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# Trajectories on real hypersurfaces of type ($A_2$) which can be seen as circles in a complex hyperbolic space

**Toshiaki Adachi** [i]
*Department of Mathematics, Nagoya Institute of Technology*
*Nagoya 466-8555, Japan*
adachi@nitech.ac.jp

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**Abstract.** We study trajectories for Sasakian magnetic fields on homogeneous tubes around totally geodesic complex submanifolds in a complex hyperbolic space. We give conditions that they can be seen as circles in a complex hyperbolic space, and show how the set of their congruence classes are contained in the set of those of circles. In view of geodesic curvatures and complex torsions of circles obtained as extrinsic shapes of trajectories, we characterize these tubes among real hypersurfaces in a complex hyperbolic space.

**Keywords:** Sasakian magnetic fields, extrinsic circular trajectories, moduli space of circles, real hypersurfaces of type ($A_2$), complex hyperbolic spaces

**MSC 2010 classification:** primary 53C22, secondary 53B35

## Introduction

When we study shapes of submanifolds of a Riemannian manfiold, it is a way to investigate how geodesics can be seen in the ambient space. For example, every geodesic on a standard sphere can be seen as a circle in a Euclidean space through an isometric embedding, and a standard sphere is characterized by this property among submanifolds in a Euclidean space. In [5] Kimura, Maeda and the author gave a characterization of homogeneous real hypersurfaces in a complex projective space by the property that some geodesics can be seen as circles in the ambient space (see also [16]), and in [6] they characterized totally $\eta$-umbilic hypersurfaces and ruled real hypersurfaces in non-flat complex space forms by the property that some geodesics locally lie on some 2-dimensional totally geodesic submanifolds of ambient spaces.

In this context it is natural to consider that if we study a family of curves containing geodesics we can get more information on shapes of Riemannian

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manifolds. In their paper [19], Nomizu-Yano characterized "extrinsic spheres" by the property that every circle on a submanifold can be seen also as a circle in the ambient manifold. To extend their result, the author proposes to investigate a family of curves associated with geometric structures of underlying submanifolds. On a real hypersurface in a Kähler manifold, we have an almost contact metric structure induced by the ambient complex structure. Associated with this structure we can define a family of dynamical systems on its unit tangent bundle which forms a perturbation of the geodesic flow (cf. [1, 8]). They are called Sasakian magnetic flows and projections of their orbits onto the real hypersurface are called trajectories for Sasakian magnetic fields. Since we have results on geodesics on homogeneous real hypersurfaces in a complex hyperbolic space $\mathbb{C}H^n$, we are interested in trajectories on these hypersurfaces.

This paper is a sequel of preceding papers [4, 9]. In these papers we studied trajectories on totally $\eta$-umbilic real hypersurfaces which are seen as circles in the ambient $\mathbb{C}H^n$, and gave a characterization of them by taking account of strengths of Sasakian magnetic fields and structure torsions of such trajectories. In this paper we mainly study trajectories on homogeneous tubes around totally geodesic complex submanifolds. We pay attention on geodesic curvatures and complex torsions of circles in $\mathbb{C}H^n$ obtained as extrinsic shapes of trajectories on homogeneous real hypersurfaces. Recalling some basic properties of trajectories on tubes around totally geodesic complex submanifolds of $\mathbb{C}H^n$ in §2, we give a condition that they are seen as circles in the ambient space. Being different from totally $\eta$-umbilic real hypersurfaces, each of these tubes has three principal curvatures. This difference gives us a viewpoint how the moduli space of such trajectories are contained in the moduli space of circles. As a consequence of our study we give characterizations of these tubes and also of totally $\eta$-umbilic real hypersurfaces by properties of geodesic curvatures and complex torsions of such trajectories.

# 1 Trajectories for Sasakian magnetic fields

Let $M$ be a real hypersurface in a Kähler manifold $\widetilde{M}$ with complex structure $J$ and Riemannian metric $\langle\ ,\ \rangle$. By use of a unit normal local vector field $\mathcal{N}$ on $M$ in $\widetilde{M}$, we define a vector field $\xi$ on $M$ by $\xi = -J\mathcal{N}$, a 1-form $\eta$ by $\eta(v) = \langle v, \xi\rangle$, and a $(1,1)$-tensor field $\phi$ by $\phi v = Jv - \eta(v)\mathcal{N}$. With the induced metric $\langle\ ,\ \rangle$ we have an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ on $M$. We call these $\xi$ and $\phi$ induced by the Kähler strucure on $\widetilde{M}$ the *characteristic vector field* and the *characteristic tensor* of $M$, respectively. In order to take a family of curves associated with the almost contact metric structure on $M$, we define a closed 2-form $\mathbb{F}_\phi$ by $\mathbb{F}_\phi(u, v) = \langle u, \phi(v)\rangle$, set $\mathbb{F}_\kappa = \kappa\mathbb{F}_\phi$ for an arbitrary constant $\kappa$, and call it a *Sasakian magnetic field* (cf. [8]). A smooth curve $\gamma$ on $M$ parameterized by its arclength is said to be a *trajectory* for $\mathbb{F}_\kappa$ if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$. When $\kappa = 0$, trajectories are geodesics. Hence we may say that trajectories for Sasakian magnetic fields are perturbations of geodesics. Though we employ physical terms, only we need is to get perturbations of the geodesic flow associated with the almost contact metric structure on $M$ (for magnetic fields, see [11, 20] for example). For readers who are not familier with magnetic fields it is enough to consider that trajectories are special curves whose first geodesic curvature are constant. To avoid physical terms, in [15] such a curve is called a Sasakian curve.

Since the tangent space $T_pM$ of a real hypersurface $M$ in $\widetilde{M}$ splits as $T_pM = T_p^0M \oplus \mathbb{R}\xi_p$ with a complex subspace $T_p^0M$ of $T_p\widetilde{M}$, for a trajectory $\gamma$ for $\mathbb{F}_\kappa$ we set $\rho_\gamma = \langle\dot\gamma, \xi_\gamma\rangle$ and call it its *structure torsion.* By use of Gauss and Weingarten formulae which are given as $\widetilde\nabla_X Y = \nabla_X Y + \langle A_M X, Y\rangle\mathcal{N}$ and $\widetilde\nabla_X\mathcal{N} = -A_M X$ with the shape operator $A_M$ for arbitrary vector fields $X, Y$ tangent to $M$, we have

$$\rho'_\gamma = \langle\kappa\phi\dot\gamma, \xi_\gamma\rangle + \langle\dot\gamma, \phi A_M\dot\gamma\rangle = \langle\dot\gamma, \phi A_M\dot\gamma\rangle.$$

Since $A_M$ is symmetric and $\phi$ is skew-symmetric, we have $2\rho'_\gamma = \langle\dot\gamma, (\phi A_M - A_M\phi)\dot\gamma\rangle$, hence $\rho_\gamma$ is constant along $\gamma$ for each trajectory $\gamma$ if $\phi A_M = A_M\phi$ holds (cf. [3]).

When $\widetilde{M}$ is a complex hyperbolic space $\mathbb{C}H^n$, such a property on the shape operator and the characteristic tensor holds for real hypersurfaces of type (A), which are a geodesic sphere $G(r)$, a horosphere $HS$, a tube $T(r)$ around $\mathbb{C}H^{n-1}$ and a tube $T_\ell(r)$ around $\mathbb{C}H^\ell$ ($\ell = 1, \ldots, n-2$), where $r$'s denote their radii. In the preceding paper [9], we studied trajectories on totally $\eta$-umbilic real hypersurfaces $G(r), HS$ and $T(r)$ whose extrinsic shapes are circles on $\mathbb{C}H^n$. Here, for a curve on $M$ we call the curve $\iota\circ\gamma$ on $\mathbb{C}H^n$ with an isometric immersion $\iota : M \to \mathbb{C}H^n$ its *extrinsic shape.* We therefore study trajectories for Sasakian magnetic fields on $T_\ell(r)$ in this paper. Tubes around $\mathbb{C}H^\ell$ ($\ell = 1, \ldots, n-2$) are called real hypersurfaces of type ($A_2$). It is known that a tube $M = T_\ell(r)$ of radius $r$ around $\mathbb{C}H^\ell$ in $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c$ has three principal curvatures. They are

$$\delta_M = \sqrt{|c|}\coth\sqrt{|c|}\,r,\quad \lambda_M = \frac{\sqrt{|c|}}{2}\coth\frac{\sqrt{|c|}}{2}\,r,\quad \mu_M = \frac{\sqrt{|c|}}{2}\tanh\frac{\sqrt{|c|}}{2}\,r.$$

The characteristic vector $\xi_p$ is a principal curvature vector associated with $\delta_M$ at each point $p \in M$. The subbundle $T^0M$ splits into subbundles $V_{\lambda_M} \oplus V_{\mu_M}$ of principal curvature vectors associated with $\lambda_M$ and $\mu_M$. It is known that each of these subbundles are invariant under the action of $\phi$. Hence the shape operator $A_M$ and the characteristic tensor $\phi$ are commutative.

We here introduce another invariant for trajectories for Sasakian magnetic fields on this tube $M = T_\ell(r)$. Let $\mathrm{Proj}_{\lambda_M} : TM \to V_{\lambda_M}$ and $\mathrm{Proj}_{\mu_M} : TM \to V_{\mu_M}$ be projections. For a trajectory $\gamma$ for $\mathbb{F}_\kappa$ we set $\omega_\gamma = \|\mathrm{Proj}_{\lambda_M}(\dot\gamma)\|$, and call it its *principal torsion*. Since we have $\|\mathrm{Proj}_{\lambda_M}(\dot\gamma)\|^2 + \|\mathrm{Proj}_{\mu_M}(\dot\gamma)\|^2 = 1 - \rho_\gamma^2$, we get $0 \le \omega_\gamma \le \sqrt{1-\rho_\gamma^2}$.

**Lemma 1** ([8])**.** *The principal torsion $\omega_\gamma$ of a trajectory $\gamma$ for $\mathbb{F}_\kappa$ on $T_\ell(r)$ in $\mathbb{C}H^n(c)$ is constant along $\gamma$.*

*Proof.* Real hypersurfaces of type (A) are characterized by a property on differentials of their shape operators. In particular, we have

$$\big\langle (\nabla_X A_M)Y, Z\big\rangle = \frac{c}{4}\big\{-\eta(Y)\langle \phi X, Z\rangle - \eta(Z)\langle \phi X, Y\rangle\big\}$$

for arbitrary vector fields $X, Y, Z$ on $T_\ell(r)$ (see [18], for example). Thus we have

$$\begin{aligned}\nabla_{\dot\gamma}\langle A_M\dot\gamma, \dot\gamma\rangle &= \big\langle (\nabla_{\dot\gamma}A_M)\dot\gamma, \dot\gamma\big\rangle + \langle \kappa A_M\phi\dot\gamma, \dot\gamma\rangle + \langle A_M\dot\gamma, \kappa\phi\dot\gamma\rangle \\ &= -\frac{c}{2}\rho_\gamma\langle \phi\dot\gamma, \dot\gamma\rangle + \kappa\big\langle (A_M\phi - \phi A_M)\dot\gamma, \dot\gamma\big\rangle = 0.\end{aligned}$$

Therefore

$$\begin{aligned}\langle A_M\dot\gamma, \dot\gamma\rangle &= \delta_M\rho_\gamma^2 + \lambda_M\omega_\gamma + \mu_M(1-\rho_\gamma^2-\omega_\gamma^2) \\ &= \mu_M + (\delta_M-\mu_M)\rho_\gamma^2 + (\lambda_M-\mu_M)\omega_\gamma^2\end{aligned}$$

is constant along $\gamma$. As $\rho_\gamma$ is constant along $\gamma$, we find that $\omega_\gamma$ is also constant along $\gamma$. $\boxed{QED}$

We say two smooth curves $\sigma_1, \sigma_2$ on a Riemannian manifold $N$ which are parameterized by their arclengths to be *congruent* to each other if there exist an isometry $\varphi$ of $N$ and a constant $t_0$ satisfying that $\sigma_2(t) = \varphi \circ \sigma_1(t+t_0)$ for all $t$. For trajectories on real hypersurfaces of type ($\mathrm{A}_2$) we have the following.

**Lemma 2** ([8])**.** *Trajectories $\gamma_1$ for a Sasakian magnetic field $\mathbb{F}_{\kappa_1}$ and $\gamma_2$ for $\mathbb{F}_{\kappa_2}$ on $T_\ell(r)$ in $\mathbb{C}H^n$ are congruent to each other if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,

ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$, $\omega_{\gamma_1} = \omega_{\gamma_2}$ *and* $|\kappa_1| = |\kappa_2|$,

iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$, $\omega_{\gamma_1} = \omega_{\gamma_2}$ *and* $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$.

## 2 Extrinsic circular trajectories

A smooth curve $\sigma$ on $\mathbb{C}H^n(c)$ which is parameterized by its arclength is said to be a *circle* if it satisfies the equation $\widetilde{\nabla}_{\dot\sigma}\widetilde{\nabla}_{\dot\sigma}\dot\sigma + \|\widetilde{\nabla}_{\dot\sigma}\dot\sigma\|^2\dot\sigma = 0$. Since its geodesic curvature $k_\sigma = \|\widetilde{\nabla}_{\dot\sigma}\dot\sigma\|$ is constant along $\sigma$, this equation is equivalent to the system of differential equations $\widetilde{\nabla}_{\dot\sigma}\dot\sigma = k_\sigma Y$, $\widetilde{\nabla}_{\dot\sigma}Y = -k_\sigma\dot\sigma$ with some field $Y$ of unit vectors along $\sigma$. For a circle $\sigma$ on $\mathbb{C}H^n(c)$ with positive geodesic curvature $k_\sigma$, we put $\tau_\sigma = \langle\dot\sigma, J\widetilde{\nabla}_{\dot\sigma}\dot\sigma\rangle/k_\sigma$. This is constant along $\sigma$, and is called its *complex torsion*. It is known that two circles $\sigma_1, \sigma_2$ on $\mathbb{C}H^n(c)$ are congruent to each other if and only if they satisfy $k_{\sigma_1} = k_{\sigma_2}$ and $|\tau_{\sigma_1}| = |\tau_{\sigma_2}|$ (see [17]). Therefore, the moduli space $\mathcal{M}(\mathbb{C}H^n)$, the set of all congruence classes, of circles of positive geodesic curvature on $\mathbb{C}H^n$ is set theoretically identified with the band $(0,\infty)\times[0,1]$.

A smooth curve $\gamma$ on $\widetilde{M}$ is said to be *extrinsic circular* if its extrinsic shape is a circle. We here give conditions that trajectories for $\mathbb{F}_\kappa$ on a tube $M = T_\ell(r)$ around $\mathbb{C}H^\ell$ in $\mathbb{C}H^n$ to be extrinsic circular. By Gauss and Weingarten formulae we have

$$
\begin{aligned}
\widetilde{\nabla}_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\dot\gamma + \langle A_M\dot\gamma,\dot\gamma\rangle\mathcal{N},\\
&= \kappa\phi\dot\gamma + \bigl\{\rho_\gamma^2\delta_M + \omega_\gamma^2\lambda_M + (1-\rho_\gamma^2-\omega_\gamma^2)\mu_M\bigr\}\mathcal{N},\\
\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma &= \kappa J\widetilde{\nabla}_{\dot\gamma}\dot\gamma - \bigl\{-\kappa\rho_\gamma + \rho_\gamma^2\delta_M + \omega_\gamma^2\lambda_M + (1-\rho_\gamma^2-\omega_\gamma^2)\mu_M\bigr\}A_M\dot\gamma\\
&= \bigl[\bigl\{\rho_\gamma^2\delta_M + \omega_\gamma^2\lambda_M + (1-\rho_\gamma^2-\omega_\gamma^2)\mu_M\bigr\}(\kappa+\rho_\gamma\delta_M) + \kappa\delta_M\rho_\gamma^2\bigr]\xi_\gamma\\
&\quad - \bigl[\kappa^2 + \bigl\{-\kappa\rho_\gamma + \rho_\gamma^2\delta_M + \omega_\gamma^2\lambda_M + (1-\rho_\gamma^2-\omega_\gamma^2)\mu_M\bigr\}\lambda_M\bigr]\mathrm{Proj}_{\lambda_M}(\dot\gamma)\\
&\quad - \bigl[\kappa^2 + \bigl\{-\kappa\rho_\gamma + \rho_\gamma^2\delta_M + \omega_\gamma^2\lambda_M + (1-\rho_\gamma^2-\omega_\gamma^2)\mu_M\bigr\}\mu_M\bigr]\mathrm{Proj}_{\mu_M}(\dot\gamma).
\end{aligned}
$$

Since $\lambda_M \neq \mu_M$, we have the following.

**Lemma 3.** *A trajectory $\gamma$ for a Sasakian magnetic field $\mathbb{F}_\kappa$ on a tube $M = T_\ell(r)$ in $\mathbb{C}H^n$ is extrinsic circular if and only if it satisfies one of the following conditions*:

i) $\rho_\gamma = \pm 1$,

ii) $\omega_\gamma = 1-\rho_\gamma^2$ *and* $\lambda_M - \kappa\rho_\gamma + (\delta_M-\lambda_M)\rho_\gamma^2 = 0$,

iii) $\omega_\gamma = 0$ *and* $\mu_M - \kappa\rho_\gamma + (\delta_M-\mu_M)\rho_\gamma^2 = 0$,

iv) $\omega_\gamma = 1-\rho_\gamma^2$ *and* $\kappa + (\delta_M-\lambda_M)\rho_\gamma = 0$,

v) $\omega_\gamma = 0$ *and* $\kappa + (\delta_M-\mu_M)\rho_\gamma = 0$.

*Corresponding to each case, the geodesic curvature and the complex torsion of the extrinsic shape are*

i) $k_\gamma = \delta_M,\ \tau_\gamma = \mp 1$,

ii) $k_\gamma = |\kappa|,\ \tau_\gamma = -\mathrm{sgn}(\kappa)$,

iii) $k_\gamma = |\kappa|,\ \tau_\gamma = -\mathrm{sgn}(\kappa)$,

iv) $k_\gamma = \sqrt{\kappa^2 - 2\lambda_M\kappa\rho_\gamma + \lambda_M^2},\ \ k_\gamma\tau_\gamma = 2\kappa\rho_\gamma^2 - \kappa - \lambda_M\rho_\gamma$,

v) $k_\gamma = \sqrt{\kappa^2 - 2\mu_M\kappa\rho_\gamma + \mu_M^2},\ \ k_\gamma\tau_\gamma = 2\kappa\rho_\gamma^2 - \kappa - \mu_M\rho_\gamma$.

As the structure torsion of a trajectory $\gamma$ satisfies $|\rho_\gamma| \le 1$ and principal curvatures satisfy $\mu_M < \sqrt{|c|}/2 < \lambda_M$, we find that $|\kappa| \ge \delta_M$ in the case ii) and $|\kappa| \ge \sqrt{|c|}$ in the case iii).

Next we study the cases iv) and v). Only in these cases, the complex torsions of extrinsic circles are not necessarily $\pm 1$. By substituting conditions, we have

$$\begin{cases} k_\gamma^2 = \kappa^2(1-\rho_\gamma^2) + \big\{\lambda_M + (\delta_M - \lambda_M)\rho_\gamma^2\big\}^2, \\ k_\gamma\tau_\gamma = -\kappa(1-\rho_\gamma^2) - \big\{\lambda_M + (\delta_M - \lambda_M)\rho_\gamma^2\big\}\rho_\gamma, \end{cases}$$

in the case iv), and have the same equalities by changing $\lambda_M$ to $\mu_M$ in the case v). Since we have $\lambda_M + \mu_M = \delta_M$ and $\mu_M = |c|/(4\lambda_M)$, we can express geodesic curvatures and complex torsions as

$$k_\gamma = \sqrt{\lambda_M^2 + \frac{|c|\rho_\gamma^2}{2} + \frac{c^2\rho_\gamma^2}{16\lambda_M^2}}, \quad \tau_\gamma = \frac{\rho_\gamma(|c| - 2|c|\rho_\gamma^2 - 4\lambda_M^2)}{4k_\gamma\lambda_M}, \tag{1}$$

$$k_\gamma = \sqrt{\mu_M^2 + \frac{|c|\rho_\gamma^2}{2} + \frac{c^2\rho_\gamma^2}{16\mu_M^2}}, \quad \tau_\gamma = \frac{\rho_\gamma(|c| - 2|c|\rho_\gamma^2 - 4\mu_M^2)}{4k_\gamma\mu_M} \tag{2}$$

As $|\rho_\gamma| \le 1$, we have $\lambda_M \le k_\gamma \le \delta_M$ in the case iv), and have $\mu_M \le k_\gamma \le \delta_M$ in the case v). By substituting the first equality in (1) to the second, we obtain

$$\tau_\gamma^2 = \frac{(k_\gamma^2 - \lambda_M^2)(32\lambda_M^2 k_\gamma^2 + 4c\lambda_M^2 - c^2)^2}{|c|(8\lambda_M^2 - c)^3 k_\gamma^2}. \tag{3}$$

We regard the right hand side of (3) as a function on $K = k_\gamma^2$, and denote it by $g(K;\lambda_M)$. We then have

$$\frac{dg(K;\lambda_M)}{dK} = \frac{\lambda_M^2(8K-c)(8K-4\lambda_M^2+c)(32\lambda_M^2 K + 4c\lambda_M^2 - c^2)}{|c|(8\lambda_M^2 - c)^3 K^2},$$

hence find that this function $g(K;\lambda_M)$ is monotone increasing with respect to $K$ and satisfies $g(\lambda_M^2;\lambda_M)=0$ and $g(\delta_M^2;\lambda_M)=1$. Similarly, by substituting the first equality in (2) to the second, we obtain

$$\tau_\gamma^2=\frac{(k_\gamma^2-\mu_M^2)(32\mu_M^2k_\gamma^2+4c\mu_M^2-c^2)^2}{|c|(8\mu_M^2-c)^3k_\gamma^2}. \tag{4}$$

Thus the right hand side $g(K;\mu_M)$ of (4) is monotone increasing with respect to $K=k_\gamma^2$ in the union of intervals $\big[\mu_M^2,(4\mu^2-c)/8\big]\cup\big[(4\lambda_M^2-c)/8,\delta_M^2\big]$, is monotone decreasing in the interval $\big[(4\mu^2-c)/8,(4\lambda_M^2-c)/8\big]$, and satisfies $g(\lambda_M^2;\mu_M)=g((4\mu^2-c)/8;\mu_M)=0,\ g(\delta_M^2;\mu_M)=1$ and $g((4\lambda^2-c)/8;\mu_M)<1$. Moreover, when $\lambda_M\le k<\delta_M$, as we have

$$\begin{aligned}&g(k^2;\mu_M)-g(k^2;\lambda_M)\\&\quad=\frac{(8k^2-c)^2(4\lambda_M^2-c)(4\lambda_M^2+4k_\gamma\lambda_M-c)(4\lambda_M^2-4k_\gamma\lambda_M-c)(4\lambda_M^2+c)^3}{|c|(2\lambda^2-c)^3(8\lambda_M^2-c)^3k_\gamma^2},\end{aligned}$$

we find $g(k^2;\mu_M)>g(k^2;\lambda_M)$.

We denote by $\mathcal{E}(M)$ the set of all congruence classes of extrinsic circular trajectories of types iv) and v) in Lemma 3. Since each isometry $\varphi$ of $M$ is equivariant, that is, there is an isometry $\hat\varphi$ of $\mathbb{C}H^n(c)$ satisfying $\iota\circ\varphi=\hat\varphi\circ\iota$ with an isometric embedding $\iota: M\to\mathbb{C}H^n$, we find that if two trajectories are congruent to each other in $M$ then their extrinsic shapes are congruent to each other in $\mathbb{C}H^n$. On the other hand, under the conditions iv) or v), the relations (1), (2) and the conditions iv), v) show that if extrinsic circular shapes of two trajectories are congruent to each other in $\mathbb{C}H^n$ then these trajectories are congruent to each other in $M$ by Lemma 2. Therefore we can regard the moduli space $\mathcal{E}(M)$ of extrinsic circular trajectories of types iv) and v) as a subset of $\mathcal{M}(\mathbb{C}H^n)$.

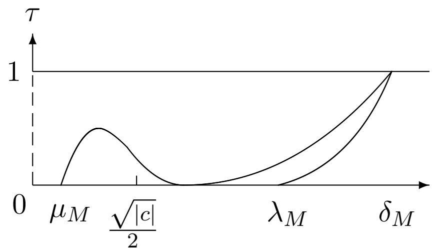

Figure 1. $\mathcal{E}(M)$ for $M=T_\ell(r)$

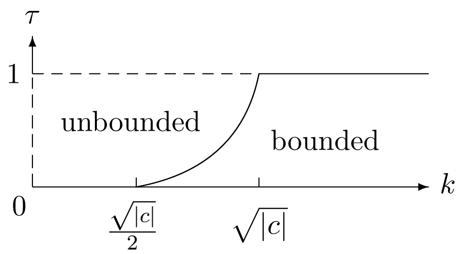

Figure 2. $\mathcal{M}(\mathbb{C}H^n)$

We here recall some properties of circles on $\mathbb{C}H^n(c)$ (see [2, 7]). We define a function $\nu : (0,\infty) \to [0,1]$ by

$$\nu(k) = \begin{cases} 0, & \text{if } 0 < k < \sqrt{|c|}/2, \\ (4k^2+c)^{3/2}/(3\sqrt{3}|c|k), & \text{if } \sqrt{|c|}/2 \le k \le \sqrt{|c|}, \\ 1, & \text{if } k > \sqrt{|c|}. \end{cases}$$

We then have

1) A circle $\sigma$ is bounded if and only if either $k_\sigma > \sqrt{|c|}$ or $\tau_\sigma < \nu(k_\sigma)$;
2) Every bounded circle $\sigma$ with $\tau_\sigma = 0$ is closed of length $4\pi/\sqrt{4k^2+c}$;
3) Every bounded circle $\sigma$ with $\tau_\sigma = \pm 1$ is closed of length $2\pi/\sqrt{k^2+c}$;
4) When $0 < |\tau_\sigma| < 1$, we have both closed circles and bounded open circles.

Coming back to the study of extrinsic circular trajectories on a tube $M = T_\ell(r)$ in $\mathbb{C}H^n(c)$, we concentrate our mind on those whose extrinsic geodesic curvatures satisfy $\lambda_M \le k_\gamma \le \delta_M$. When $\lambda_M < \sqrt{|c|}$, we have

$$\begin{aligned} &\nu(k_\gamma)^2 - g(k_\gamma^2;\mu_M) \\ &\qquad = \frac{(8k_\gamma^2-c)^2(4\lambda_M^2+c)^2(16k_\gamma^2\lambda_M^2+16c\lambda_M^2-32ck_\gamma^2-5c^2)}{27c^2(8\lambda_M^2-c)^3k_\gamma^2} > 0 \end{aligned}$$

for $\lambda_M \le k_\gamma \le \sqrt{|c|}$ $(< \delta_M)$. Therefore we see that those trajectories are bounded. Summarizing up we obtain the following.

**Theorem 1.** *Let $M = T_\ell(r)$ be a tube around totally geodesic $\mathbb{C}H^\ell$ in $\mathbb{C}H^n(c)$.*

(1) *The moduli space of extrinsic shapes of trajectories on $M$ of types* i), ii) *and* iii) *in Lemma 3 forms a half-line $\{(k,1) \mid k \ge \sqrt{|c|}\}$ in $\mathcal{M}(\mathbb{C}H^n) = (0,\infty)\times[0,1]$.*

(2) *The moduli space $\mathcal{E}(M)$ of extrinsic shapes of trajectories on $M$ of types* iv) *and* v) *in Lemma 3 forms a curve in $\mathcal{M}(\mathbb{C}H^n)$. It does not have self-intersections, and is smooth except at $(\delta_M,1)$.*

(3) *For each $k$ with $\lambda_M \le k < \delta_M$, we have two congruence classes of circles of geodesic curvature $k$ on $\mathbb{C}H^n$ which are obtained as extrinsic shapes of trajectories for some Sasakian magnetic fields on $M$.*

(4) *The curve $\mathcal{E}(M)\cap\big([\lambda_M,\delta_M]\times[0,1]\big)$ in $\mathcal{M}(\mathbb{C}H^n)$ is contained in the moduli space of bounded circles on $\mathbb{C}H^n$.*

## 3 Characterizations of real hypersurfaces of type (A)

In this section we give a characterization of tubes around totally geodesic complex submanifolds by some properties given in Theorem 1. A real hypersurface $M$ in $\mathbb{C}H^n(c)$ is said to be *Hopf* if its characteristic vector field gives a principal curvature vector at each point. It is known that for a Hopf hypersurface in $\mathbb{C}H^n$ the principal curvature associated with its characteristic vector field is locally constant [13, 14]). In $\mathbb{C}H^n$ a homogeneous Hopf hypersurface is one of the following with some radius $r$; a horosphere $HS$, a geodesic sphere $G(r)$, a tube $T(r)$ around totally geodesic $\mathbb{C}H^{n-1}$, our tube $T_\ell(r)$ with some $1 \le \ell \le n-2$, and a tube $R(r)$ around totally geodesic $\mathbb{R}H^{n-1}$ (see [10]). These hypersurfaces have constant principal curvature functions, and every Hopf hypersurface with constant principal curvatures is locally congruent to one of these homogeneous real hypersurfaces. When $M$ is one of $HS, G(r)$ and $T(r)$, it has two principal curvatures $\delta_M, \lambda_M$, which are for its characteristic vectors and for vectors orthogonal to them. Their values are

$$\delta_M = \begin{cases} \sqrt{|c|}, \\ \sqrt{|c|}\coth\sqrt{|c|}\,r, \\ \sqrt{|c|}\coth\sqrt{|c|}\,r, \end{cases} \qquad \lambda_M = \begin{cases} \frac{\sqrt{|c|}}{2}, & \text{when } M = HS, \\ \frac{\sqrt{|c|}}{2}\coth\frac{\sqrt{|c|}}{2}\,r, & \text{when } M = G(r), \\ \frac{\sqrt{|c|}}{2}\tanh\frac{\sqrt{|c|}}{2}\,r, & \text{when } M = T(r). \end{cases}$$

A trajectory for $\mathbb{F}_\kappa$ on one of these real hypersurfaces is extrinsic circular if and only if one of the following conditions holds ([4]):

i) $\rho_\gamma = \pm 1$,

ii) $\lambda_M - \kappa\rho_\gamma + (\delta_M - \lambda_M)\rho_\gamma^2 = 0$,

iii) $\kappa + (\delta_M - \lambda_M)\rho_\gamma = 0$.

When $M = R(r)$, it has three principal curvatures $\delta_M, \lambda_M, \mu_M$. The subbundle $T^0M$ which consists of tangent vectors orthogonal to $\xi$ splits into a direct sum of subbundles of principal curvature vectors associated with $\lambda_M$ and $\mu_M$. The values of principal curvatures are

$$\delta_M = \sqrt{|c|}\tanh\sqrt{|c|}\,r, \quad \lambda_M = \frac{\sqrt{|c|}}{2}\coth\frac{\sqrt{|c|}}{2}\,r, \quad \mu_M = \frac{\sqrt{|c|}}{2}\tanh\frac{\sqrt{|c|}}{2}\,r.$$

In order to show that a trajectory is extrinsic circular, we need to check that it satisfies a differential equation of a circle at its each point. To give a characterization we weaken the extrinsic circular condition and consider a pointwise condition. A smooth curve $\gamma$ on a submanifold $M$ in $\widetilde{M}$ which is parameterized by its arclength is said to be *tangentially of order* 2 at $\gamma(t_0)$ if the vector $\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma(t_0) + \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma(t_0)\|^2\dot\gamma(t_0)$ does not have a component tangent to $M$. Trivially, when $\gamma$ is extrinsic circular, then it is tangentially of order 2 at each point of its extrinsic shape. For a trajectory $\gamma$ for a Sasakian magnetic field $\mathbb{F}_\kappa$ on a real hypersurface $M$ in $\mathbb{C}H^n$, we have

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma + \langle A_M\dot\gamma,\dot\gamma\rangle\mathcal{N}, \tag{5}$$

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = -\kappa^2\dot\gamma - \big(\langle A_M\dot\gamma,\dot\gamma\rangle - \kappa\rho_\gamma\big)\big(A_M\dot\gamma + \kappa\xi_\gamma\big) + \frac{d}{dt}\big(\langle A_M\dot\gamma,\dot\gamma\rangle - \kappa\rho_\gamma\big)\mathcal{N}, \tag{6}$$

hence the first geodesic curvature $k_\gamma = \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ and the first complex torsion $\tau_\gamma = \langle\dot\gamma, J\widetilde{\nabla}_{\dot\gamma}\dot\gamma\rangle/k_\gamma$ of its extrinsic shape are given as

$$k_\gamma = \sqrt{\kappa^2(1-\rho_\gamma^2) + \langle A_M\dot\gamma,\dot\gamma\rangle^2}, \qquad \tau_\gamma = -\big\{\kappa(1-\rho_\gamma^2) + \langle A_M\dot\gamma,\dot\gamma\rangle\rho_\gamma\big\}/k_\gamma. \tag{7}$$

If the extrinsic shape of $\gamma$ is tangentially of order 2 at $\gamma(t_0)$, we find by (6) that

$$(k_\gamma(t_0)^2 - \kappa^2)\dot\gamma(t_0) = \big(\langle A_M\dot\gamma(t_0),\dot\gamma(t_0)\rangle - \kappa\rho_\gamma(t_0)\big)\big(A_M\dot\gamma(t_0) + \kappa\xi_{\gamma(t_0)}\big) \tag{8}$$

holds.

We say a unit tangent vector $v \in U_pM$ of a real hypersurface $M$ satisfies Condition $(k,\tau)$ if the extrinsic shape of a trajectory $\gamma$ for some $\mathbb{F}_\kappa$ with $|\kappa| \neq k$ and of initial vector $v$ is tangentially of order 2 at $p$ and satisfies $k_\gamma(0) = k$, $\tau_\gamma(0) = \tau$.

**Theorem 2.** *A connected real hypersurface $M$ in $\mathbb{C}H^n(c)$ is locally congruent to a tube $T_\ell(r)$ around totally geodesic $\mathbb{C}H^\ell$ with $1 \le \ell \le n-2$ if and only if it satisfies the following conditions*:

i) *At each point $p \in M$, the extrinsic shape of a trajectory $\gamma_p$ for some $\mathbb{F}_{\kappa_p}$ with $\dot\gamma_p(0) = \xi_p$ is tangentially of order 2 at $p$ and its first geodesic curvature at $p$ satisfies $k_p(0) \neq |\kappa_p|$;*

ii) *There exist positive constants $k, \tau_1, \tau_2$ with $k \ge \sqrt{|c|}$ and $\tau_1 < \tau_2 < 1$ such that at each point $p \in M$ we can choose linearly independent unit tangent vectors $v_1, \ldots, v_{2n-2} \in U_pM$ which satisfy*

   a) *either Condition $(k,\tau_1)$ or Condition $(k,\tau_2)$ holds, but not all of them satisfy one of these conditions,*

   b) *their components $v_i - \langle v_i,\xi_p\rangle\xi_p$ $(i = 1, \ldots, 2n-2)$ span the tangent space $T_p^0M$ orthogonal to $\xi_p$.*

*Proof.* Since $\delta_M > \sqrt{|c|}$, we see that every tube $T_\ell(r)$ satisfies the conditions by Lemma 3 and Theorem 1. We hence need to show the "if" part.

By the first condition and by (7) and (8), we have

$$k_{\gamma_p}(0) = |\langle A_M\dot\gamma_p(0), \dot\gamma_p(0)\rangle|,$$

$$\big(k_{\gamma_p}(0)^2 - \kappa\langle A_M\dot\gamma_p(0), \dot\gamma_p(0)\rangle\big)\xi_p = \big(\langle A_M\dot\gamma_p(0), \dot\gamma_p(0)\rangle - \kappa\big)A_M\dot\gamma_p(0).$$

As $k_{\gamma_p}(0) \neq |\kappa|$ by the condition, we find that $\xi_p$ is a principal curvature vector. Thus $M$ is a Hopf hypersurface.

We denote by $\delta_M$ the principal curvature associated with the characteristic vector field of $M$. This is locally constant. If a trajectory $\gamma$ for $\mathbb{F}_\kappa$ is tangentially of order 2 at $p = \gamma(0)$ and $\rho_\gamma(0) \neq \pm 1$, by considering the components which are orthogonal to $\xi_p$ and tangent to $\xi_p$ in (8) we find

$$\big(k_\gamma^2(0) - \kappa^2\big)w = \big\{\langle A_M w, w\rangle + \rho_\gamma(0)^2\delta_M - \kappa\rho_\gamma(0)\big\}A_M w, \tag{9}$$

$$\big(k_\gamma^2(0) - \kappa^2\big)\rho_\gamma(0) = \big\{\langle A_M w, w\rangle + \rho_\gamma(0)^2\delta_M - \kappa\rho_\gamma(0)\big\}\big(\delta_M\rho_\gamma(0) + \kappa\big), \tag{10}$$

where $w = \dot\gamma(0) - \rho_\gamma(0)\xi_p \in T_p^0 M$. Thus, if $k_\gamma(0) \neq |\kappa|$, we find by (9) that $w = \dot\gamma(0) - \rho_\gamma(0)\xi_p$ is principal. If we denote its principal curvature by $\alpha$, (9) and (10) show

$$\kappa + (\delta_M - \alpha)\rho_\gamma(0) = 0, \tag{11}$$

because $k_\gamma(0) \neq |\kappa|$. Moreover, in this case by (7) we have

$$k_\gamma(0)^2 = \kappa^2\{1 - \rho_\gamma(0)^2\} + \big\{\alpha + (\delta_M - \alpha)\rho_\gamma(0)^2\big\}^2, \tag{12}$$

$$k_\gamma(0)\tau_\gamma(0) = -\kappa\{1 - \rho_\gamma(0)^2\} - \big\{\alpha + (\delta_M - \alpha)\rho_\gamma(0)^2\big\}\rho_\gamma(0). \tag{13}$$

We regard equalities (11), (12), (13) as a system of equations on $\kappa, \alpha, \rho = \rho_\gamma(0)$ when we give $k = k_\gamma(0)$ and $\tau = \tau_\gamma(0)$. Then the second condition of our theorem guarantees that for $(k, \tau_j)$ this system of equations has some solutions satisfying $|\kappa| \neq k$. Since the solutions of this system is finite and $\delta_M$ is locally constant, by perturbation theory (cf. [12]) we find that the principal curvatures of $M$ are constant. Hence $M$ is locally congruent to one of $HS, G(r), T(r), T_\ell(r)$ and $R(r)$.

We shall now check whether these homogeneous real hypersurfaces satisfy the conditions. As we listed above, when $M$ is one of $HS, G(r), T(r)$, every tangent vector orthogonal to $\xi$ is a principal curvature vector associated with $\lambda_M$, and $\delta_M - \lambda_M = |c|/(4\lambda_M)$. By substituting (11) with $\alpha = \lambda_M$ to (12) and (13) we have

$$k_\gamma(0)^2 = \lambda_M^2 + \Big(\frac{|c|}{2} + \frac{c^2}{16\lambda^2}\Big)\rho_\gamma(0)^2,$$

$$k_\gamma(0)\tau_\gamma(0) = \rho_\gamma(0)\Big(\frac{|c|}{4\lambda_M} - \lambda_M - \frac{|c|}{2\lambda_M}\rho_\gamma(0)^2\Big).$$

By the first equality $\rho_\gamma(0)^2$ is determined. Hence we can not take two positive $\tau_1, \tau_2$ satisfying the condition ii) in our assertion.

When $M = R(r)$ we have two principal curvatures $\lambda_M, \mu_M$ for vectors orthogonal to $\xi$. We note $\delta_M = |c|/(\lambda_M + \mu_M)$ and $\mu_M = |c|/(4\lambda_M)$. When $\dot\gamma(0) - \rho_\gamma(0)\xi_p$ is a principal curvature vector associated with $\lambda_M$, by substituting (11) with $\alpha = \lambda_M$ to (12) we have

$$k_\gamma(0)^2 = \lambda_M^2 + (\delta_M^2 - \lambda_M^2)\rho_\gamma(0)^2.$$

Thus we obtain the following:

1) if $\delta_M > \lambda_M$, that is, if $\lambda_M^2 < 3|c|/4$, we find that $k_\gamma(0)$ varies monotone increasingly with respect to $|\rho_\gamma(0)|$ and takes values in the interval $\big[\lambda_M, \delta_M\big]$;

2) if $\delta_M = \lambda_M$, that is $\lambda_M^2 = |c|/4$, we have $k_\gamma(0) = \lambda_M$;

3) if $\delta_M < \lambda_M$, that is, if $\lambda_M^2 > 3|c|/4$, we find that $k_\gamma(0)$ varies monotone decreasingly with respect to $|\rho_\gamma(0)|$ and takes values in the interval $\big[\delta_M, \lambda_M\big]$.

When $\dot\gamma(0) - \rho_\gamma(0)\xi_p$ is a principal curvature vector associated with $\mu_M$, by substituting (11) with $\alpha = \mu_M$ to (12) we have

$$k_\gamma(0)^2 = \mu_M^2 + (\delta_M^2 - \mu_M^2)\rho_\gamma(0)^2.$$

Hence $k_\gamma(0)$ is monotone increasing and takes values in the interval $\big[\mu_M, \delta_M\big]$. As $\delta_M < \sqrt{|c|}$, we find that $R(r)$ does not satisfies the condition ii) in our assertion. Therefore we get the conclusion. $\boxed{QED}$

**Remark 1.** If we drop the condition $k \geq \sqrt{|c|}$ in Theorem 2, then its proof shows that tubes $R(r)$ of radius $r \geq \big(1/\sqrt{|c|}\,\big)\log\big(\sqrt{3}+1)/(\sqrt{3}-1)\big)$ satisfy the conditions.

We note that the existence of two points $(k, \tau_1), (k, \tau_2)$ in $\mathcal{M}(\mathbb{C}H^n)$ satisfying extrinsic circular condition is important. We shall say that a real hypersurface $M$ satisfies Condition (TC) at $p \in M$ if there exists a Sasakian magnetic field $\mathbb{F}_\kappa$ having the following property: The trajectory $\gamma$ for $\mathbb{F}_\kappa$ of initial vector $\dot\gamma(0) = \xi_p$ is tangentially of order 2 and the geodesic curvature of its extrinsic shape satisfies $k_\gamma(0) \neq |\kappa|$. The condition i) in Theorem 2 means that $M$ satisfies Condition (TC) holds at each point. Our proof of Theorem 2 also shows the following:

**Theorem 3.** *A connected real hypersurface $M$ in $\mathbb{C}H^n(c)$ is locally congruent to a totally $\eta$-umbilic real hypersurface if and only if it satisfies the following conditions*:

i) *Condition* $(TC)$ *holds at each point* $p \in M$;

ii) *There exist positive constants* $k, \tau$ *with* $\tau < 1$ *such that at each point* $p \in M$ *we can choose linearly independent unit tangent vectors* $v_1, \dots, v_{2n-2} \in U_pM$ *satisfying Condition* $(k, \tau)$ *whose components* $v_i - \langle v_i, \xi_p \rangle \xi_p$ $(i = 1, \dots, 2n-2)$ *span the tangent space* $T_p^0 M$ *orthogonal to* $\xi_p$.

If we take care of more on geodesic curvatures of circular trajectories, we can classify totally $\eta$-umbilic real hypersurfaces.

**Corollary 1.** *A connected real hypersurface* $M$ *in* $\mathbb{C}H^n(c)$ *is locally congruent to a tube around totally geodesic* $\mathbb{C}H^{n-1}$ *if and only if it satisfies the following conditions*:

i) *Condition* $(TC)$ *holds at each point* $p \in M$;

ii) *There exist positive constants* $k, \tau$ *with* $k < \sqrt{|c|}/2$ *and* $\tau < 1$ *such that at each point* $p \in M$ *we can choose linearly independent unit tangent vectors* $v_1, \dots, v_{2n-2} \in U_pM$ *satisfying Condition* $(k, \tau)$ *whose components* $v_i - \langle v_i, \xi_p \rangle \xi_p$ $(i = 1, \dots, 2n-2)$ *span the tangent space* $T_p^0 M$ *orthogonal to* $\xi_p$.

**Corollary 2.** *A connected real hypersurface* $M$ *in* $\mathbb{C}H^n(c)$ *is locally congruent to either a geodesic sphere or a tube around totally geodesic* $\mathbb{C}H^{n-1}$ *if and only if it satisfies the following conditions*:

i) *Condition* $(TC)$ *holds at each point* $p \in M$;

ii) *There exist positive constants* $k, \tau$ *with* $k \geq \sqrt{|c|}$ *and* $\tau < 1$ *such that at each point* $p \in M$ *we can choose linearly independent unit tangent vectors* $v_1, \dots, v_{2n-2} \in U_pM$ *satisfying Condition* $(k, \tau)$ *whose components* $v_i - \langle v_i, \xi_p \rangle \xi_p$ $(i = 1, \dots, 2n-2)$ *span the tangent space* $T_p^0 M$ *orthogonal to* $\xi_p$.

In the above we paid attention to complex torsions of extrinsic shapes of circular trajectories which have a given geodesic curvature, we can get similar characterizations by geodesic curvatures of those which have a given complex torsion.

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