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A note on groups with restrictions on centralizers of infinite index

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Abstract. A group G is said to be an *AFC-group* if for each element x of G, either x has finitely many conjugates or the factor group $C_G(x)/\langle x\rangle$ is finite. In this survey article some results concerning AFC-groups and minimal-non-AFC groups are collected.

Keywords: AFC-group; minimal-non-AFC group; FC-centre

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Introduction

An element x of a group G is said to be an *FC-element* if it has finitely many conjugates in in G, i.e. its centralizer has finite index in G. In any group G the set $FC(G)$ of all FC-elements is a subgroup, called the *FC-centre* of G, and a group is said to be an *FC-group* if it coincides with its FC-centre. Obvious examples of FC-groups are all finite groups and all abelian groups, and the first results obtained in the theory of FC-groups essentially say that some properties common to these two relevant classes also hold for FC-groups. Many authors have developed the theory of FC-groups in the last century; we refer to [19] and to [4] for the main results concerning this class of groups.

In a paper published in 1994, A. Shalev [16] has studied locally finite groups in which an extremal condition is imposed to centralizers; more precisely, he considered locally finite groups G in which for each element x, the centralizer $C_G(x)$ either is finite or has finite index, proving that in such a group the FC-centre has finite index.

More recently, in a paper which is submitted for publication, M. De Falco, F. de Giovanni, C. Musella and N. Trabelsi [7] have studied a class which generalize the one studied by Shalev. In this paper a group G is said to be an *AFC-group* if for each element x of G, either x is an FC-element or the factor group $C_G(x)/\langle x\rangle$ is finite. Using this terminology, the above quoted Shalev's theorem can be reformulated saying that if G is a locally finite AFC-group, then the factor group $G/FC(G)$ is finite. A similar problem has been considered in [15], where groups in which every non-normal cyclic subgroup has finite index in its centralizer and groups in which every non-normal subgroup has finite index in its normalizer are characterized within the universe of locally soluble-by-finite groups.

If $\mathfrak{X}$ is a class of groups, a group G is said to be minimal-non-$\mathfrak{X}$ if G is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. Minimal-non-$\mathfrak{X}$ groups have been investigated for several different choices of the group class $\mathfrak{X}$. In particular, V.V. Belyaev and N.F. Sesekin [2] completely described the structure of a minimal-non-FC group G, under the assumption that G admits a non-trivial homomorphic image which is either finite or abelian; in particular, it turns out that if a minimal non-FC group G has a non-trivial abelian homomorphic image, then it has a non-trivial finite one. In the paper [5] M. De Falco, F. de Giovanni, M. Kuzucuoğlu and C. Musella have studied minimal non-AFC groups. In particular, they have proved that any minimal non-AFC group admitting proper subgroups of finite index has a large FC-centre, like AFC-groups themselves. Moreover, it turns out that imperfect minimal non-AFC groups have a restricted structure, although it is possible that they do not contain proper subgroups of finite index.

The aim of this paper is to provide an overview of the results concerning the structure of AFC-groups and that of minimal-non-AFC groups.

1 The FC-centre of AFC-groups

The class of AFC-groups is clearly closed with respect subgroups, but it will be shown now that it is not closed with respect to homomorphic images. Consider the group $D(A) = \langle x \rangle \ltimes A$, where A is a torsion-free abelian group and $a^x = a^{-1}$; it is easy to see that $D(A)$ is an AFC-group. Assume now that A has infinite rank, and let B be a free-abelian subgroup of A such that A/B is periodic; then in the factor group $D(A)/B^4$ the centralizer of the coset xB^4 is infinite and has infinite index, so that $D(A)/B^4$ is not an AFC-group.

Any way, in some cases, factoring AFC-groups with respect to suitable types of subgroups, we obtain AFC-groups, as next two results show; the first of them has been proved in [7].

Proposition 1. *Let G be a an AFC-group. Then the factor group $G/Z(G)$ is likewise an AFC-group. Moreover, if G is not an FC-group, then for every positive integer n, $Z_n(G)/Z(G)$ is finite.*

Proposition 2. *Let N be a finite normal subgroup of a group G. Then G is an AFC-group if and only if G/N is an AFC-group.*

Proof. Clearly, it is enough to show that for any element x of G, the index $|C_G(x) : \langle x \rangle|$ is finite if and only if the index $|C_{G/N}(xN) : \langle xN \rangle|$ is finite.

Put $H/N = C_{G/N}(xN)$. Assume first that $|C_G(x) : \langle x \rangle|$ is finite, and put $C = C_H(N)$. The map

$$\varphi : y \in C \longrightarrow [y, x] \in N$$

is a homomorphism with kernel $C_C(x)$, so that $C_C(x)$ has finite index in C, and hence $C_G(x)$ has finite index in H. It follows that $\langle xN \rangle$ has finite index in H/N.

Conversely, assume that $\langle xN \rangle = \langle x \rangle N/N$ has finite index in H/N; then $\langle x \rangle$ has finite index in H and hence in $C_G(x)$. QED

Many authors have investigated the influence that the existence of centralizers which are "small" in some sense has on the structure of a group. In particular, there are in literature many results which show that in many cases, if a group G contains an element whose centralizer satisfies a suitable finiteness condition, this fact has a strong impact on the structure of G. One of the most famous results obtained on this matter is due to Šunkov [18], who proved that if G is a periodic group G containing an involution with finite centralizer, then G is soluble-by-finite. Later this result has been improved by B. Hartley and T. Meixner [8], who showed that, under the same hypotheses, the group G has a nilpotent subgroup of index at most two. In 1982 Hartley posed in the Kourovka notebook [46, 8.81] a question which is equivalent to ask if there exist functions f and g such that any finite group G admitting an automorphism of prime order p whose centralizer has finite order m, must contain a nilpotent subgroup of index at most $f(p, m)$ and class at most $g(p)$. This problem can be reduced to the nilpotent case in view of a result proved by Hartley and Meixner [9] and independently by M.R. Pettet [14], which states that any finite group admitting an automorphism of prime order p whose centralizer has finite order m, must contain a nilpotent subgroup whose index is bounded by a function of m and p; a positive answer for all locally finite groups was finally obtained in 1992, when E.I. Khukhro [10] solved the nilpotent case. The following theorem, which summarize the results obtained in order to solve the question posed by Hartley, is considered one of the most important goals reached in the theory of locally finite groups (see for instance [17]):

Theorem 1. *Let G be a locally finite group admitting an automorphism of prime order p whose centralizer has finite order m. Then G contains a nilpotent subgroup whose index is finite and bounded by a function of m and p and whose nilpotency class is bounded by a function of p.*

In [16], Shalev has proved the following theorem, which gives a relevant information about the structure of locally finite AFC-groups, and is also a crucial step of the proof of the theorem about the finiteness of the factor group of such groups over the FC-centre.

Theorem 2. *Let G be a locally finite AFC-group. Then G is either an FC-group or nilpotent-by-finite.*

Proof. Assume that G is not an FC-group. Then there exists an element $x \in G \setminus FC(G)$ such that the coset $xFC(G)$ has prime order p.

Let $H = (C_G(x^p))_G$ be the core of the centralizer of x^p, so that x acts on H as an automorphism of prime order and the centralizer of x in H is finite; it follows from Theorem 1 that H is nilpotent-by-finite. Therefore G is likewise nilpotent-by-finite. QED

Theorem 3. *Let G be a locally finite AFC-group. Then the FC-centre F of G has finite index in G.*

Proof. Assume that G is not an FC-group, and let N be a nilpotent normal subgroup of finite index of G. It can be proved by induction on the nilpotency class c of N that N is contained in the FC-centre F of G. If N is abelian, then clearly every element of N is an FC-element, and so $N \leq F$. Assume that $c > 1$; if the centre A of N is finite, then G/A is an AFC-group, and it follows from the induction hypothesis that N/A is contained in the FC-centre F/A of G/A, so that N is contained in F. Therefore it can be assumed that A is infinite, so that every element of N has infinite centralizer, and hence also in this case N is contained in F. QED

The proof of the previous result consists essentially in showing that every nilpotent normal subgroup of an AFC-group is contained in the FC-centre. This property cannot be extended to locally nilpotent normal subgroups, since there exist locally nilpotent AFC-groups which are not FC-groups, as the following example, given in [13], shows:

Let A be the direct product of a countably infinite collection $(< a_n >)_{n \in \mathbb{N}}$ of groups of order p, where p is a fixed prime, and let x be the automorphism of A defined by putting $a_1^x = a_1$ and $a_n^x = a_{n-1}a_n$ if $n > 1$. Then G is clearly locally nilpotent, and for every element a of A, the normal closure $\langle a \rangle^G$ is finite, so that a is an FC-element of G; moreover for every positive integer k the centralizer $C_G(x^k)$ is contained in $\langle x, a_1, \ldots, a_k \rangle$, so that the index $|C_G(x^k) : \langle x^k \rangle|$ is finite. It follows that G is an AFC-group, but not an FC-group.

On the other hand, it has been proved in [7] that nilpotent subnormal subgroups of any AFC-group are contained in its FC-centre. Recall here that the subgroup generated by all subnormal cyclic subgroups of any group G is called the *Baer radical of G*, so that the fact that in any AFC-goup all nilpotent subnormal subgroups are contained in the FC-centre can be formulated in the following way:

Theorem 4. *Let G be an AFC-group. Then the FC-centre of G contains the Baer radical.*

Theorem 3 cannot be extended to non-periodic AFC-groups; in fact, the following example shows that there exists a metabelian AFC-group such that the factor group $G/FC(G)$ is infinite.

For each odd prime p let $\langle a_p \rangle$ be a group of order p, and let A be the direct product of the collection $(\langle a_p \rangle)_{p \in P}$. Consider an automorphism x of A inducing an automorphism of order $p-1$ on every $\langle a_p \rangle$, and let G be the semidirect product of A by $\langle x \rangle$.

Clearly, x has infinite order, and A is the set of all elements of finite order of G. Moreover, $C_{\langle x \rangle}(a_p) = \langle x^{p-1} \rangle$, for each positive integer k and we have that $C_A(x^k) = \langle a_p \mid p-1 \textit{ divides } k \rangle$. In particular, the index $|C_G(x^k) : \langle x^k \rangle|$ is finite for each positive integer k, and it follows that every infinite cyclic subgroup of G has finite index in its centralizer. Moreover, for any $a \in A$, the subgroup $< a >$ is normal in G, and so $A = FC(G)$. Therefore G has the AFC-property, but $G/FC(G)$ is infinite cyclic.

In any AFC-group G, the size of the factor group $G/FC(G)$ is related with the torsion-free rank of the abelian subgroups of G. In fact, if G contains a direct product of the form $\langle a \rangle \times \langle b \rangle$ where $\langle a \rangle$ and $\langle b \rangle$ are infinite cyclic, and x is any element of $C_G(a) \cap C_G(b)$, then the index $|C_G(x) : \langle x \rangle|$ is infinite, so that x is an FC-element of G. Moreover, it is clear that a and b are both FC-elements of G, so that $C_G(a) \cap C_G(b)$ has finite index in G, and hence the FC-centre of G has likewise finite index. So we have the following result (see [7]).

Proposition 3. *Let G be an AFC-group such that the factor group $G/F(G)$ is infinite. Then every abelian subgroup of G has torsion-free rank at most* 1.

As a consequence of the previous proposition, it can also be proved that the factor group of any AFC-group with respect to its FC-centre satisfies some finiteness conditions.

Proposition 4. *Let G be an AFC-group, and let F be the FC-centre of G.*

(1) If F is periodic, then every locally finite subgroup of G/F is finite and every abelian non-periodic subgroup of G/F is torsion-free of rank 1*;*

(2) if F is not periodic, then G/F is periodic.

The main result proved in [7] about the size of the factor group of any AFC-group with respect to its FC-centre gives a strong information on this factor group within the universe of locally (soluble-by-finite) groups. Recall here that a group G is said to have *finite (Prüfer) rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property; if such an r does not exists, G is said to have *infinite rank*.

Theorem 5. *Let G be a locally (soluble-by-finite) AFC-group. Then the factor group $G/FC(G)$ is a soluble-by-finite group of finite rank.*

It is well-known that locally soluble FC-groups are hyperabelian with length at most ω; combining this fact with the solubility of the factor group of any locally soluble AFC-group over its FC-centre, it is easy to prove the following corollary:

Corollary 1. *Let G be a locally soluble AFC-group. Then G is hyperabelian with length at most $\omega + k$ for some non-negative integer k.*

It is well-known that the commutator subgroup of any FC-group is locally finite, so that, in particular torsion-free FC-groups are abelian; the last results of this section are taken from [7] and give some information about the behaviour of torsion-free AFC-groups. The first result assures that in the case of locally (soluble-by finite) groups, it turns out that the FC-centre has finite index, while the other two results restrict the structure of torsion-free AFC-groups which are not abelian-by-finite.

Theorem 6. *Let G be a torsion-free locally (soluble-by-finite) AFC-group. Then G is abelian-by-finite.*

Theorem 7. *Let G be a torsion-free AFC-group which is not abelian-by-finite. Then:*

(a) The centre and the FC-centre of G coincide, and all abelian subgroups of G are cyclic.

(b) If the FC-centre of G is not trivial, then $G/Z(G)$ is a periodic group whose non-trivial elements have finite centralizer.

Examples of torsion-free AFC-groups which are not abelian-by-finite can be found in [11] and in [1]; more precisely, A.Y. Ol'shanskiĭ in [11] constructed a simple group in which all proper non-trivial subgroups are infinite cyclic, while in the proof of Theorem 2 of [1] it is possible to find the description of a group G in which the centre is infinite cyclic and the factor group $G/Z(G)$ is a Tarski group.

2 Minimal-non-AFC groups

In this section, some general properties of minimal-non-AFC groups will be presented; it will be shown that the class of minimal-non-FC groups and the one of of minimal-non-AFC groups have some common properties. Remark here that easy examples show that these two classes are incomparable; in fact, the locally dihedral 2-group $D(2^\infty)$ is an AFC-group which is a minimal-non-FC group since its proper subgroups are either finite or abelian. On the other hand, in order to construct a minimal-non-AFC group which is not minimal-non-FC, consider a prime number p and two Prüfer p-groups

$$A = \langle a_n \mid n \in \mathbb{N}_0\rangle \text{ and } B = \langle b_n \mid n \in \mathbb{N}_0\rangle$$

with the usual relations

$$a_0 = b_0 = 1,\ \ a_{n+1}^p = a_n,\ \ b_{n+1}^p = b_n\ \ \forall n \in \mathbb{N}_0.$$

Let M be the direct product $A\times B$, let x be the automorphism of M defined by putting $a_n^x = b_n$ and $b_n^x = a_n$, and let $G = \langle x\rangle \ltimes M$ be the semidirect product of M by $\langle x\rangle$. If X is any infinite proper non-abelian subgroup of G, the intersection $X\cap M$ is an infinite proper subgroup of M, and so X contains a Prüfer subgroup of finite index. Therefore every subgroup of X either is finite or has finite index in X, and hence X is an AFC-group. On the other hand, G is not an AFC-group, since x is not an FC-element of G, but its centralizer contains the subgroup $\langle a_nb_n \mid n \in \mathbb{N}_0\rangle$, so that the index $|C_G(x) : \langle x\rangle|$ is infinite. Thus the group G is minimal-non-AFC. Finally, G is not minimal-non-FC, since the subgroup $\langle a_n^{-1}b_n \mid n \in \mathbb{N}_0\rangle$ is a normal Prüfer subgroup of G which is not centralized by x, so that $\langle U, x\rangle$ is a proper subgroup of G for which the property FC does not hold.

As we quoted in the introduction, V.V. Belyaev and N.F. Sesekin completely described minimal-non-FC groups having a non-trivial finite homomorphic image. Recall here that the *finite residual* of a group G is the intersection of all (normal) subgroups of finite index of G, and that G is *residually finite* if its finite residual is trivial.

Lemma 1. *Let G be a minimal-non FC group. Then there exists a prime p such that every finite homomorphic image of G is a cyclic p-group.*

Proof. Let N be a normal subgroup of finite index of G. If x is any element of G with infinite conjugacy class, then x has infinitely many conjugates also in $\langle N, x\rangle$, so that $\langle N, x\rangle = G$, and G/N is cyclic. Assume for a contradiction that $|G/N|$ is divisible by two different primes p and q; then $\langle N, x^p\rangle$ and $\langle N, x^q\rangle$

are proper subgroups of G, so that they are FC-groups. It follows that x^p and x^q are FC-elements of G, and hence x belongs to the FC-centre of G; this contradiction shows that G/N has prime-power order. The statement is proved, since the intersection of two subgroups of finite index has likewise finite index. QED

Lemma 2. *Let G be a group and let J be the finite residual of G. If there exists a prime p such that every finite homomorphic image of G is a cyclic p-group, then the factor group G/J is either a cyclic group of prime-power order or a torsion-fee abelian group.*

Proof. Obviously, G/J is an abelian group; let T/J be the subgroup of G/J consisting of all elements of finite order. As every finite homomorphic image of G is cyclic of prime-power order, T/J is a cyclic p-group for some prime p, and hence it is a direct factor of G/J. Let K/J be a subgroup of G/J such that $G/J = T/J \times K/J$. Again the hypothesis on the finite homomorphic images of G yields that at least one of the subgroups T/J and K/J must be trivial. If $K = J$, G/J is a cyclic p-group, while if $T = J$, G/J is a torsion-free abelian group. QED

Theorem 8. (V.V. Belyaev, N.F. Sesekin [2]) *Let G be a minimal-non-FC group admitting a proper subgroup of finite index, and let J be the finite residual of G. Then the following conditions hold:*

(i) there exist an element x of G and a prime p such that $G = \langle J, x\rangle$, $x^{p^n} \in J$ for some positive integer n and $x^p \in Z(G)$;

(ii) J is a divisible q-group of finite rank for some prime q;

(iii) J coincides with the commutator subgroup of G;

(iv) J has no infinite proper G-invariant subgroups.

Proof. It follows from Lemma 1 and Lemma 2 that the factor group G/J is either a cyclic group of prime-power order or a torsion-fee abelian group. Assume for a contradiction that G/J is torsion-free; in this situation it is known that G/J contains three subgroups H_1, H_2, H_3 such that $G = H_1H_2H_3$ and $H_iH_j < G$ for all i, j, so that G is an FC-group (see [6], Corollary 2.2). This contradiction shows that G/J is a cyclic group of prime power-order, so that there exist an element x of G and a prime p such that $G = \langle J, x\rangle$ and $x^{p^n} \in J$ for some positive integer n. Moreover, x^p centralizes J since $\langle J, x^p\rangle$ is an FC-group; therefore x^p belongs to the centre of G, and (i) is proved.

As G/J is finite, the FC-group J has no proper subgroup of finite index, so that it is a divisible abelian group. In order to prove statement (ii), it remains to show that J is a q-group for some prime q. Let T be the subgroup consisting of all elements of finite order of J. Assume for a contradiction that T is central in G, and let a be any element of infinite order of J. Choose an element b of J such that $a = b^p$; the finitely generated abelian-by-finite group $\langle\langle b\rangle^G, x\rangle$ is properly contained in G, so that $[b, x] \in T$, and

$$[a, x] = [b^p, x] = [b, x]^p = [b, x^p] = 1.$$

Therefore $J \leq Z(G)$, and G is abelian; this contradiction shows that T is not contained in the centre of G, and hence there exists, for some prime q, a Prüfer q-subgroup Q of G such that Q does not centralizes x. It follows that $\langle Q, x\rangle$ is not an FC-group, and hence $\langle Q, x\rangle = G$; in particular, $G = \langle Q^G, x\rangle$, so that $J = Q^G$ is generated by finitely many conjugates of Q, and hence it is a divisible q-group of finite rank.

Assume now for a contradiction that G' is properly contained in J, so that the abelian-by-finite group $H = \langle G', x\rangle$ is an FC-group, and hence H' is finite; moreover G/H' is nilpotent of class 2, so that if u is any element of J, and v is an element of J such that $u = v^p$, we have

$$[u, x]H' = [v^p, x]H' = [v, x]^p H' = [v, x^p]H' = H'.$$

Therefore G/H' is abelian, and G is finite-by-abelian; this contradiction shows that $J = G'$, and (iii) holds.

Finally, assume for a contradiction that J contains an infinite proper G-invariant subgroup N, so that N contains a Prüfer subgroup P, and hence P is contained in the centre of the FC-group $\langle N, x\rangle$, so that P is contained in the centre of G. Let y be any element of P; then there exixts an element z of G such that $y = [z, x]$. Therefore $y^p = [z, x]^p = [z, x^p] = 1$, and P has exponent p; this last contradiction completes the proof. QED

Using similar arguments, it has been shown in in [5] that the property described in Lemma 1 also holds for minimal-non-AFC groups.

Lemma 3. *Let G be a minimal-non-AFC group. Then there exists a prime p such that every finite homomorphic image of G is a cyclic p-group.*

Using the previous lemma, a description has been obtained in [5] of locally (soluble-by-fnite) minimal-non-AFC-groups; in particular, it turns out that in the universe of locally (soluble-by-finite) groups, minimal-non-AFC groups with a non-trivial finite homomorphic image have large FC-centre.

Theorem 9. *Let G be a locally (soluble-by-finite) minimal-non-AFC group admitting a proper subgroup of finite index. Then the FC-centre of G is abelian and has prime index.*

In [2], Belyaev and Sesekin proved that any minimal-non-FC group with an abelian non-trivial homomorphic image also has a finite non-trivial homomorphic image.

Theorem 10. (V.V. Belyaev, N.F. Sesekin [2]) *Let G be a non-perfect minimal-non-FC group. Then G admits a proper subgroup of finite index.*

Proof. Assume for a contradiction that G has no proper subgroups of finite index. Suppose first that there exist two proper subgroups H and N such that $G = HN$ and N is normal in G, and let h be any element of H; then $\langle N, h\rangle$ is a proper subgroups of G, since G/N cannot be cyclic, and hence $\langle N, h\rangle$ is an FC-group. It follows that $C_H(h)$ has finite index in H and $C_N(h)$ has finite index in N, so that $\langle C_H(h), C_N(h)\rangle$ has finite index in $G = HN$; thefore $G = \langle C_H(h), C_N(h)\rangle$, and H is contained in the centre of G. In particular, $G = Z(G)N$, and hence, for any element $x \in N$, we have that $C_G(N) = Z(G)C_N(N)$ has finite index in G, so that $C_G(x) = G$ and also N is contained in $Z(G)$. It follows that G is abelian, and this contradiction shows that any proper normal subgroup N of G cannot be supplemented by a proper subgroup.

Since G has no proper subgroups of finite index, the factor group G/G' is a divisible abelian group. It follows that G/G' is a group of type p^∞ for some prime p, since it cannot be the product of two proper subgroups.

For each positive integer n, let P_n/G' the subgroup of G/G' of order p^n, so that G is the set-theoretic union of the collection $\{P_n \mid n \in \mathbb{N}\}$, which consists of FC-subgroups; in particular, G' is locally finite, and hence G is locally finite. Let P be a Sylow p-subgroup of G; then $G = PG'$ (see [19], Theorem 5.4), so that $G = P$ is a p-group. In particular, G is locally nilpotent, so that the FC-group G' is hypercentral, and hence G'' is properly contained in G'; therefore G/G'' is not abelian, and hence it is not an FC-group. We can therefore replace G with G/G'', and assume that G' is abelian. Therefore for each positive integer n, P_n is abelian-by-finite, and hence central-by-finite, so that P_n' is a finite normal subgroup of G; it follows that P_n' is contained in $Z(G)$, so that G' is contained in $Z(G)$, and hence $G/Z(G)$ is locally cyclic. It follows that G is abelian, and this last contradiction completes the proof. QED

The previous theorem cannot be extended to minimal-non-AFC groups. To see this, consider the following group constructed by V.S. Čarin [3]. Fix a prime number p and a non-empty set π of primes different from p. Let $\mathcal{K}$ be the algebraic closure of the field of order p; then the multiplicative group $\mathcal{K}^*$ of $\mathcal{K}$ is a direct product of groups of type q^∞, one for each prime $q \neq p$. Let Q be the π-component of $\mathcal{K}^*$, and let A be the additive group of the subfield F of $\mathcal{K}$ generated by Q; finally, let $G(p, \pi)$ be the semidirect product $Q \ltimes A$, where the action of an element $x \in Q$ on A is the multiplication by x (in F). The group $G(p, \pi)$ is called the *Čarin group of type* (p, π).

In [5], it has been proved the following proposition, from which it follows that the Čarin group $G(p, q)$ is a minimal-non-AFC group for each pair (p, q) of different primes. On the other hand, $G(p, q)$ is metabelian, but it has no non-trivial finite homomorphic images.

Proposition 5. *Let Q be a group of type p^∞ for some prime number p, and let A be an infinite irreducible Q-module. Then every extension of A by Q is a minimal-non-AFC group.*

Even if it is possible that a minimal-non-AFC group with no proper subgroups of finite index is not perfect, the abelianization of such a group cannot be too large. In fact (see [5]):

Proposition 6. *Let G be a minimal-non-AFC group which has no proper subgroups of finite index. Then either $G = G'$ or G/G' is a group of type p^∞ for some prime number p.*

3 The locally nilpotent case

It is well known that any locally nilpotent FC-group is hypercentral and its upper central series has length at most ω. In [13] D.I. Zaitsev and V.A. Onishchuk have proved that if a locally nilpotent group G contains a finitely generated subgroup whose centralizer has finite rank, then the centre of G is non-trivial. Moreover, Onishchuk [12] has proved that if a locally nilpotent group G contains a cyclic subgroup whose centralizer is a group of finite rank whose Sylow subgroups are finite, then G is hypercentral; in particular, it follows from this result that locally nilpotent AFC-groups are hypercentral; in [7] it has been obtained also an information on the hypercentral length of such groups.

Theorem 11. *Let G be a locally nilpotent AFC-group. Then G is is hypercentral and its upper central series has length at most $\omega+k$ for some non-negative integer k.*

The following theorem, proved in [7], gives more information on the structure of locally nilpotent AFC-groups in which the FC-centre has infinite index.

Theorem 12. *Let G be a locally nilpotent AFC-group with FC-centre F, and let T be the subgroup consisting of all elements of G of finite order. Then $G' \leq T \leq F$. Moreover, if the factor group G/F is infinite, then F is periodic and G/F is a torsion-free abelian group of rank* 1.

Also for minimal-non-AFC groups there are some relevant properties in the locally nilpotent case. In particular, locally nilpotent minimal-non-AFC groups either are minimal-non-FC groups or are hypercentral.

Theorem 13. (see [5]) *Let G be a locally nilpotent group minimal-non-AFC-group. Then G is either hypercentral or minimal-non-FC.*

Finally a remarkable property of locally nilpotent minimal-non-AFC groups is the fact that the study of such groups can be reduced either to that of minimal-non-FC groups or to the case of groups admitting proper subgroups of finite index. In fact, it follows from Theorem 13 that any locally nilpotent minimal non-AFC group which is perfect must be minimal-non-FC; on the other hand the following proposition shows that (as it happens for minimal-non-FC groups) the presence of a non-trivial abelian homomorphic image forces a locally nilpotent minimal non-AFC-group to have a non-trivial finite homomorphic image.

Proposition 7. (see [5]) *Let G be a locally nilpotent non-perfect minimal-non-AFC group. Then G admits a proper subgroup of finite index.*

References

[1] I.S. Ashmanov, A.Y. Ol'shanskiĭ: *Abelian and central extensions of aspherical groups*, Soviet. Math. (Iz. VUZ) **29** (1985), 65–82.

[2] V.V. Belyaev, N.F. Sesekin: *On infinite groups of Miller-Moreno type*, Acta Math. Acad. Sci. Hungar. **26** (1975), 369-376.

[3] V.S. Čarin: *A remark on the minimal condition for subgroups*, Dokl. Akad. Nauk SSSR **66** (1949), 575-576.

[4] F. Catino, F. de Giovanni: Some topics in the theory of groups with finite conjugacy classes, Algoritmica, 1, Aracne Editrice, Rome, 2015.

[5] M. De Falco, F. de Giovanni, M. Kuzucuoğlu, C. Musella: *Groups whose proper subgroups have restricted infinite conjugacy classes*, submitted.

[6] M. De Falco, F. de Giovanni, C. Musella, N. Trabelsi: *Groups whose proper subgroups of infinite rank have finite conjugacy classes*, Bull. Austral. Math. Soc. **89** (2014), 41-48.

[7] M. De Falco, F. de Giovanni, C. Musella, N. Trabelsi: *Groups with restrictions on infinite conjugacy classes*, submitted.

[8] B. Hartley, T. Meixner: *Periodic groups in which the centralizer of an involution has bounded order*, J. Algebra **64** (1980), 285–291.

[9] B. Hartley, T. Meixner: *Finite soluble groups in which the centralizer of an element of prime order is small*, Arch. Math. (Basel) **36** (1981), 21–213.

[10] E.I. Khukhro: *Groups and Lie rings admitting an almost regular automorphism of prime order*, Math. USSR Sbornik **71** (1992), 51–63.

[11] A.Y. Ol'shanskiĭ: *An infinite simple torsion-free noetherian group*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 1328–1393.

[12] V.A. Onishchuk: *Locally nilpotent groups with a centralizer of finite rank*, Ukrain. Math. J. **44** (1992), 1390-1394.

[13] V.A. Onishchuk, D.I. Zaicev: *Locally nilpotent groups with a centralizer of finite rank*, Ukrain. Math. J. **43** (1991), 1018-1021.

[14] M.R. Pettet: *Automorphisms and Fitting factors of finite groups*, J. Algebra **72** (1981), 404–412.

[15] D.J.S. Robinson: *On groups with extreme centralizers and normalizers*, Adv. Group Theory Appl. **1** (2016), 97-112.

[16] A. Shalev: *Profinite groups with restricted centralizers*, Proc. Amer. Math. Soc. **132** (1994), 1279-1284.

[17] P. Shumyatsky: *Centralizers in locally finite groups*, Turk. J. Math. **31** (2007) Suppl., 149–170.

[18] V.P. Šunkov: *On periodic groups with an almost regular involution*, Algebra i Logic **11** (1972), 470–493.

[19] M.J. Tomkinson: FC-groups, Pitman, Boston (1984).