Note di Matematica Vol.IV, 57-64(1984)

# ON THE DERIVATIVES OF THE WEIERSTRASS FUNCTIONS WITH RESPECT TO THEIR INVARIANTS

E. FERRARI (*)

ABSTRACT. *On the basis of earlier results on the differentiation of elliptic functions with respect to the parameter, and by using the homogeneity property of the Weierstrass functions, a closed formula is obtained for the derivatives of the Weierstrass functions with respect to the invariants* $g_2, g_3$ *at fixed argument. The treatment includes also the limiting case* $\Delta = 0$.

In this paper I want to show that it is possible, by elementary methods, to obtain a set of simple, closed formulae for the derivatives of the Weierstrass functions $\wp(u|g_2,g_3)$, $\zeta(u|g_2,g_3)$ and $\sigma(u|g_2,g_3)$ with respect to the invariants $g_2, g_3$ at fixed u. The following proof of the above statement will be partly based on the results of the analysis carried out in a previous paper [1], where a closed formula for the derivative with respect to the parameter has been obtained for all elliptic functions depending on a single parameter, such as Jacobi functions, generali-

(*) Dipartimento di Fisica Università "La Sapienza" - Roma
I.N.F.N. - Sezione di Roma - (Italy)

zed trigonometric functions (GTF) (*), and also a subclass of Weierstrass functions (with a particular choice of the periods) that can be directly expressed in terms of GTF (cf.[2]). Let me recall the result obtained in the case of Jacobi functions: by denoting an arbitrary combination of such functions by G, the following formula has been proved to hold:

$$\frac{\partial G}{\partial k^2} = G_o - H(x,k^2)\frac{\partial G}{\partial x} \qquad (1)$$

where $G_0$ is another suitable combination of Jacobi functions (to be determined case by case) and $H(x,k^2)$ is a transcendental function defined as follows:

$$H(x,k^2) = \frac{1}{2}\int_o^x \frac{sn^2(x',k^2)}{dn^2(x',k^2)}\,dx' \qquad (2)$$

In order to apply the method used in [1] to the case of Weierstrass functions, one has first to consider the well-known homogeneity property of the Weierstrass function $\wp(u|g_2,g_3)$, which can be written in the following form

$$\wp(u|g_2,g_3) = t^2\,\wp(x|G_2,G_3) \qquad (3)$$

(*) Generalized trigonometric functions, introduced in an earlier work [2], provide a new representation of elliptic functions obtained through an algebraic-geometrical definition of unconventional type. As discussed in [1], this definition allows one to establish a general procedure of differentiation with respect to the parameter, which has been straightforwardly applied to elliptic functions.

where the argument and the invariants on the l.h.s. and on the r.h.s. of eq.(1) are linked by the following relations:

$$
\begin{aligned}
u &= x/t \\
g_2 &= t^4 G_2 \\
g_3 &= t^6 G_3
\end{aligned}
\tag{4}
$$

Relations similar to eq. (3) can be written also for the other basic Weierstrass functions, by changing the exponent of t on the r.h.s. (which becomes 3 for $\wp'(u)$, 1 for $\zeta(u)$, -1 for $\sigma(u)$, and so on).

As shown in [2], by a suitable choice of the invariants $G_2, G_3$ as functions of a single parameter $\lambda$, it is possible to express $\wp(x|G_2,G_3)$ in terms of GTF of argument x and parameter $\lambda$: the use of eq.(3) can then provide an expression in terms of the same GTF for all Weierstrass functions having the same $g_3^2/g_2^3$ ratio. If one restricts to real invariants for simplicity, it is possible to keep $\lambda$ in the interval between -1 and 1, and two different expressions (involving the same GTF) are obtained for the representation of the $\wp$'s with $\Delta < 0$ and the $\wp$'s with $\Delta > 0$, where the discriminant $\Delta$ is defined as follows (*):

(*) This is the definition given in [2]; many texts define $\Delta$ with a change of sign. The two $\wp$'s (with $\Delta < 0$ and $\Delta > 0$) corresponding to the same $\lambda$ are connected by a second-order transformation of the periods. The two functions match at $\lambda = \pm 1$, corresponding to degenerate Weierstrass functions (with $\Delta = 0$).

$$\Delta = g_3^2 - g_2^3/27 \tag{5}$$

Furthermore, an explicit expression for the derivative of $\wp(x|G_2(\lambda), G_3(\lambda))$ with respect to $\lambda$ at fixed x can be easily obtained on the basis of the procedure outlined in [1]. At this point, eqs. (4) can be considered as defining a change of variables, from u, $g_2, g_3$ to $x, \lambda, t$; since the derivatives of the r.h.s. of eq.(3) with respect to $x, \lambda, t$ can be calculated explicitly, it is clear that the derivative of any Weierstrass function with respect to $g_2$ or $g_3$ at fixed u can be obtained by elementary methods. Let me refer to the case $\Delta > 0$ for definiteness: since the final result will be expressed in terms of u, $g_2$ and $g_3$ only, it will have a general validity.

In order to help the reader to get to the final result, it will be useful to recall a few preliminary formulae from [1],[2], namely, the expression of $G_2$ and $G_3$ as functions of $\lambda$ :

$$G_2 = \frac{4}{3}(4\lambda^2-3);\; G_3 = \frac{8\lambda}{27}(9-8\lambda^2) \tag{6}$$

and the derivative of $\wp(x|G_2(\lambda), G_3(\lambda))$ with respect to $\lambda$ at fixed x (*):

(*) The second equality in eq. (7) has been obtained by expressing the transcendental function $\mathfrak{J}(x)$ introduced in [2] in terms of $\zeta(x|G_2,G_3)$: the formula has been given in a previous work (in Italian)[3]. Repeated use of eqs. (47) of [2] has been made in order to convert GTF into Weierstrass functions and to obtain the r.h.s. of eq. (7) in its present form.

$$\frac{\partial \wp(x)}{\partial \lambda} = \frac{2}{3} + \mathcal{J}(x|\lambda)\wp'(x) =$$

$$= \frac{2}{3} + \frac{1}{2(1-\lambda^2)} \left[\zeta(x)\wp'(x)+2(\wp(x) - \frac{2\lambda}{3})(\wp(x) + \frac{\lambda}{3}) - \frac{1}{3}\lambda\, x\, \wp'(x)\right] \tag{7}$$

By using eqs.(3) to (7) at different stages, one arrives at the following closed expressions for the derivatives of $\wp(u|g_2,g_3)$ with respect to the invariants:

$$\frac{\partial \wp(u)}{\partial g_2} = \frac{1}{18\,\Delta}\left[-g_2 g_3+3g_3(\zeta(u)\wp'(u)+2\wp^2(u)) - \frac{1}{6}\, g_2^2(u\,\wp'(u) + 2\,\wp(u))\right]$$

$$\frac{\partial \wp(u)}{\partial g_3} = \frac{1}{18\Delta}\left[\frac{2}{3}\, g_2^2-2g_2(\zeta(u)\wp'(u)+2\wp^2(u)) + 3g_3(u\wp'(u)+2\wp(u))\right] \tag{8}$$

By integrating eqs. (8) between 0 and u two times, and by taking account that, at the normalization point u=0, the derivatives of $\zeta(u)$ and $\log\sigma(u)$ with respect to $g_2, g_3$ must vanish (*),

(*) This is not in contrast with the fact that $\zeta(u)$ and $\log\,\sigma(u)$ are singular at u=0 (cf. a discussion on the argument in [1]).It should be pointed out that it is the explicit presence of u in the last term of eqs. (8) to (10) that cancels the singularity at the origin, but not at the other poles of the Weierstrass functions (as must be expected, since the positions of the poles other than the origin are parameter-dependent). However, nonsingular derivatives can occur also at those poles, when $g_2$ or $g_3$ is zero.

one obtains the following expressions for the derivatives of $\zeta(u)$ and $\log\sigma(u)$:

$$\frac{\partial\zeta(u)}{\partial g_2} = \frac{1}{18\Delta}\left[\frac{1}{4} g_2 g_3 u - 3g_3(\zeta(u)\wp(u)+\frac{1}{2}\wp'(u))+\frac{1}{6} g_2^2(u\wp(u)-\zeta(u))\right] \quad (9)$$

$$\frac{\partial\zeta(u)}{\partial g_3} = \frac{1}{18\Delta}\left[-\frac{1}{6} g_2^2 u+2g_2(\zeta(u)\wp(u)+\frac{1}{2}\wp'(u))-3g_3(u\wp(u)-\zeta(u))\right]$$

$$\frac{\partial\log\sigma(u)}{\partial g_2} = \frac{1}{18\Delta}\left[\frac{1}{8} g_2 g_3 u^2 + \frac{3}{2}g_3(\zeta^2(u)-\wp(u))-\frac{1}{6} g_2^2(u\zeta(u)-1)\right]$$

(10)

$$\frac{\partial\log\sigma(u)}{\partial g_3} = \frac{1}{18\Delta}\left[-\frac{1}{12} g_2^2 u^2-g_2(\zeta^2(u)-\wp(u)) + 3g_3(u\zeta(u) - 1)\right]$$

Eqs. (8) to (10), which provide the requested closed formulae for the derivatives of the Weierstrass functions with respect to the invariants, are already essentially contained in one of the Weierstrass papers [4], where, although not explicitly written, they can be easily deduced (*). However, Weierstrass arguments are rather involved, complicated and of difficult comprehension, whereas the present approach is simple and direct.

---

(*) If one takes eqs. (A) and (B) of the Weierstrass text together, they constitute a linear system in $\partial\sigma/\partial g_2$ and $\partial\sigma/\partial g_3$, which, for $\Delta \neq 0$, can be solved and yields eqs. (10).

Furthermore, it allows the evaluation of the limiting case $\Delta = 0$, which is excluded in the Weierstrass paper. Indeed, in the previous analysis it is sufficient to look at the $\lambda$ behaviour of all quantities in the vicinity of the values $\lambda = \pm 1$. For the function $\log \sigma(u)$ (the other cases being easily obtained by differentiation with respect to u) eqs. (10) in this limit reduce to:

$$\frac{\partial \log \sigma(u)}{\partial g_2} = \frac{1}{64t^4}\left[2(tu)^2 + 7\, tu \cot(tu) - \frac{1}{3}\cos^2(tu) - \frac{20}{3}\right]$$

$$\frac{\partial \log \sigma(u)}{\partial g_3} = \frac{1}{64t^6}\left[6(tu)^2 + 15\, tu \cot(tu) + \cos^2(tu) - 16\right] \qquad (11)$$

for

$$g_2 = \frac{4}{3}t^4,\ g_3 = \frac{8}{27}t^6,\ \wp(u) = t^2\left[\frac{2}{3} + \cot^2(tu)\right]$$

and

$$\frac{\partial \log \sigma(u)}{\partial g_2} = \frac{1}{64t^4}\left[-2(tu)^2 + 7tu\, \mathrm{cth}\,(u) - \frac{1}{3}\cosh^2(tu) - \frac{20}{3}\right]$$

$$(12)$$

$$\frac{\partial \log \sigma(u)}{\partial g_3} = \frac{1}{64t^6}\left[6(tu)^2 - 15\, tu\, \mathrm{cth}(tu) - \cosh^2(tu) + 16\right]$$

for

$$g_2 = \frac{4}{3}t^4,\ g_3 = -\frac{8}{27}t^6,\ \wp(u) = t^2\left[-\frac{2}{3} + \mathrm{cth}^2\,(tu)\right]$$

It can be seen that eqs. (11) go into eqs. (12) and viceversa if one replaces $t$ by $i \cdot t$.

The limit $t \to 0$ yields the ultra-degenerate case $g_2=g_3=0$, for which, however, the derivatives can be more easily obtained from the well-known power-series expansions around $u = 0$:

$$\left.\frac{\partial \wp(u)}{\partial g_2}\right|_{g_2=g_3=0} = \frac{u^2}{20} \; ; \; \left.\frac{\partial \wp(u)}{\partial g_3}\right|_{g_2=g_3=0} = \frac{u^4}{28} \tag{13}$$

I would like to thank Prof. L.Gatteschi for reading the manuscript and for his interest in this work.

## REFERENCES

[1] E.FERRARI, Boll. Un.Mat.Ital.2-A (1983), 215.

[2] E.FERRARI, Boll.Un.Mat.Ital.18-B(1981),933.

[3] E.FERRARI, Quaderni Ist.Mat.Univ.Lecce,Q.3 -1980.

[4] K.WEIERSTRASS, Math.Werke, Vol.2 (1895), 245.

Ricevuto il 7/11/1983