alpha}$ and $M^2_{r,\alpha,}$ have been also generated by the authors.

Motivated by the results presented in [2, 5, 12], in this paper, we introduce new $w-$weighted concepts for continuous random variables that have $p.d.f.$ defined on some finite real intervals. Then, we obtain new integral inequalities for the fractional $w-$weighted expectation and the fractional $w-$weighted variance functions. We also present new integral inequalities for the fractional $(r,w)-$weighted moments. At the last section, some applications on the uniform random distribution are given. For our results, some classical and fractional results can be deduced as some special cases.

2 Preliminaries

The following notations, definitions and preliminary facts will be used throughout this paper.

Definition 1. [15] The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[a,b]$ is defined as

$$J_a^\alpha\left[f(t)\right]=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau,\ \alpha>0,\ a<t\leq b. \tag{1}$$

For $\alpha > 0$, $\beta > 0$, we have:

$$J_a^{\alpha} J_a^{\beta} \left[f\left(t\right) \right] = J_a^{\alpha+\beta} \left[f\left(t\right) \right], \tag{2}$$

and

$$J_a^{\alpha} J_a^{\beta} \left[f\left(t\right) \right] = J_a^{\beta} J_a^{\alpha} \left[f\left(t\right) \right]. \tag{3}$$

Let us consider a positive continuous function w defined on $[a,b]$. We introduce the concepts:

Definition 2. The fractional $w-$weighted expectation function of order $\alpha > 0$, for a random variable X with a positive $p.d.f.$ f defined on $[a,b]$ is defined as

$$E_{X,\alpha,w}\left(t\right) := \tfrac{1}{\Gamma(\alpha)} \int_a^t \left(t-\tau\right)^{\alpha-1} \tau w\left(\tau\right) f\left(\tau\right) d\tau,\ a \le t < b,\ \alpha > 0, \tag{4}$$

where $w : [a,b] \to \mathbb{R}^+$ is a positive continuous function.

Definition 3. The fractional $w-$weighted expectation function of order $\alpha > 0$ for the random variable $X - E\left(X\right)$ is given by

$$E_{X-E(X),\alpha,w}\left(t\right) := \frac{1}{\Gamma\left(\alpha\right)} \int_a^t \left(t-\tau\right)^{\alpha-1} \left(\tau - E\left(X\right)\right) w\left(\tau\right) f\left(\tau\right) d\tau,\ a \le t < b,\ \alpha > 0, \tag{5}$$

where $f : [a,b] \to \mathbb{R}^+$ is the $p.d.f.$ of X.

We introduce also the following definitions:

Definition 4. The fractional $w-$weighted variance function of order $\alpha > 0$ for a random variable X having a positive $p.d.f.$ f on $[a,b]$ is defined as

$$\sigma^2_{X,\alpha,w}\left(t\right) := \frac{1}{\Gamma\left(\alpha\right)} \int_a^t \left(t-\tau\right)^{\alpha-1} \left(\tau - E\left(X\right)\right)^2 w\left(\tau\right) f\left(\tau\right) d\tau,\ a \le t < b,\ \alpha > 0. \tag{6}$$

Definition 5. The fractional $w-$weighted moment function of orders $r > 0$, $\alpha > 0$ for a continuous random variable X having a $p.d.f.$ f defined on $[a,b]$ is defined as

$$M_{r,\alpha,w}\left(t\right) := \tfrac{1}{\Gamma(\alpha)} \int_a^t \left(t-\tau\right)^{\alpha-1} \tau^r w\left(\tau\right) f\left(\tau\right) d\tau,\ a \le t < b,\ \alpha > 0. \tag{7}$$

For the particular case $t = b$, we list the following definitions:

Definition 6. The fractional $w-$weighted expectation of order $\alpha > 0$ for a random variable X with a positive $p.d.f.$ f defined on $[a, b]$ is defined as

$$E_{X,\alpha,w} := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau w(\tau) f(\tau) \, d\tau, \ \alpha > 0. \tag{8}$$

Definition 7. The fractional $w-$weighted variance of order $\alpha > 0$ for a random variable X having a positive $p.d.f.$ f on $[a, b]$ is given by

$$\sigma^2_{X,\alpha,w} := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \left(\tau - E(X)\right)^2 w(\tau) f(\tau) \, d\tau, \ \alpha > 0. \tag{9}$$

Definition 8. The fractional $w-$weighted moment of orders $r > 0$, $\alpha > 0$ for a continuous random variable X having a $p.d.f.$ f defined on $[a, b]$ is defined by

$$M_{r,\alpha,w} := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau^r w(\tau) f(\tau) \, d\tau, \ \alpha > 0. \tag{10}$$

Based on the above definitions, we give the following remark:

Remark 1. (1:) If we take $\alpha = 1, w(t) = 1, t \in [a, b]$ in Definition 4, we obtain the classical expectation: $E_{X,1,1} = E(X)$.
(2:) If we take $\alpha = 1, w(t) = 1, t \in [a, b]$ in Definition 8, we obtain the classical variance: $\sigma^2_{X,1,1} = \sigma^2(X) = \int_a^b \left(\tau - E(X)\right)^2 f(\tau) \, d\tau$.
(3:) For $\alpha > 0$, we have $J_a^\alpha \left[f(t)\right] \leq \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}$.
(4:) If we take $\alpha = 1, w(t) = 1, t \in [a, b]$ in Definition 8, we obtain the classical moment of order $r > 0$ given by $M_r := \int_a^b \tau^r f(\tau) \, d\tau$.

3 Main Results

In this section, we present new $w-$weighted integral inequalities for random variables with probability density functions defined on some finite real intervals. We begin by the following theorem:

Theorem 1. *Let X be a continuous random variable having a p.d.f. $f : [a, b] \to \mathbb{R}^+$, and let $w : [a, b] \to \mathbb{R}^+$ be a positive continuous function. Then for all $\alpha > 0$, $a < t \leq b$, the following inequalities for fractional integrals hold:*

$$\begin{aligned} & J_a^\alpha \left[(wf)(t)\right] \sigma^2_{X,\alpha,w}(t) - \left(E_{X-E(X),\alpha,w}(t)\right)^2 \\ & \leq \|f\|^2_\infty \left[J_a^\alpha \left[w(t)\right] J_a^\alpha \left[t^2 w(t)\right] - \left(J_a^\alpha \left[t w(t)\right]\right)^2\right], \ f \in L_\infty [a, b], \end{aligned} \tag{11}$$

and

$$J_a^{\alpha}\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t)-\left(E_{X-E(X),\alpha,w}(t)\right)^2\leq \frac{1}{2}(t-a)^2\left(J_a^{\alpha}\left[(wf)(t)\right]\right)^2. \quad (12)$$

PROOF. We define the quantities:

$$H(\tau,\rho):=\left(g(\tau)-g(\rho)\right)\left(h(\tau)-h(\rho)\right),\ \tau,\rho\in(a,t),\ a<t\leq b, \quad (13)$$

and

$$\varphi_\alpha(t,\tau):=\tfrac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}p(\tau),\ \tau\in(a,t),\ a<t\leq b, \quad (14)$$

where $p:[a,b]\to\mathbb{R}^+$ is a continuous function.

Using (13) by (14), we can write

$$\int_a^t\varphi_\alpha(t,\tau)H(\tau,\rho)\,d\tau=\int_a^t\varphi_\alpha(t,\tau)\left(g(\tau)-g(\rho)\right)\left(h(\tau)-h(\rho)\right)d\tau. \quad (15)$$

And then,

$$\begin{aligned}&\int_a^t\int_a^t\varphi_\alpha(t,\tau)\varphi_\alpha(t,\rho)H(\tau,\rho)\,d\tau d\rho\\ &=\int_a^t\int_a^t\varphi_\alpha(t,\tau)\varphi_\alpha(t,\rho)\left(g(\tau)-g(\rho)\right)\left(h(\tau)-h(\rho)\right)d\tau d\rho.\end{aligned} \quad (16)$$

Hence,

$$\begin{aligned}&\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)p(\rho)\left(g(\tau)-g(\rho)\right)\left(h(\tau)-h(\rho)\right)d\tau d\rho\\ &=2J_a^{\alpha}\left[p(t)\right]J_a^{\alpha}\left[(pgh)(t)\right]-2J_a^{\alpha}\left[(pg)(t)\right]J_a^{\alpha}\left[(ph)(t)\right].\end{aligned} \quad (17)$$

Now, replacing $p(t)=w(t)f(t)$, $g(t)=h(t)=t-E(X)$, $w:[a,b]\to\mathbb{R}^+$, $a<t\leq b$ in (17), we obtain

$$\begin{aligned}&\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}(\tau-\rho)^2w(\tau)w(\rho)f(\tau)f(\rho)\,d\tau d\rho\\ &=2J_a^{\alpha}\left[(wf)(t)\right]J_a^{\alpha}\left[(wf)(t)(t-E(X))^2\right]-2(J_a^{\alpha}\left[(wf)(t)(t-E(X)\right])^2\\ &=2J_a^{\alpha}\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t)-2\left(E_{X-E(X),\alpha,w}(t)\right)^2.\end{aligned} \quad (18)$$

Since $f\in L_\infty\left([a,b]\right)$, we have

$$\begin{aligned}&\|f\|_\infty^2\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}w(\tau)w(\rho)(\tau-\rho)^2\,d\tau d\rho\\ &\leq 2\|f\|_\infty^2\left[J_a^{\alpha}\left[w(t)\right]J_a^{\alpha}\left[t^2w(t)\right]-(J_a^{\alpha}\left[tw(t)\right])^2\right].\end{aligned} \quad (19)$$

On the other hand, for τ, $\rho \in [a,t]$, $a < t \leq b$, we obtain

$$\begin{aligned}&\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} w(\tau)w(\rho)(\tau-\rho)^2 f(\tau) f(\rho)\, d\tau d\rho \\ &\leq \sup_{\tau,\rho\in[a,t]} \left|(\tau-\rho)\right|^2 \frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} w(\tau)w(\rho) f(\tau) f(\rho)\, d\tau d\rho \\ &= (t-a)^2 (J_a^\alpha\left[(wf)(t)\right])^2.\end{aligned} \tag{20}$$

Combining (18) and (19), we conclude that

$$\begin{aligned}&J_a^\alpha\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t) - \Big(E_{X-E(X),\alpha,w}(t)\Big)^2 \\ &\leq \|f\|_\infty^2 \Big[J_a^\alpha\left[w(t)\right] J_a^\alpha\left[t^2 w(t)\right] - (J_a^\alpha\left[tw(t)\right])^2\Big],\end{aligned} \tag{21}$$

and thanks to (18) and (20), it yields that

$$J_a^\alpha\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t) - \Big(E_{X-E(X),\alpha,w}(t)\Big)^2 \leq \tfrac{1}{2}(t-a)^2 (J_a^\alpha\left[(wf)(t)\right])^2. \tag{22}$$

Theorem 1 is thus proved. QED

Remark 2. If we take $w(t) = 1$, $a < t \leq b$ in Theorem 1, we obtain Theorem 3.1 of [12].

We prove also the following theorem.

Theorem 2. *Suppose that X is a continuous random variable with a p.d.f. $f : [a,b] \to \mathbb{R}^+$ and let $w : [a,b] \to \mathbb{R}^+$ be a continuous function.*
(I): If $f \in L_\infty([a,b])$, then for all $\alpha > 0, \beta > 0$, $a < t \leq b$,

$$\begin{aligned}&J_a^\alpha\left[(wf)(t)\right]\sigma^2_{X,\beta,w}(t) + J_a^\beta\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t) \\ &\quad - 2E_{X-E(X),\alpha,w}(t)\, E_{X-E(X),\beta,w}(t) \\ &\leq \|f\|_\infty^2 \Big[J_a^\alpha\left[w(t)\right] J_a^\beta\left[t^2 w(t)\right] + J_a^\beta\left[w(t)\right] J_a^\alpha\left[t^2 w(t)\right] \\ &\quad -2J_a^\alpha\left[tw(t)\right] J_a^\beta\left[tw(t)\right]\Big].\end{aligned} \tag{23}$$

(II): For $a < t \leq b$, the inequality

$$\begin{aligned}&J_a^\alpha\left[(wf)(t)\right]\sigma^2_{X,\beta,w}(t) + J_a^\beta\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t) \\ &\quad - 2E_{X-E(X),\alpha,w}(t)\, E_{X-E(X),\beta,w}(t) \\ &\leq (t-a)^2 J_a^\alpha\left[(wf)(t)\right] J_a^\beta\left[(wf)(t)\right]\end{aligned} \tag{24}$$

is also valid for any $\alpha > 0, \beta > 0$.

PROOF. Thanks to (15), yields the following identity

$$\begin{aligned}&\int_a^t\int_a^t \varphi_\alpha(t,\tau)\,\varphi_\beta(t,\rho)\,H(\tau,\rho)\,d\tau d\rho\\ =&\int_a^t\int_a^t \varphi_\alpha(t,\tau)\,\varphi_\beta(t,\rho)\big(g(\tau)-g(\rho)\big)\big(h(\tau)-h(\rho)\big)\,d\tau d\rho.\end{aligned}\tag{25}$$

This implies that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)\,p(\rho)\big(g(\tau)-g(\rho)\big)\big(h(\tau)-h(\rho)\big)\,d\tau d\rho\\ &= J_a^\alpha\big[p(t)\big]J_a^\beta\big[(pgh)(t)\big]+J_a^\beta\big[p(t)\big]J_a^\alpha\big[(pgh)(t)\big]\\ &-J_a^\alpha\big[(ph)(t)\big]J_a^\beta\big[(pg)(t)\big]-J_a^\beta\big[(ph)(t)\big]J_a^\alpha\big[(pg)(t)\big].\end{aligned}\tag{26}$$

In (26), if we take $p(t)=w(t)f(t)$, $g(t)=h(t)=t-E(X)$, then we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}(\tau-\rho)^2\,w(\tau)w(\rho)f(\tau)\,f(\rho)\,d\tau d\rho\\ &=J_a^\alpha\big[(wf)(t)\big]J_a^\beta\Big[(wf)(t)(t-E(X))^2\Big]\\ &\quad+J_a^\beta\big[(wf)(t)\big]J_a^\alpha\Big[(wf)(t)(t-E(X))^2\Big]\\ &\quad-2J_a^\alpha\big[(wf)(t)(t-E(X))\big]J_a^\beta\big[(wf)(t)(t-E(X))\big]\\ &=J_a^\alpha\big[(wf)(t)\big]\sigma^2_{X,\beta,w}(t)+J_a^\beta\big[(wf)(t)\big]\sigma^2_{X,\alpha,w}(t)\\ &\quad-2E_{X-E(X),\alpha,w}(t)\,E_{X-E(X),\beta,w}(t).\end{aligned}\tag{27}$$

Let $f\in L_\infty\big([a,b]\big)$. Then,

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}(\tau-\rho)^2\,w(\tau)w(\rho)f(\tau)\,f(\rho)\,d\tau d\rho\\ &\le\|f\|_\infty^2\,\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}(\tau-\rho)^2\,w(\tau)w(\rho)d\tau d\rho\\ &=\|f\|_\infty^2\Big[J_a^\alpha\big[w(t)\big]J_a^\beta\Big[t^2w(t)\Big]+J_a^\beta\big[w(t)\big]J_a^\alpha\Big[t^2w(t)\Big]\\ &\quad-2J_a^\alpha\big[tw(t)\big]J_a^\beta\big[tw(t)\big]\Big].\end{aligned}\tag{28}$$

Using the fact that $\sup\limits_{\tau,\rho\in[a,t]}\left|(\tau-\rho)\right|^2=(t-a)^2$, it follows that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}w(\tau)w(\rho)(\tau-\rho)^2f(\tau)f(\rho)\,d\tau d\rho\\ &\leq\sup_{\tau,\rho\in[a,t]}\left|(\tau-\rho)\right|^2\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}w(\tau)w(\rho)f(\tau)f(\rho)d\tau d\rho\\ &\qquad\leq(t-a)^2J_a^\alpha\left[(wf)(t)\right]J_a^\beta\left[(wf)(t)\right].\end{aligned}\tag{29}$$

Thanks to (27) and (28), we obtain

$$\begin{aligned}&J_a^\alpha\left[(wf)(t)\right]\sigma^2_{X,\beta,w}(t)+J_a^\beta\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t)\\ &\qquad-2E_{X-E(X),\alpha,w}(t)\,E_{X-E(X),\beta,w}(t)\\ &\qquad\leq\|f\|_\infty^2\Big[J_a^\alpha\left[w(t)\right]J_a^\beta\left[t^2w(t)\right]+J_a^\beta\left[w(t)\right]J_a^\alpha\left[t^2w(t)\right]\\ &\qquad\qquad-2J_a^\alpha\left[tw(t)\right]J_a^\beta\left[tw(t)\right]\Big].\end{aligned}\tag{30}$$

By (27) and (29), we have

$$\begin{aligned}&J_a^\alpha\left[(wf)(t)\right]\sigma^2_{X,\beta,w}(t)+J_a^\beta\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t)\\ &\qquad-2E_{X-E(X),\alpha,w}(t)\,E_{X-E(X),\beta,w}(t)\\ &\qquad\qquad\leq(t-a)^2J_a^\alpha\left[(wf)(t)\right]J_a^\beta\left[(wf)(t)\right].\end{aligned}\tag{31}$$

QED

Remark 3. (i) : Applying Theorem 2 for $\alpha=\beta$, we obtain Theorem 1.
(ii) : Taking $w(t)=1$, $a<t\leq b$ in Theorem 2, we obtain theorem 3.2 of [12].

The third main result is the following theorem which generalizes the second part of Theorem 1. We have:

Theorem 3. *Let f be the p.d.f. of X on $[a,b]$ and $w:[a,b]\to\mathbb{R}^+$. Then the following fractional inequality holds:*

$$J_a^\alpha\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t)-\Big(E_{X-E(X),\alpha,w}(t)\Big)^2\leq\frac{1}{4}(b-a)^2\Big(J_a^\alpha\left[(wf)(t)\right]\Big)^2,\tag{32}$$

for $\alpha>0$ and $a<t\leq b$.

Proof. Let $l\leq h(t)\leq L$ and $m\leq g(t)\leq M$, with $l,L,m,M\in\mathbb{R}^+$. For $\alpha>0$ and for each $a<t\leq b$, by Theorem 3.1 of [9], we have

$$\begin{aligned}&\Big|J_a^\alpha\left[p(t)\right]J_a^\alpha\left[(phg)(t)\right]-J_a^\alpha\left[(ph)(t)\right]J_a^\alpha\left[(pg)(t)\right]\Big|\\ &\qquad\leq\frac{1}{4}\Big(J_a^\alpha\left[p(t)\right]\Big)^2(L-l)(M-m).\end{aligned}\tag{33}$$

In this above inequality, we replace h by g, we will have

$$\left| J_a^\alpha \left[p\left(t\right)\right] J_a^\alpha \left[\left(pg^2\right)\left(t\right)\right] - \left(J_a^\alpha \left[pg\left(t\right)\right]\right)^2 \right| \leq \frac{1}{4}\left(J_a^\alpha \left[p(t)\right]\right)^2 \left(M-m\right)^2 . \quad (34)$$

By taking $p(t) = w(t)f(t)$, $g(t) = t - E(X)$, $a < t \leq b$ in (34), we obtain:

$$\left| J_a^\alpha \left[\left(wf\right)\left(t\right)\right] J_a^\alpha \left[\left(wf\right)\left(t\right)\left(t-E(X)\right)^2\right] - \left(J_a^\alpha \left[\left(wf\right)\left(t\right)\left(t-E(X)\right)\right]\right)^2 \right| \\ \leq \frac{1}{4}\left(J_a^\alpha \left[\left(wf\right)\left(t\right)\right]\right)^2 \left(M-m\right)^2 . \quad (35)$$

In (35), we take $M = b - E(X)$ and $m = a - E(X)$, then we have

$$0 \leq J_a^\alpha \left[\left(wf\right)\left(t\right)\right] J_a^\alpha \left[\left(wf\right)\left(t\right)\left(t-E(X)\right)^2\right] - \left(J_a^\alpha \left[\left(wf\right)\left(t\right)\left(t-E(X)\right)\right]\right)^2 \\ \leq \frac{1}{4}\left(b-a\right)^2 \left(J_a^\alpha \left[\left(wf\right)\left(t\right)\right]\right)^2 , \quad (36)$$

which is clearly equivalent to the following inequality

$$J_a^\alpha \left[\left(wf\right)\left(t\right)\right] \sigma^2_{X,\alpha,w}\left(t\right) - \left(E_{X-E(X),\alpha,w}\left(t\right)\right)^2 \leq \frac{1}{4}\left(b-a\right)^2 \left(J_a^\alpha \left[\left(wf\right)\left(t\right)\right]\right)^2 . \quad (37)$$

QED

Remark 4. Taking $w(t) = 1$, $a < t \leq b$ in Theorem 3, we obtain Theorem 3.3 of [12].

Another result is the following:

Theorem 4. *Let f be the p.d.f. of the random variable X on $[a,b]$ and $w : [a,b] \to \mathbb{R}^+$. Then for all $\alpha > 0, \beta > 0$, $a < t \leq b$, the inequality*

$$J_a^\alpha \left[\left(wf\right)\left(t\right)\right] \sigma^2_{X,\beta,w}\left(t\right) + J_a^\beta \left[\left(wf\right)\left(t\right)\right] \sigma^2_{X,\alpha,w}\left(t\right) + 2\left(a - E\left(X\right)\right) \\ \times \left(b - E\left(X\right)\right) J_a^\alpha \left[\left(wf\right)\left(t\right)\right] J_a^\beta \left[\left(wf\right)\left(t\right)\right] \\ \leq \left(a + b - 2E(X)\right) \\ \times \left(J_a^\alpha \left[\left(wf\right)\left(t\right)\right] E_{X-E(X),\beta,w}\left(t\right) + J_a^\beta \left[\left(wf\right)\left(t\right)\right] E_{X-E(X),\alpha,w}\left(t\right)\right) , \quad (38)$$

is valid.

Proof. We take $p(t) = w(t)f(t)$, $g(t) = t - E(X)$. Then, thanks to Theorem

3.4 of [9], we can write

$$\begin{aligned}
&\Big[J_a^{\alpha}\left[(wf)(t)\right]J_a^{\beta}\left[(wf)(t)(t-E(X))^2\right]+J_a^{\beta}\left[(wf)(t)\right]J_a^{\alpha}\left[(wf)(t)(t-E(X))^2\right]\\
&\qquad -2J_a^{\alpha}\left[(wf)(t)(t-E(X))\right]J_a^{\beta}\left[(wf)(t)(t-E(X))\right]\Big]^2\\
&\qquad \leq\Big[\Big(MJ_a^{\alpha}\left[(wf)(t)\right]-J_a^{\alpha}\left[(wf)(t)(t-E(X))\right]\Big)\\
&\qquad \times\Big(J_a^{\beta}\left[(wf)(t)(t-E(X))\right]-mJ_a^{\beta}\left[(wf)(t)\right]\Big)\\
&\qquad +\Big(J_a^{\alpha}\left[(wf)(t)(t-E(X))\right]-mJ_a^{\alpha}\left[(wf)(t)\right]\Big)\\
&\qquad \times\Big(MJ_a^{\beta}\left[(wf)(t)\right]-J_a^{\beta}\left[(wf)(t)(t-E(X))\right]\Big)\Big]^2.
\end{aligned}\tag{39}$$

By (27) and (39) and taking into account the fact that the left-hand side of (27) is positive, we can write

$$\begin{aligned}
&J_a^{\alpha}\left[(wf)(t)\right]J_a^{\beta}\left[(wf)(t)(t-E(X))^2\right]+J_a^{\beta}\left[(wf)(t)\right]J_a^{\alpha}\left[(wf)(t)(t-E(X))^2\right]\\
&\qquad -2J_a^{\alpha}\left[(wf)(t)(t-E(X))\right]J_a^{\beta}\left[(wf)(t)(t-E(X))\right]\\
&\qquad \leq\Big(MJ_a^{\alpha}\left[(wf)(t)\right]-J_a^{\alpha}\left[(wf)(t)(t-E(X))\right]\Big)\\
&\qquad \times\Big(J_a^{\beta}\left[(wf)(t)(t-E(X))\right]-mJ_a^{\beta}\left[(wf)(t)\right]\Big)\\
&\qquad +\Big(J_a^{\alpha}\left[(wf)(t)(t-E(X))\right]-mJ_a^{\alpha}\left[(wf)(t)\right]\Big)\\
&\qquad \times\Big(MJ_a^{\beta}\left[(wf)(t)\right]-J_a^{\beta}\left[(wf)(t)(t-E(X))\right]\Big).
\end{aligned}\tag{40}$$

Therefore,

$$\begin{aligned}
&J_a^{\alpha}\left[(wf)(t)\right]\sigma^2_{X,\beta,w}(t)+J_a^{\beta}\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t)\\
&\qquad -2E_{X-E(X),\alpha,w}(t)\,E_{X-E(X),\beta,w}(t)\\
&\qquad \leq\Big(MJ_a^{\alpha}\left[(wf)(t)\right]-E_{X-E(X),\alpha,w}(t)\Big)\\
&\qquad \times\Big(E_{X-E(X),\beta,w}(t)-mJ_a^{\beta}\left[(wf)(t)\right]\Big)\\
&\qquad +\Big(E_{X-E(X),\alpha,w}(t)-mJ_a^{\alpha}\left[(wf)(t)\right]\Big)\\
&\qquad \times\Big(MJ_a^{\beta}\left[(wf)(t)\right]-E_{X-E(X),\beta,w}(t)\Big).
\end{aligned}\tag{41}$$

This implies that

$$\begin{aligned}
&J_a^\alpha\left[(wf)(t)\right]\sigma^2_{X,\beta,w}(t)+J_a^\beta\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t)\\
&\quad+2mMJ_a^\alpha\left[(wf)(t)\right]J_a^\beta\left[(wf)(t)\right]\\
&\quad\leq(M+m)\\
&\quad\times\Big(J_a^\alpha\left[(wf)(t)\right]E_{X-E(X),\beta,w}(t)+J_a^\beta\left[(wf)(t)\right]E_{X-E(X),\alpha,w}(t)\Big).
\end{aligned}\tag{42}$$

In (42), we take $M=b-E(X)$, $m=a-E(X)$. We obtain:

$$\begin{aligned}
&J_a^\alpha\left[(wf)(t)\right]\sigma^2_{X,\beta,w}(t)+J_a^\beta\left[(wf)(t)\right]\sigma^2_{X,\alpha,w}(t)+2\left(a-E(X)\right)\\
&\qquad\times\left(b-E(X)\right)J_a^\alpha\left[(wf)(t)\right]J_a^\beta\left[(wf)(t)\right]\\
&\qquad\leq\left(a+b-2E(X)\right)\\
&\quad\times\Big[J_a^\alpha\left[(wf)(t)\right]E_{X-E(X),\beta,w}(t)+J_a^\beta\left[(wf)(t)\right]E_{X-E(X),\alpha,w}(t)\Big].
\end{aligned}\tag{43}$$

QED

Remark 5. If we take $w(t)=1, t\in[a,b]$ in Theorem 4, we obtain Theorem 3.4 of [12].

Next, we present the following five results for fractional $w-$weighted moments, where w is a positive continuous function defined on $[a,b]$.

Theorem 5. *Let X be a continuous random variable having a p.d.f. $f:[a,b]\to\mathbb{R}^+$. Then, for any $a<t\leq b$ and $\alpha>0$, the following two inequalities hold:*

$$\begin{aligned}
&J_a^\alpha\left[(wf)(t)\right]E_{X^{r-1}(X-E(X)),\alpha,w}(t)-E_{X-E(X),\alpha,w}(t)\,M_{r-1,\alpha,w}(t)\\
&\leq\|f\|_\infty^2\Big[J_a^\alpha\left[w(t)\right]J_a^\alpha\left[t^rw(t)\right]-J_a^\alpha\left[tw(t)\right]J_a^\alpha\left[t^{r-1}w(t)\right]\Big],\ f\in L_\infty[a,b]
\end{aligned}\tag{44}$$

and

$$\begin{aligned}
&J_a^\alpha\left[(wf)(t)\right]E_{X^{r-1}(X-E(X)),\alpha,w}(t)-E_{X-E(X),\alpha,w}(t)\,M_{r-1,\alpha,w}(t)\\
&\leq\tfrac{1}{2}(t-a)\left(t^{r-1}-a^{r-1}\right)\Big(J_a^\alpha\left[(wf)(t)\right]\Big)^2,\ \alpha>0,\ a<t\leq b.
\end{aligned}\tag{45}$$

Proof. In (17), we choose $p(t)=w(t)f(t)$, $g(t)=t-E(X)$ and $h(t)=t^{r-1}$.

So, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)w(\tau)w(\rho)f(\tau)f(\rho)d\tau d\rho \\
&\quad = 2J_a^\alpha\left[(wf)(t)\right]J_a^\alpha\left[t^{r-1}(t-E(X))(wf)(t)\right] \\
&\quad\quad - 2J_a^\alpha\left[(t-E(X))(wf)(t)\right]J_a^\alpha\left[t^{r-1}(wf)(t)\right] \\
&\quad = 2J_a^\alpha\left[(wf)(t)\right]E_{X^{r-1}(X-E(X)),\alpha,w}(t) \\
&\quad\quad - 2\left(E_{X-E(X),\alpha,w}(t)\right)M_{r-1,\alpha,w}(t).
\end{aligned} \tag{46}$$

We use the fact $f\in L_\infty([a,b])$, we can write

$$\begin{aligned}
&\tfrac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)w(\tau)w(\rho)f(\tau)f(\rho)d\tau d\rho \\
&\le \|f\|_\infty^2\tfrac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)w(\tau)w(\rho)d\tau d\rho \\
&\quad = \|f\|_\infty^2\left[2J_a^\alpha[w(t)]J_a^\alpha[t^r w(t)] - 2J_a^\alpha[tw(t)]J_a^\alpha[t^{r-1}w(t)]\right].
\end{aligned} \tag{47}$$

By (46) and (47), we have

$$\begin{aligned}
&J_a^\alpha\left[(wf)(t)\right]E_{X^{r-1}(X-E(X)),\alpha,w}(t) - \left(E_{X-E(X),\alpha,w}(t)\right)M_{r-1,\alpha,w}(t) \\
&\quad\le \|f\|_\infty^2\left[J_a^\alpha[w(t)]J_a^\alpha[t^r w(t)] - J_a^\alpha[tw(t)]J_a^\alpha[t^{r-1}w(t)]\right].
\end{aligned} \tag{48}$$

Since $\sup_{\tau,\rho\in[a,t]}\left[|\tau-\rho|\left|\tau^{r-1}-\rho^{r-1}\right|\right] = (t-a)\left(t^{r-1}-a^{r-1}\right)$, then we observe that

$$\begin{aligned}
&\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)w(\tau)w(\rho)f(\tau)f(\rho)d\tau d\rho \\
&\quad\le \sup_{\tau,\rho\in[a,t]}\left[|\tau-\rho|\left|\tau^{r-1}-\rho^{r-1}\right|\right] \\
&\quad\quad\times\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}w(\tau)w(\rho)f(\tau)f(\rho)d\tau d\rho \\
&\quad = (t-a)\left(t^{r-1}-a^{r-1}\right)\left(J_a^\alpha\left[(wf)(t)\right]\right)^2.
\end{aligned} \tag{49}$$

Thanks to (46) and (49), we obtain

$$\begin{aligned} &J_a^{\alpha}\left[(wf)(t)\right]E_{X^{r-1}\left(X-E(X)\right),\alpha,w}(t)-\left(E_{X-E(X),\alpha,w}(t)\right)M_{r-1,\alpha,w}(t)\\ &\qquad\leq(t-a)\left(t^{r-1}-a^{r-1}\right)\left(J_a^{\alpha}\left[(wf)(t)\right]\right)^2. \end{aligned} \tag{50}$$

QED

Theorem 6. *Let X be a continuous random variable having a p.d.f. $f:[a,b]\to\mathbb{R}^+$. Then we have:*

(I^*): For any $\alpha>0,\beta>0$,

$$\begin{aligned} &J_a^{\alpha}\left[(wf)(t)\right]E_{X^{r-1}\left(X-E(X)\right),\beta,w}(t)+J_a^{\beta}\left[(wf)(t)\right]E_{X^{r-1}\left(X-E(X)\right),\alpha,w}(t)\\ &\qquad-E_{X,\alpha,w}(t)M_{r-1,\beta,w}(t)-E_{X,\beta,w}(t)M_{r-1,\alpha,w}(t)\\ &\qquad\leq\|f\|_{\infty}^2\Big[J_a^{\alpha}\left[w(t)\right]J_a^{\beta}\left[t^rw(t)\right]+J_a^{\beta}\left[w(t)\right]J_a^{\alpha}\left[t^rw(t)\right]\\ &\qquad-J_a^{\alpha}\left[tw(t)\right]J_a^{\beta}\left[t^{r-1}w(t)\right]-J_a^{\beta}\left[tw(t)\right]J_a^{\alpha}\left[t^{r-1}w(t)\right]\Big],\ a<t\leq b, \end{aligned} \tag{51}$$

where $f\in L_{\infty}[a,b]$.

(II^*): The inequality

$$\begin{aligned} &J_a^{\alpha}\left[(wf)(t)\right]E_{X^{r-1}\left(X-E(X)\right),\beta,w}(t)+J_a^{\beta}\left[(wf)(t)\right]E_{X^{r-1}\left(X-E(X)\right),\alpha,w}(t)\\ &\qquad-E_{X,\alpha,w}(t)M_{r-1,\beta,w}(t)-E_{X,\beta,w}(t)M_{r-1,\alpha,w}(t)\\ &\qquad\leq(t-a)\left(t^{r-1}-a^{r-1}\right)J_a^{\alpha}\left[(wf)(t)\right]J_a^{\beta}\left[(wf)(t)\right],\ a<t\leq b, \end{aligned} \tag{52}$$

is also valid for any $\alpha>0$, $\beta>0$.

Proof. In (26), we take $p(t)=w(t)f(t)$, $g(t)=t-E(X)$, $h(t)=t^{r-1}$. So, we get

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^t\int_a^t(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1})(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)w(\tau)w(\rho)f(\tau)f(\rho)\,d\tau d\rho\\ &\qquad=J_a^{\alpha}\left[(wf)(t)\right]J_a^{\beta}\left[t^{r-1}(t-E(X))(wf)(t)\right]\\ &\qquad\quad+J_a^{\beta}\left[(wf)(t)\right]J_a^{\alpha}\left[t^{r-1}(t-E(X))(wf)(t)\right]\\ &\qquad\quad-J_a^{\alpha}\left[(t-E(X))(wf)(t)\right]J_a^{\beta}\left[t^{r-1}wf(t)\right]\\ &\qquad\quad-J_a^{\beta}\left[(t-E(X))(wf)(t)\right]J_a^{\alpha}\left[t^{r-1}(wf)(t)\right]\\ &\qquad=J_a^{\alpha}\left[(wf)(t)\right]E_{X^{r-1}\left(X-E(X)\right),\beta,w}(t)\\ &\qquad\quad+J_a^{\beta}\left[(wf)(t)\right]E_{X^{r-1}\left(X-E(X)\right),\alpha,w}(t)\\ &\qquad\quad-E_{X,\alpha,w}(t)M_{r-1,\beta,w}(t)-E_{X,\beta,w}(t)M_{r-1,\alpha,w}(t). \end{aligned} \tag{53}$$

We have also

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)w(\tau)w(\rho)f(\tau)f(\rho)d\tau d\rho\\
&\leq \|f\|_\infty^2\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)w(\tau)w(\rho)d\tau d\rho\\
&\quad= \|f\|_\infty^2\Big[J_a^\alpha\left[w(t)\right]J_a^\beta\left[t^r w(t)\right]+J_a^\beta\left[w(t)\right]J_a^\alpha\left[t^r w(t)\right]\\
&\quad- J_a^\alpha\left[tw(t)\right]J_a^\beta\left[t^{r-1}w(t)\right]-J_a^\beta\left[tw(t)\right]J_a^\alpha\left[t^{r-1}w(t)\right]\Big].\end{aligned}\tag{54}$$

By (53) and (54), we obtain (51).

To prove (52), we remark that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}w(\tau)w(\rho)(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)d\tau d\rho\\
&\leq \sup_{\tau,\rho\in[a,t]}\left|(\tau-\rho\right|\left|\tau^{r-1}-\rho^{r-1}\right|\\
&\quad\times\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}w(\tau))w(\rho)f(\tau f(\rho)d\tau d\rho\\
&= (t-a)\left(t^{r-1}-a^{r-1}\right)J_a^\alpha\left[(wf)(t)\right]J_a^\beta\left[(wf)(t)\right].\end{aligned}\tag{55}$$

Therefore, by (53) and (55), we get (52). This ends the proof of Theorem 6. QED

Theorem 7. *Let X be a continuous random variable having a p.d.f. $f : [a,b]\to\mathbb{R}^+$. Then, for all $\alpha>0$, we have:*

$$J_a^\alpha\left[(wf)(t)\right]M_{2r,\alpha,w}(t)-M_{r,\alpha,w}^2(t)\leq\frac{1}{4}\left(b^r-a^r\right)^2\left(J_a^\alpha\left[(wf)(t)\right]\right)^2,\quad a<t\leq b.\tag{56}$$

Proof. We use the same arguments as in the proof of Theorem 3 by taking $p(t)=w(t)f(t)$, $g(t)=t^r$, $a<t\leq b$, $m=a^r$ and $M=b^r$. QED

Theorem 8. *Let X be a continuous random variable having a p.d.f. f :*

$[a,b]\to\mathbb{R}^+$. *Then, for all* $\alpha>0,\beta>0$, *we have:*

$$\begin{aligned} J_a^{\alpha}\left[(wf)(t)\right]M_{2r,\beta,w}(t)+J_a^{\beta}\left[(wf)(t)\right]M_{2r,\alpha,w}(t) \\ +2a^rb^rJ_a^{\alpha}\left[(wf)(t)\right]J_a^{\beta}\left[(wf)(t)\right] \\ \le (a^r+b^r)\,J_a^{\alpha}\left[(wf)(t)\right]M_{r,\beta,w}(t)+J_a^{\beta}\left[(wf)(t)\right]M_{r,\alpha,w}(t), \\ a<t\le b. \end{aligned} \quad (57)$$

Proof. We use the same techniques as in the proof of Theorem 4 by letting $p(t)=w(t)f(t)$, $g(t)=t^r$, $a<t\le b$, $m=a^r$ and $M=b^r$. QED

Theorem 9. *Let* X *be a continuous random variable having a p.d.f.* $f:[a,b]\to\mathbb{R}^+$ *such that* $m\le f\le M$, m,M, *are positive real numbers. Then for* $\alpha>0$, *the following inequality holds:*

$$\begin{aligned} \left|\tfrac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}M_{r,\alpha,w}(t)-J_a^{\alpha}\left[f(t)\right]J_a^{\alpha}\left[t^rw(t)\right]\right| \\ \le \tfrac{(t-a)^{\alpha}}{2\Gamma(\alpha+1)}(M-m)\left(\tfrac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\alpha}\left[t^{2r}w^2(t)\right]-\left(J_a^{\alpha}\left[t^rw(t)\right]\right)^2\right)^{\frac{1}{2}},\ a<t\le b. \end{aligned} \quad (58)$$

Proof. Using Theorem 3.1 and lemma 3.2 of [11], we can write

$$\begin{aligned} \left|\tfrac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\alpha}\left[(fg)(t)\right]-J_a^{\alpha}\left[f(t)\right]J_a^{\alpha}\left[g(t)\right]\right| \\ \le \tfrac{(t-a)^{\alpha}}{2\Gamma(\alpha+1)}(M-m)\left(\tfrac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\alpha}\left[g^2(t)\right]-\left(J_a^{\alpha}\left[g(t)\right]\right)^2\right)^{\frac{1}{2}}. \end{aligned} \quad (59)$$

Taking $g(t)=w(t)t^r$, $a<t\le b$, we obtain

$$\begin{aligned} \left|\tfrac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\alpha}\left[t^r(fw)(t)\right]-J_a^{\alpha}\left[f(t)\right]J_a^{\alpha}\left[t^rw(t)\right]\right| \\ \le \tfrac{(t-a)^{\alpha}}{2\Gamma(\alpha+1)}(M-m)\left(\tfrac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\alpha}\left[t^{2r}w^2(t)\right]-\left(J_a^{\alpha}\left[t^rw(t)\right]\right)^2\right)^{\frac{1}{2}}. \end{aligned} \quad (60)$$

This implies that

$$\begin{aligned} \left|\tfrac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}M_{r,\alpha,w}(t)-J_a^{\alpha}\left[f(t)\right]J_a^{\alpha}\left[t^rw(t)\right]\right| \\ \le \tfrac{(t-a)^{\alpha}}{2\Gamma(\alpha+1)}(M-m)\left(\tfrac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\alpha}\left[t^{2r}w^2(t)\right]-\left(J_a^{\alpha}\left[t^rw(t)\right]\right)^2\right)^{\frac{1}{2}}. \end{aligned} \quad (61)$$

QED

To end this section, we give the following theorem.

Theorem 10. *Let X be a continuous random variable having a p.d.f. $f : [a,b] \to \mathbb{R}^+$, $m \le f \le M$. Then, for all $\alpha > 0, \beta > 0$, $a < t \le b$, we have:*

$$\begin{aligned}
&\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} M_{r,\beta,w}(t) + \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} M_{r,\alpha,w}(t) \\
&\quad - J_a^{\alpha}\left[f(t)\right] J_a^{\beta}\left[t^r w(t)\right] - J_a^{\beta}\left[f(t)\right] J_a^{\alpha}\left[t^r w(t)\right] \\
&\le \left[\left(M\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} - J_a^{\alpha}\left[f(t)\right] \right) \left(J_a^{\beta}\left[f(t)\right] - m\frac{(t-a)^{\beta}}{\Gamma(\beta+1)} \right) \right. \\
&\quad \left. + \left(J_a^{\alpha}\left[f(t)\right] - m\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \left(M\frac{(t-a)^{\beta}}{\Gamma(\beta+1)} - J_a^{\beta}\left[f(t)\right] \right) \right] \qquad (62) \\
&\quad \times \left[\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} J_a^{\beta}\left[t^{2r} w^2(t)\right] + \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} J_a^{\alpha}\left[t^{2r} w^2(t)\right] \right. \\
&\quad \left. - 2J_a^{\alpha}\left[t^r w(t)\right] J_a^{\beta}\left[t^r w(t)\right] \right]^{\frac{1}{2}}.
\end{aligned}$$

Proof. Taking $g(t) = w(t)t^r$, $a < t \le b$, and using Theorem 3.3 and Lemma 3.4 of [11], we can obtain (62). QED

Remark 6. Applying Theorem 10 for $\alpha = \beta$, we obtain Theorem 9.

4 Applications

We present some fractional applications for the uniform random variable X whose *p.d.f.* is defined for any $x \in [a,b]$ by $f(x) = (b-a)^{-1}$.

Case 1: Taking $w(x) = 1, x \in [a,b]$, we can obtain:

a1: Fractional Expectation of Order α:

$$E_{X,\alpha,1} = (b-a)^{-1}\left[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{a(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right]; \alpha \ge 1 \qquad (63)$$

Note that if we take $\alpha = 1$, then we get:

$$E_{X,1,1} = \frac{b+a}{2} = E(X). \qquad (64)$$

b1: Fractional Moment of Orders $(2,\alpha)$:

$$E_{X^2,\alpha,1} = \frac{2(b-a)^{\alpha+1}}{\Gamma(\alpha+3)} + 2a\left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+2)} + \frac{a(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}\right) - \frac{a^2(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}; \alpha \ge 1 \qquad (65)$$

Taking $\alpha = 1$, we obtain the classical moment of order 2:

$$E_{X^2,1,1} = \frac{a^2+b^2+ab}{3} = E(X^2). \qquad (66)$$

c1: Fractional Variance of Order α:
By simple calculations, we obtain

$$\sigma^2_{X,\alpha,1} = \frac{2(b-a)^{\alpha+1}}{\Gamma(\alpha+3)} + 2a\left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+2)} + \frac{a(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}\right) - \frac{a^2(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}; \alpha \geq 1, \tag{67}$$

which corresponds, for $\alpha = 1$, to the classical variance of the uniform distribution X.
Case 2: Consider w as an arbitrary positive function on $[a, b]$ and apply Theorem 1, we obtain the following fractional estimation on $\sigma_{X,\alpha,w}$:

$$\frac{1}{b-a} J_a^{\alpha}\left[w(b)\right] \sigma^2_{X,\alpha,w} \leq \frac{1}{\Gamma^2(\alpha)(b-a)^2}\left[\int_a^b (b-\tau)^{\alpha-1}\left(\tau - E\left(X\right)\right) w\left(\tau\right) d\tau\right]^2 + \frac{1}{2}\left(J_a^{\alpha}\left[w(b)\right]\right)^2. \tag{68}$$

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