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Non-existence of smooth rational curves of degree $d = 13, 14, 15$ contained in a general quintic hypersurface of $\mathbb{P}^4$ and in some quadric hypersurface

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Abstract. Let $W \subset \mathbb{P}^4$ be a general quintic hypersurface. We prove that W contains no smooth rational curve $C \subset \mathbb{P}^4$ with degree $d \in \{13, 14, 15\}$, $h^0(\mathcal{I}_C(1)) = 0$ and $h^0(\mathcal{I}_C(2)) > 0$.

Keywords: General quintic 3-fold, rational curve, Clemens' conjecture

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Introduction

For any positive integer d let M_d be the set of all smooth and rational curves $C \subset \mathbb{P}^4$ with $\deg(C) = d$. Let Γ_d be the set of all non-degenerate $C \in M_d$ with $h^0(\mathcal{I}_C(2)) > 0$. Clemens conjecture asks if for each d a general quintic hypersurface $W \subset \mathbb{P}^4$ contains only finitely many elements of M_d (a stronger form asks the same also for singular rational curves of degree $d > 5$) ([1], [2], [4], [12], [13], [14], [15], [19], [20], [24], [25]). For higher genera cases (and also for more general Calabi-Yau 3-folds), see [16], [17].

All the quoted finiteness results work for very low d, say $d \leq 12$. Here we add a very strong condition (to be contained in an integral quadric hypersurface) and prove the following result.

Theorem 1. *If* $13 \leq d \leq 15$*, then a general quintic hypersurface of* $\mathbb{P}^4$ *contains no element of* Γ_d*.*

The proof requires a result on the splitting type of the normal bundle of a smooth rational curve $C \subset \mathbb{P}^4$ ([3], [23]) and its use when C is contained in quadric hypersurface.

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Concerning elements of $C \in M_d$ contained in a hyperplane we prove the following result.

Proposition 1. *Let $C \in M_d$ be a degenerate curve, say contained in a hyperplane H, and let α be the the minimal degree of a surface of H containing C. Assume that C is contained in a general quintic hypersurface. If $13 \le d \le 17$, then $\alpha \in \{4, 5\}$. If $d \ge 18$, then $\alpha = 5$.*

1 Preliminaries

Let $\mathcal{W}$ denote the set of all smooth quintic hypersurfaces $W \subset \mathbb{P}^4$ satisfying the thesis of [4]. In particular each $W \in \mathcal{W}$ contains only finitely many smooth rational curves D of degree ≤ 11 and all of them have as normal bundle $N_{D,W}$ the direct sum of two line bundles of degree -1, i.e. $h^i(N_{D,W}) = 0$, $i = 0, 1$.

For any scheme $A \subset \mathbb{P}^4$ let $\mathcal{I}_A$ denote the ideal sheaf of A in $\mathbb{P}^4$.

Let X be any projective scheme, $N \subset X$ an effective Cartier divisor and $Z \subset X$ any closed subscheme. The residual scheme $\mathrm{Res}_N(Z)$ of Z with respect to N is the closed subscheme of X with $\mathcal{I}_{Z,X} : \mathcal{I}_{N,X}$ as its ideal sheaf. We always have $\mathrm{Res}_N(Z) \subseteq Z$. If Z is zero-dimensional, we have $\deg(Z) = \deg(Z \cap N) + \deg(\mathrm{Res}_N(Z))$. For any line bundle $\mathcal{L}$ on X we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_N(Z),X} \otimes \mathcal{L}(-N) \to \mathcal{I}_{Z,X} \otimes \mathcal{L} \to \mathcal{I}_{Z\cap N,N} \otimes \mathcal{L}_{|N} \to 0 \qquad (1)$$

(the residual exact sequence of N in X).

Lemma 1. *Take any $C \in \Gamma_d$, $d \ge 6$, and any $Q \in |\mathcal{I}_C(2)|$. Let $a_1 \ge a_2$ be the splitting type of the normal sheaf $N_{C,Q}$ of C in Q. Then $a_1 \le 3d - 8$.*

Proof. Since C is a smooth curve and $N_{C,Q}$ is the dual of the conormal sheaf of C in Q, $N_{C,Q}$ is a rank 2 vector bundle and hence by the classification of vector bundles on $\mathbb{P}^1$ it has a splitting type. Let $b_1 \ge b_2 \ge b_3$ be the splitting type of the normal bundle $N_{C,\mathbb{P}^4}$ of C in $\mathbb{P}^4$. We have $b_1 + b_2 + b_3 = 5d - 2$. By [3, case $r = 1$ of Lemma 4.3] we have $b_3 \ge d + 3$ and hence $b_1 \le 3d - 8$. The injective map $N_{C,Q} \to N_{C,\mathbb{P}^4}$ gives $a_1 \le 3d - 8$. QED

Remark 1. Obviously $\Gamma_d \neq \emptyset$ if and only if $d \ge 4$. The aim of this remark is to prove that $\dim \Gamma_d = 3d + 14$ and to prove a more precise result for the part associated to quadric hypersurfaces with a line as their singular locus. Let $Q \subset \mathbb{P}^4$ be an integral quadric and let $C \subset Q$ be a smooth and non-degenerate rational curve of degree d. Let $N_{C,Q}$ be the normal sheaf of C in Q and $N_{C,\mathbb{P}^4}$ the normal bundle of C in $\mathbb{P}^4$. Since C is a smooth curve and by its definition $N_{C,Q}$ is the dual of the conormal sheaf of C in Q, $N_{C,Q}$ is locally free. Since C is not contained in the singular locus of Q, $N_{C,Q}$ has rank 2. There is a natural map $j : N_{C,Q} \to N_{C,\mathbb{P}^4}$, which is injective outside the finite set $C \cap \mathrm{Sing}(Q)$. Hence j is injective. We have an exact sequence

$$0 \to N_{C,Q} \xrightarrow{j} N_{C,\mathbb{P}^4} \xrightarrow{u} \mathcal{O}_C(2) \tag{2}$$

with $\Delta := \mathrm{coker}(u)$ supported on the finite set $\mathrm{Sing}(Q) \cap C$. Set $e := \deg(\Delta)$. If $\mathrm{Sing}(Q)$ is a point, o, then blowing up it we get that $e = 1$ if $o \in C$ and $e = 0$ if $o \notin C$. Since $Q \setminus \mathrm{Sing}(Q)$ is homogeneous, $N_{C,Q}$ is spanned. Since $p_a(C) = 0$, we get $h^1(N_{C,Q}) = 0$ and hence $h^0(N_{C,Q}) = 3d + e$. Hence the subset of all Γ parametrizing curves contained either in smooth quadrics or in quadric cones with 0-dimensional vertex has dimension $3d + 14$. Let $\Gamma'_{d,e}$ be the subset of all non-degenerate $C \in M_d$ contained in some integral quadric hypersurface with singular locus a line R with $\deg(R \cap C) = e$. Now assume that Q has the line R as its singular locus. We consider only the part of the Hilbert scheme of Q formed by curves C' with $\deg(R \cap C') = e$ (it contains C by assumption). Let $a_1 \geq a_2$ be the splitting type of $N_{C,Q}$. Since $a_1 \leq 3d - 8$ (Lemma 1), we have $a_2 \geq e + 6$. Hence $h^1(N_{C,Q}(-Z)) = 0$ and so $\dim H(Q, Z, d) = 3d + e - 2e$. Since R has ∞^e subschemes of degree e and $\mathbb{P}^4$ has ∞^{11} rank 3 quadrics, we get that the part coming from quadrics with rank 3 has dimension $\leq 3d + 11$.

Lemma 2. *Let $\Gamma_{d,2}$ be the set of all non-degenerate $C \in M_d$, $d > 12$, with $h^0(\mathcal{I}_C(2)) = 2$ and contained in a smooth quintic hypersurface. Then* $\dim \Gamma_{d,2} \leq d + 25$.

Proof. Fix $C \in \Gamma_{d,2}$ and let $T \subset \mathbb{P}^4$ be the intersection of two different elements of $|\mathcal{I}_C(2)|$. Let S be the irreducible component of T containing C. Since C is non-degenerate, we have $\deg(S) \geq 3$. Hence either $\deg(S) = 3$ or $S = T$ and T is irreducible.

(a) Assume $\deg(S) = 3$. Since S spans $\mathbb{P}^4$, it is a minimal degree surface, i.e. either a cone over a rational normal curve of $\mathbb{P}^3$ or an embedding of the Hirzebruch surface F_1.

(a1) Assume that S is a cone with vertex o and let $m : U \to S$ be its minimal desingularization. U is isomorphic to the Hirzebruch surface F_3 and m is induced by the complete linear system $|\mathcal{O}_{F_3}(h + 3f)|$, where h is the section of the ruling of F_3 with negative self-intersection and f is a fiber of the ruling of F_3. We have $f^2 = 0$, $f \cdot h = 1$ and $h^2 = -3$. Let C' be the strict transform of C in U and take positive integers a, b with $b \geq 3a$ and $C' \in |ah + bf|$. Since m is induced by $|h + 3f|$, we have $b = d$. Since $\omega_{F_3} \cong \mathcal{O}_{F_3}(-2h - 5f)$, the adjunction formula gives $\omega_{C'} \cong \mathcal{O}_{C'}((a-2)h + (d-5)f)$. Since C is smooth, we have $C' \cong C$ and in particular $p_a(C') = 0$. Hence $-2 = (ah + df) \cdot ((a-2)h + (d-5)f) = (a-2)(d-3a) + a(d-5)$. Hence $a = 1$. Since $d \geq 7$, the curve $C = f(C')$ has a singular point at o, a contradiction.

(a2) Assume $S \cong F_1$ and take integers a, b with $b \geq a > 0$ and $C \in |ah + bf|$, where h is the section of the ruling of F_1 with negative intersection and f is a ruling of F_1. We have $|\mathcal{O}_{F_1}(1)| = |\mathcal{O}_{F_1}(h+2f)|$ and $\omega_{F_1} \cong \mathcal{O}_{F_1}(-2h-3f)$ and so $\omega_C \cong \mathcal{O}_C((a-2)h+(b-3)f)$. Hence $d = a+b$ and $-2 = (a-2)(b-a)+a(b-3)$. Hence $a = 1$ and $b = d-1$. Since $d-1 > 5$, every quintic hypersurface W containing C contains S. If W is smooth, then its Picard group is generated by $\mathcal{O}_W(1)$, by the Lefschetz theorem and so it contains only surfaces whose degree is divisible by 5. Hence $S \nsubseteq W$, a contradiction.

(b) Assume $S = T$, i.e. assume that T is irreducible. For a general hyperplane $H \subset \mathbb{P}^4$, $T \cap H$ is an integral curve with $p_a(T \cap H) = 1$ and hence it has at most one singular point. Hence the one-dimensional part of $\mathrm{Sing}(T)$ is either empty or a line.

(b1) Assume that $\mathrm{Sing}(T)$ contains a line L. A general hyperplane section of T is an irreducible and singular curve with arithmetic genus 1. Hence if T is a cone with vertex o, then T is the image of a minimal degree cone T' of $\mathbb{P}^5$ by a birational, but not isomorphic linear projection. If T is not a cone, then it is the image of a minimal degree smooth surface F of $\mathbb{P}^5$ by a birational, but not isomorphic linear projection ([8, Theorem 19.5]).

(b1.1) Assume that T is the image of a minimal degree non-degenerate cone $T' \subset \mathbb{P}^5$ and let $u : U \to T'$ be its minimal desingularization. We have $U \cong F_4$ and u is induced by the complete linear system $|\mathcal{O}_{F_4}(h + 4f)|$. Let $D \subset U$ be the strict transform of the curve, whose image in $\mathbb{P}^4$ is C. Write $D \in |ah + bf|$ with $b \geq 4a > 0$. As in step (a1) we first get $b = d$ and then $a = 1$. We get that $u(D)$ is singular and hence C is singular, a contradiction.

(b1.2) Assume that T is the image of a minimal degree smooth surface F of $\mathbb{P}^5$ and let $D \subset F$ be the curve with image C. Since C is smooth, D is smooth. There is $e \in \{0, 2\}$ such that $F \cong F_e$ embedded by the complete linear system $|h + (e+1)f|$. Take positive integers a, b such that $D \in |\mathcal{O}_{F_e}(ah + bf)|$ and $b \geq ea$. As in step (a) we first get $a = 1$ and then $b = d-1$. If $e = 0$ we get that every quintic hypersurface containing D contains F and hence every quintic hypersurface containing C contains T, contradicting the Lefschetz theorem as in step (a2). Now assume $e = 2$. F_2 has no smooth plane conic and its lines are either the elements of $|f|$ or h. Since $h \cdot (h + (d-1)f) = d-3$, we have $\deg(L \cap C) = d-3$. Since 3 is a prime integer, the linear projection $\ell_L : \mathbb{P}^4 \setminus L \to \mathbb{P}^2$ maps C birationally onto an integral plane cubic. Hence C is contained in the intersection of T with a cubic hypersurface, contradicting the assumption $d > 12$ by Bezout.

(b2) Assume that $\mathrm{Sing}(T)$ is finite. Since T is a complete intersection, it is a locally complete intersection. Hence T is a normal Del Pezzo surface of degree 4. Let $u : V \to T$ be a minimal desingularization and D the strict transform of C in V. Since D is smooth and rational, the adjunction formula gives $-2 = \omega_V \cdot D + D^2$. V is rational and it is classified ([6]). Since V is a weak del Pezzo, u is induced by the complete linear system $|\omega_V^\vee|$. Hence $d = \mathcal{O}_T(C) \cdot \mathcal{O}_T(1) = u^*(C) \cdot \omega_C^\vee$. Write $u^*(C) = D + \sum c_i D_i$ with $c_i \geq 0$ and D_i contracted by u. Since $\omega_V^\vee$ is spanned ([6, IV, §3, Théorème 1]), we get $\omega_V^\vee \cdot D_i = 0$. Hence $\omega_V \cdot D = -d$. Hence $D^2 = d - 2$. Hence $h^0(\mathcal{O}_D(D)) = d - 1$. Thus the set of all $C \subset T$ depends on $d-1$ parameters. Since the Grassmannian $G(2, 15)$ of all lines of $|\mathcal{O}_{\mathbb{P}^4}(2)|$ has dimension 26, this part of $\Gamma_{d,2}$ has dimension at most $d + 25$. QED

Lemma 3. *There is no non-degenerate $C \in M_d$, $d > 12$, with $h^0(\mathcal{I}_C(2)) \geq 3$ and contained in a smooth quintic hypersurface.*

Proof. Take a non-degenerate $C \in M_d$, $d > 12$, with $h^0(\mathcal{I}_C(2)) \geq 3$. Let T be the intersection of two general elements of $|\mathcal{I}_C(2)|$ and let S be the irreducible component of T containing C. Since C is non-degenerate, we have $\deg(S) \geq 3$. Hence either $\deg(S) = 3$ or $S = T$ and T is irreducible. We exclude the case $S = T$, because $d > 8$ and $h^0(\mathcal{I}_T(2)) = 2$. We exclude the case $\deg(S) = 3$ as in step (a) of the proof of Lemma 2. QED

Lemma 4. *Let $\Delta(d)$ be the set of all $C \in \Gamma_d$ for which there exists a line $L \subset \mathbb{P}^4$ with $\deg(L \cap C) \geq 5$. Then $\dim \Delta(d) \leq 12 + 3d$.*

Proof. We take $C \in \Gamma_d$ and a line $L \subset \mathbb{P}^4$ such that $\deg(L \cap C) \geq 5$. Take $Q \in |\mathcal{I}_C(2)|$. Bezout implies $L \subset Q$. If Q has a line as its singular locus, then we use Remark 1. Hence we may assume that either Q is smooth or it is a cone with vertex a single point, o. We write $e = 1$ if Q is singular and $o \in C$ and $e = 0$ otherwise. Take $Z \subseteq C \cap L$ with $\deg(Z) = 5$. Let $a_1 \geq a_2$ be the splitting type of $N_{C,Q}$. Since $a_1 \leq 3d - 8$ (Lemma 1), we have $a_2 \geq 4$. Hence $h^1(N_{C,Q}(-Z)) = 0$. Use that L has ∞^5 subschemes of degree 5 and that Q has ∞^3 lines. QED

2 Proof of Theorem 1

Fix any non-degenerate $C \in M_d$ and let $H \subset \mathbb{P}^4$ be any hyperplane. We often use the exact sequence

$$0 \to \mathcal{I}_C(t-1) \to \mathcal{I}_C(t) \to \mathcal{I}_{C \cap H, H}(t) \to 0 \qquad (3)$$

Lemma 5. *Let $Z \subset \mathbb{P}^3$ be a degree d curvilinear scheme spanning $\mathbb{P}^3$. Assume $d \leq 15$ and $h^1(\mathbb{P}^3, \mathcal{I}_Z(5)) > 0$. Then either there is a line $L \subset \mathbb{P}^3$ with $\deg(L \cap Z) \geq 7$ or there is a conic D with $\deg(D \cap C) \geq 12$.*

Proof. Since Z spans $\mathbb{P}^3$, we have $\deg(Z\cap N)\le 14$ for every plane N. Assume for the moment the existence of a plane $N\subset\mathbb{P}^3$ such that $h^1(N,\mathcal{I}_{Z\cap N,N}(5))>0$, then N contains either a line $L\subset\mathbb{P}^3$ with $\deg(L\cap Z)\ge 7$ or a conic D with $\deg(D\cap C)\ge 12$ ([7, Corollaire 2]). Now assume $h^1(N,\mathcal{I}_{Z\cap C,N}(t))=0$ for all planes $N\subset\mathbb{P}^3$. We may assume $h^1(\mathcal{I}_{Z'}(5))=0$ for all $Z'\subsetneq Z$ (taking if necessary a smaller non-degenerate Z), because $h^1(N,\mathcal{I}_{Z\cap C,N}(t))=0$ for all planes N. Set $Z_0:=Z$. Let $N_1\subset\mathbb{P}^3$ be a plane such that $e_1:=\deg(Z_0\cap N_1)$ is maximal. Set $Z_1:=\mathrm{Res}_{N_1}(Z_0)$. Define recursively for each integer $i\ge 2$ the plane $N_i\subset\mathbb{P}^3$, the integer e_i and the scheme Z_i in the following way. Let N_i be any plane such that $e_i:=\deg(Z_{i-1}\cap N_i)$ is maximal. Set $Z_i:=\mathrm{Res}_{N_i}(Z_{i-1})$. We have $e_i\le e_{i-1}$ for all $i\ge 2$. For each $i\ge 1$ we have the exact sequence

$$0\to\mathcal{I}_{Z_i}(5-i)\to\mathcal{I}_{Z_{i-1}}(6-i)\to\mathcal{I}_{Z_{i-1}\cap N_i,N_i}(6-i)\to 0 \tag{4}$$

If $e_i\le 2$, then $Z_{i-1}\subset N_i$ and hence $Z_i=\emptyset$. Since $\deg(Z)\le 15$, we get $\deg(Z_6)\le 0$, i.e. $Z_6=\emptyset$. Since $h^1(N_6,\mathcal{O}_{N_6})=0$, there is an integer i such that $1\le i\le 5$ and $h^1(\mathcal{I}_{Z_{i-1}\cap N_i,N_i}(6-i))>0$. We call f such a minimal integer. Since $h^1(N,\mathcal{I}_{Z\cap C,N}(5))=0$ for all planes N, we have $f\ge 2$. Hence $f\in\{2,3,4,5\}$. We have $e_f\ge 8-f$. Since the sequence $\{e_i\}$ is non-increasing, we get $f(8-f)\le 15$. Since $f\ge 2$, we get that $f\in\{2,3,5\}$.

(a) Assume $f=3$. Since $e_1\ge e_2\ge e_3\ge 5$, we get $e_1=e_2=e_3=5$. Since $e_3\le 7$ and $h^1(N_3,\mathcal{I}_{Z_2\cap N_3,N_3}(3))>0$, there is a line $R\subset N_3$ with $\deg(R\cap Z_2)\ge 5$. Taking a plane F containing R and with maximal $\deg(M\cap Z_1)$ we get $e_2\ge 6$, a contradiction.

(b) Assume $f=2$. We have $e_2\ge 6$. Since $e_1\ge e_2$ and $e_1+e_2\le 15$, we have $e_2\le 7$. Hence there is a line $R\subset N_2$ such that $\deg(R\cap Z_1)\ge 6$. Assuming that L does not exists, then $\deg(R\cap Z)=6$. Let $M_1\subset\mathbb{P}^3$ be a plane containing R and with maximal $g_1:=\deg(M_1\cap Z)$ among the planes containing R. Since Z spans $\mathbb{P}^3$, we have $g_1\ge 7$. Set $W_1:=\mathrm{Res}_{M_1}(Z)$. By assumption $h^1(M_1,\mathcal{I}_{Z\cap M_1,M_1}(5))=0$. Hence the residual sequence of $M_1\subset\mathbb{P}^3$ gives $h^1(\mathbb{P}^3,\mathcal{I}_{W_1}(4))>0$. Let $M_2\subset\mathbb{P}^3$ be a plane with maximal $g_2:=\deg(W_1\cap M_2)$. Set $W_2:=\mathrm{Res}_{M_2}(W_1)$. Let $M_3\subset\mathbb{P}^3$ be a plane with maximal $g_3:=\deg(W_2\cap M_3)$. Set $W_3:=\mathrm{Res}_{M_3}(W_2)$. In this way we get a non-decreasing sequence $\{g_i\}_{i\ge 2}$ with $\sum_{i\ge 2}g_i=d-g_1\le 8$. We get an integer $h\in\{2,3\}$ with $h^1(M_h,\mathcal{I}_{M_h\cap W_{h-1},M_h}(6-h))>0$ and $g_h\ge 8-h$. As in step (a) we exclude the case $h=3$. Hence $h=2$. As in the first part of step (b) we get a line $D\subset\mathbb{P}^3$ such that $\deg(D\cap W_1)=6$.

(b1) Assume $D\cap R=\emptyset$. Let $T\subset\mathbb{P}^3$ be a general quadric surface containing $D\cup R$. Since $\mathcal{I}_{D\cup R}(2)$ is spanned and Z is curvilinear, T is smooth and $T\cap Z=(D\cup R)\cap Z$ (as schemes). Hence $h^1(T,\mathcal{I}_{Z\cap T,T}(5))=0$. Since $\deg(\mathrm{Res}_T(Z))=d-12\le 3$, we have $h^1(\mathcal{I}_{\mathrm{Res}_T(Z)}(3))=0$. The residual sequence of T gives a contradiction.

(b2) Assume $D \cap R \neq \emptyset$ and $D \neq R$. Let N be the plane spanned by $D \cup R$. Since $\deg(\mathrm{Res}_N(Z)) \le d - 11$, we have $h^1(N, \mathcal{I}_{\mathrm{Res}_N(Z),N}(4)) = 0$. The residual sequence of N gives $h^1(N, \mathcal{I}_{Z\cap N,N}(5)) > 0$, contradicting one of our assumptions.

(b3) Assume $D = R$. Let $H, M \subset \mathbb{P}^3$ be general planes containing R. Since $\mathrm{Res}_{H\cup M}(Z) = \mathrm{Res}_H(\mathrm{Res}_M(Z))$, we have $\deg(\mathrm{Res}_{H\cup R}(Z)) \le d - 12 \le 3$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_{H\cup M}(Z)}(3)) = 0$. The residual sequence of $H \cup M$ gives $h^1(H \cup M, \mathcal{I}_{Z\cap(H\cup M),H\cup M}(5)) > 0$. The minimality condition of Z gives $Z \cap (H \cup R) = Z$. Hence $d = 12$. For any $q \in Z_{\mathrm{red}}$ let Z_q be the connected component of Z containing q. Since $\mathrm{Res}_H(Z)$ has degree 6 and it is supported by D, we have $2\deg(\mathrm{Res}_H(Z_q)) = \deg(Z_q)$ for all q. In particular we may take q with $Z_q \nsubseteq R$. Since Z is curvilinear, we may find a plane $N \supset R$ with $\deg(N \cap Z_q) > \deg(R \cap Z_q)$. Since $\deg(\mathrm{Res}_N(Z)) \le 12 - 7$, we have $h^1(N, \mathcal{I}_{\mathrm{Res}_N(Z),N}(4)) = 0$. The residual sequence of N gives $h^1(N, \mathcal{I}_{Z\cap N,N}(5)) > 0$, contradicting one of our assumptions.

(c) Assume $f = 5$. Since $\deg(Z_{t-1}) \le 4$, we get the existence of a line $R \subset N_5$ such that $\deg(R \cap Z_4) \ge 3$. Since $\deg(R \cap Z_3) \ge 3$, the maximality property of N_4 implies $e_4 \ge 4$. Hence $15 \ge 4 \cdot 4 + 3$, a contradiction. QED

Lemma 6. *Fix a non-degenerate $C \in M_d$ contained in some $W \in \mathcal{W}$ and assume the existence of a conic $D \subset \mathbb{P}^4$ with $\deg(D \cap C) \ge 12$ and that $\deg(L \cap C) \le 6$ for each line $L \subseteq D_{\mathrm{red}}$. Then D is smooth.*

Proof. Take $W \in \mathcal{W}$ containing C. Let N be the plane spanned by D. First assume that $D \subset N$ is a double line. Set $L := D_{\mathrm{red}}$. Since $\deg(L \cap C) \le 6$ by assumption, we have $\deg(L \cap C) = 6$. Bezout implies $L \subset W$. Since $W \in \mathcal{W}$, we have $N_{L,W} \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1)$. Bezout implies that $D \subseteq W \cap N$. Fix a general hyperplane $H \supset N$. Since W is smooth $W \cap H$ has isolated singularities. We have an injective map $N_{L,H\cap W} \to N_{L,W}$, contradicting the inclusion $D \subset H \cap W$. Now assume that $D = R \cup L$ with R, L lines and $R \neq L$. Since $\deg(L \cap C) \le 6$ and $\deg(R \cap C) \le 6$ by assumption, we have $\deg(L \cap C) = \deg(R \cap C) \le 6$. Hence $L \cup R \subset W$, contradicting the fact that any two lines of W are disjoint. QED

Lemma 7. *Take a non-degenerate $C \in M_d$ contained in some $W \in \mathcal{W}$. Assume $h^1(\mathcal{I}_C(5)) > 0$ and that there is either a line L with $\deg(L \cap C) \ge 7$ or a conic D with $\deg(D \cap C) \ge 12$. Then $h^1(\mathcal{I}_C(4)) > h^1(\mathcal{I}_C(5))$.*

Proof. Let S_1 be the set of all lines L with $\deg(L \cap C) \ge 7$ and let S_2 be the set of all conics D such that $\deg(D \cap C) \ge 10$. Assume for the moment that the sets S_1 and S_2 are finite. Let $N \subset \mathbb{P}^4$ be a general plane and let $M \subset \mathbb{P}^4$ be any hyperplane containing N. Set $V := H^0(\mathcal{I}_N(1))$. We have $\dim(V) = 2$. Since S_1

is finite and N is general, then $N \cap L = \emptyset$ for all $L \in S_1$ and hence $L \nsubseteq M$ for all $L \in S_2$. Since S_2 is finite, then N contains a unique point of the plane spanned by any $D \in S_2$ and hence $D \nsubseteq M$. Lemma 5 gives $h^1(M, \mathcal{I}_{C\cap M,M}(5))) = 0$. Hence the bilinear map $H^0(\mathcal{I}_C(5))^\vee \times V \to H^0(\mathcal{I}_C(4))^\vee$ is non-degenerate in the second variable. By the bilinear lemma we have $h^1(\mathcal{I}_C(4)) \geq h^1(\mathcal{I}_C(5)) - 1 + \dim V$.

Now assume that S_1 is infinite and call Δ an irreducible positive dimensional family of its elements. Take a general $(R, L) \in \Delta$. We have $L \cap R = \emptyset$, unless either there is $o \in \mathbb{P}^4$ with $o \in J$ for all $J \in \Delta$ or there is a plane N with $J \subset N$ for all $J \in \Delta$. The second case is not possible, because $C \nsubseteq N$. The first case is excluded, because the linear projection from o would map C onto a non-degenerate curve of $\mathbb{P}^3$ with degree $\leq (d-1)/6 < 3$.

Now assume that S_2 is infinite. Let S_2' be the set of all $D \in S_2$ with D a smooth conic. As in the proof just given we find that the set of all lines R with $\deg(R \cap C) = 6$ and supporting a component of some $D \in S_2$ is finite. Hence it is sufficient to prove that S_2' is finite. For each $D \in S_2'$ let $\langle D \rangle$ be the plane spanned by D. If $D_1 \neq D_2$, no hyperplane contains $D_1 \cup D_2$ by Bezout and hence $\langle D_1 \rangle \cap \langle D_2 \rangle = \emptyset$. Since any two planes of $\mathbb{P}^4$ meet, we have $\sharp(S_2') \leq 1$. QED

Proof of Theorem 1: Fix $C \in M_d$, $d \leq 15$.

By Remark 1 we may assume $h^1(\mathcal{I}_C(5)) \geq 2d - 13$.

(a) Assume $h^0(\mathcal{I}_C(2)) = 1$, say $\{Q\} = |\mathcal{I}_C(2)|$. Fix a general hyperplane $H \subset \mathbb{P}^4$.

(a1) Assume that there is no line $L \subset \mathbb{P}^4$ with $\deg(L \cap C) \geq 7$ and no conic D with $\deg(D \cap C) \geq 12$. Lemma 5 gives $h^1(H, \mathcal{I}_{C\cap H,H}(5)) = 0$ for every hyperplane $H \subset \mathbb{P}^4$. Hence the bilinear lemma gives $h^1(\mathcal{I}_C(4)) \geq h^1(\mathcal{I}_C(5)) + 4 \geq 2d - 9$. Since $C \cap H$ is in uniform position, we have $h^1(H, \mathcal{I}_{C\cap H,H}(4)) \leq d - 13 \leq 2$ ([10, Lemma 3.9]). By (3) we have $h^1(\mathcal{I}_C(3)) \geq 2d - 11$. Hence $h^0(\mathcal{I}_C(3)) \geq 35 - 3d - 1 + 2d - 11 \geq 8$. Since $h^0(\mathcal{I}_C(2)) = 1$, the general $M \in |\mathcal{I}_C(3)|$ has not Q as a component. Set $F := Q \cap M$. First assume that F is irreducible. The curve $D := F \cap H$ is a complete intersection curve with degree 6 and arithmetic genus 4. In particular $h^1(H, \mathcal{I}_{C,H}(3)) = 0$. Thus $h^1(H, \mathcal{I}_{C\cap H,H}(3)) = h^1(D, \mathcal{I}_{C\cap H,D}(3))$. We have $h^1(D, \mathcal{I}_{C\cap H,D}(3)) \leq 1$, because $\deg(\mathcal{I}_{C\cap H,D}(3)) = 18 - d \geq 3$. Hence $h^1(H, \mathcal{I}_{C\cap H,H}(3)) \leq 1$. Since $h^1(\mathcal{I}_C(2)) \geq 2d - 12$, we have $h^0(\mathcal{I}_C(2)) = 15 - 2d - 1 + h^1(\mathcal{I}_C(2)) \geq 2$, contradicting the assumption of step (a).

Now assume that F is not irreducible. Call T the irreducible component of F containing C. T is a non-degenerate surface and hence $\deg(T) \geq 3$. Since $h^0(\mathcal{I}_C(2)) = 1$, we have $h^0(\mathcal{I}_T(2)) = 1$ and hence neither $\deg(T) = 3$ nor T is the complete intersection of two quadrics.

Assume $\deg(T) = 4$. Since T is not a complete intersection, a general hyperplane section of T is a smooth rational curve of degree 4. Since $h^1(H, \mathcal{I}_{C\cap H,H}(t)) =$

0 for all $t \geq 2$ and $h^0(\mathcal{O}_{C\cap H}(t)) = 4t+1$, $t = 3,4$, we get $h^1(H, \mathcal{I}_{C\cap H,H}(3)) \leq d-13$ and $h^1(\mathcal{I}_{C\cap H,H}(4)) = 0$. We get $h^1(\mathcal{I}_C(3)) \geq h^1(\mathcal{I}_C(4))$ and $h^1(\mathcal{I}_C(2)) \geq h^1(\mathcal{I}_C(3)) + 13 - d \geq d+4$. Hence $h^0(\mathcal{I}_C(2)) \geq 18-d$, a contradiction.

Now assume $\deg(T) = 5$. In this case T is linked to a plane by the complete intersection T and hence $T \cap H$ is linked to a line by a complete intersection of a quadric and a cubic. Hence $T \cap H$ is arithmetically Cohen-Macaulay with degree 5 and arithmetic genus 2 ([18, Theorem 1.1 (a)], [22], [21, Proposition 3.1]). Thus $h^1(H, \mathcal{I}_{C\cap H,H}(4)) = h^1(T\cap H, \mathcal{I}_{C\cap H,H}(4)) = 0$ and $h^1(H, \mathcal{I}_{C\cap H,H}(3)) \leq 2$. We get $h^1(\mathcal{I}_C(2)) \leq 2d-11$ and hence $h^0(\mathcal{I}_C(2)) \geq 3$, a contradiction.

(a2) Now assume that there is a line $L \subset \mathbb{P}^4$ with $\deg(L\cap C) \geq 7$. By Lemma 4 we may assume $h^1(\mathcal{I}_C(5)) \geq 2d-11$. Lemma 7 gives $h^1(\mathcal{I}_C(4)) \geq 2d-10$ and hence $h^1(\mathcal{I}_C(3)) \geq 2d-12 \geq 7$. We get $h^0(\mathcal{I}_C(3)) > 5$. We repeat the proof of step (a1) with a loss of 1; for instance, if $\deg(T) = 4$ (resp. $\deg(T) = 5$) we get $h^1(\mathcal{I}_C(2)) \geq d+3$ and $h^0(\mathcal{I}_C(2)) \geq 17-d$ (resp. $h^1(\mathcal{I}_C(2)) \geq 2d-12$ and hence $h^0(\mathcal{I}_C(2)) \geq 2$), a contradiction.

(a3) Assume the existence of a conic D with $\deg(D\cap C) \geq 12$, but that there is no line $L \subset \mathbb{P}^4$ with $\deg(L\cap C) \geq 7$. By Lemma 6 we may assume that D is smooth.

(a3.1) Assume for moment $h^1(\mathcal{I}_C(5)) \geq 2d-12$. Lemma 7 gives $h^1(\mathcal{I}_C(4)) \geq 2d-11$. The case $t=4$ of (3) and [10, Lemma 3.9] give $h^1(\mathcal{I}_C(3)) \geq 2d-13$. Hence $h^0(\mathcal{I}_C(3)) \geq 35-14-d > 5$. As in step (a1) we first get $h^1(\mathcal{I}_C(3)) \geq h^1(\mathcal{I}_C(4))$ and then $h^1(\mathcal{I}_C(2)) \geq h^1(\mathcal{I}_C(3)) - 1$. Thus $h^0(\mathcal{I}_C(2)) \geq 2$, contradicting our assumption.

(a3.2) Now we justify the assumption made in step (a3.1). If Q is a quadric with vertex a line, then we may assume $h^1(\mathcal{I}_C(5)) \geq 2d-10$ by Remark 1. If Q is a quadric cone with vertex a point o and $o \notin C$, then we may assume $h^1(\mathcal{I}_C(3)) \geq 2d-12$ by Remark 1. Now assume that C is a contained in a quadric cone Q with vertex a point $o \in C$. It is sufficient to prove that for each irreducible component Δ of the set of all non-degenerate $Y \in M_d$ with $Y \subset Q$ and $o \in Y$ a general $Y \in \Delta$ has no conic D with $\deg(D\cap Y) \geq 12$ or that if $C \in \Delta$, then it may be deformed to $Y \in \Delta$ with no offending conic. Bezout gives $D \subset Q$. We need to distinguish the case $o \in D$ and $o \notin D$. First assume $o \in D$. Fix $Z \subseteq D\cap C$ with $\deg(Z) = 12$ and $o \in Z_{\mathrm{red}}$. Since D has ∞^{12} zero-dimensional schemes with degree 12 and Q has ∞^5 conics through o, it is sufficient to prove that $h^0(N_{C,Q}(-Z)) < 3d+1-5-12$. We have $h^0(N_{C,Q}(-Z)) \leq 3d+1-12-7$ by Lemma 1. If $o \notin D$ we use the same proof, just using that Q has ∞^6 conics.

The case of a smooth Q is similar.

(b) Now assume $h^0(\mathcal{I}_C(2)) \geq 2$. By Lemmas 2 and 3 C is contained in an integral complete intersection of 2 quadrics and we may assume that

$h^1(\mathcal{I}_C(5)) \geq 4d-24$. Hence as in step (a) we get $h^1(\mathcal{I}_C(3)) \geq 4d-24$, $h^1(\mathcal{I}_C(2)) \geq 3d-13$ and hence $h^0(\mathcal{I}_C(2)) > 2$, contradicting Lemma 3. QED

3 Proof of Proposition 1

Remark 2. Fix an integer $d \geq 13$ and $C \in M_d$ contained in a hyperplane $H \subset \mathbb{P}^4$. Since $h^0(H, \mathcal{I}_C(5)) = 56$, we have $h^1(\mathcal{I}_C(5)) \geq 5(d-11) > 0$.

Proof of Proposition 1: Take $C \in M_d$ contained in a hyperplane $H \subset \mathbb{P}^4$ and contained in some $W \in \mathcal{W}$. Let $S \subset H$ be a degree α hypersurface. Since α is the minimal degree of a surface of H containing C and C is irreducible, S is irreducible. Since $C \subset W \cap H$, we have $\alpha \leq 5$.

(a) Assume $\alpha = 2$. If S is smooth, then up to a change of the ruling of S we may assume $C \in |\mathcal{O}_S(1, d-1)|$. Since $d-1 > 5$, $W \supset S$, contradicting the Lefschetz theorem which implies that all surfaces contained in W have degree divisible by 5. If S is a cone, then any smooth curve on it is projectively normal ([11, Ex. V.2.9]), contradicting Remark 2.

(b) Assume $\alpha = 3$. Bezout implies $h^0(H, \mathcal{I}_C(3)) = 1$. By the Lefschetz theorem we have $S \nsubseteq W$. Since $C \subseteq S \cap W$, we get $d \leq 15$. The case $d = 15$ is excluded, because the $\omega_{S\cap W} \cong \mathcal{O}_{S\cap W}(4)$ and so $S \cap W \neq C$. The case $d = 14$ is excluded, because it would give that the complete intersection $S \cap W$ would link C to a line and hence it is arithmetically normal ([18], [21], [22]), contradicting Remark 2. Now assume $d = 13$. In this case $S \cap W$ links C to a degree 2 locally Cohen-Macaulay curve D. If D is a plane curve, then C is arithmetically Cohen-Macaulay, contradicting Remark 2. If D is a disjoint union of 2 lines, then $p_a(D) = -1$, contradicting [21, Proposition 3.1]. Now assume that D is a double structure on a line L, but it is not a conic, i.e. that D is not a conic. Since $S \cap W$ links $C \cup L$ to L, $C \cup L$, we have $p_a(C \cup L) - p_a(L) = 2(11-1)$ ([21, Proposition 3.1]), i.e. $p_a(C \cup L) = 20$, and hence $\deg(C \cap L) = 21$, contradicting the inequality $d < 21$.

(c) Assume $\alpha = 4$. Since $C \subseteq W \cap S$, we have $d \leq 20$. We exclude the cases $d = 20$ and $d = 19$ as in step (b). Now assume $d = 18$. $S \cap W$ links C to a degree 2 locally Cohen-Macaulay curve D. If D is a plane curve, then C is arithmetically Cohen-Macaulay, contradicting Remark 2. Now assume that D is a double structure on a line L, but it is not a conic, i.e. that D is not a conic. Since $S \cap W$ links $C \cup L$ to L, $C \cup L$, we have $p_a(C \cup L) - p_a(L) = (17-1)5/2$ ([21, Proposition 3.1]) and hence $\deg(C \cap L) > 40$, a contradiction. QED

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