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SOME RESULTS ON TITS' GEOMETRIES OF TYPE F_4

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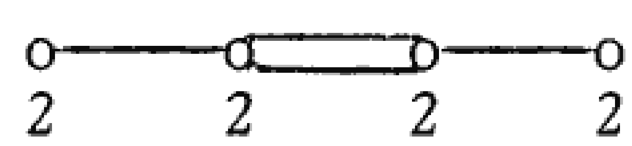
Summary. *It is known that all finite thick geometries of type C_n $(n\geq 4)$ with known parameters are buildings (see* [10] *and* [6]*). Several facts suggest the conjecture that the same holds in general. Moreover, a finite thick geometry of type F_4 with known parameters is a building unless its parameters are as below*

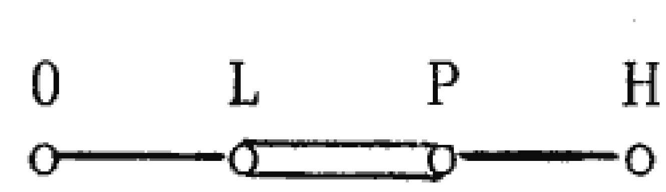


(see [6]*). It is sensible to conjecture that all finite thick geometries of type F_4 are buildings. I am not able to prove this conjecture. But I collect in this paper some partial results related to this problem. They improve other results given in* [9] *and* [6].

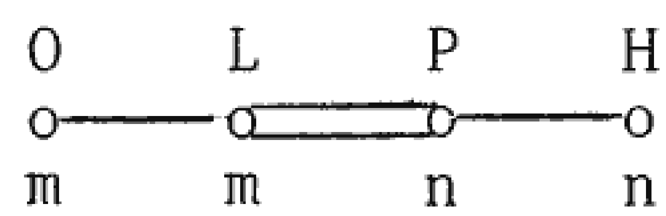
1. DEFINITIONS AND PRELIMINARY RESULTS.

All geometries considered in this paper are understood to be residually connected (that is, strongly connected, by [5], because we deal with geometries of finite rank).

In this section the symbol Γ always denotes a geometry belonging to the diagram



where the letters 0,L,P and H denote types. We say that the elements of type 0 are *points*, the elements of type L are *lines*, those of type P are *planes* and those of type H are *hyperlines*. We denote the incidence relation of Γ by the symbol $*$. If Γ admits parameters m,n where $m \geq n$, we always adopt the convention to mark types as below



Given an element x of Γ and a type i=O,L,P or H, the set of elements of type i incident with x is the i-*shadow* of x and it is denoted by $\sigma_i(x)$ (see [3]). Given two distinct points a,b of Γ ,we say that they are *collinear* and we write $a \perp b$ if there is some line incident with both them. Given a point a of Γ , the set of all points collinear with a is denoted by $a^{\perp}$ and, given a set X of points of Γ , the symbol $X^{\perp}$ denotes the set of points collinear with all of the points in X. That is, $X^{\perp} = \bigcap_{x \in X} x^{\perp}$. The *collinearity graph* is the graph defined by the collinearity relation on the set of points of Γ . Given two points a and b, the symbol $d_{\perp}(a,b)$ denotes the distance from a to b in the collinearity graph. Given two distinct hyperlines u and v, we say that they are *cocollinear* and we write $u \top v$ if there is some plane incident with both them. The definitions of the symbols $u^{\top}, X^{\top}$, of the cocollonearity graph and of the sympbol $d_{\top}(u,v)$ are similar to those given for $a^{\perp}, X^{\perp}$, for the collinearity graph and for $d_{\perp}(a,b)$.

In several statements the words "point" and "hyperline", the words "collinear" and "cocollinear" and the words "line" and "plane" can be permuted without any loss of sense. When those permutations are allowed in a given statement, the statement got by them is said to be the *dual* of the previous one.

If the geometry Γ is a building, then the following assertions hold:

(LL) (see [17], Sect.6). Given two collinear points a and b, there is exactly one line incident with both them.

(LH) (see [17], Sect.6). Let x be a line and let u be a hyperline. If $\sigma_0(x) \cap \sigma_0(u)$ contains more than one point, then we have $x * u$.

(HH) (see [17], Sect.6). Let a,b be distinct points. If there are two distinct hyperlines incident with both them, then we have $a \perp b$.

Tits considers in [17] another property besides LL,LH and HH, namely the property 0 of Sect.6 of [17]. That property turns out to be rather weak. On the other hand, for geometries of type F_4, it is a consequence of LL. Thus, we shall not make any use of it.

The reader can see [17] for the definition of *covering*, 2- *covering, simple connectedness* and *2-connectedness*.

The following results are well known:

PROPOSITION A (Tits [17]). *The universal 2-cover of Γ is a building if all residues of Γ of type C_3 are 2-covered by buildings.* (This proposition is a specialization of Theorem 1 of [17]).

PROPOSITION B (Brouwer and Cohen [2]). *Let Γ be finite and thick. Then Γ is a building if it is 2-covered by a building.*
(This proposition is a specialization of Proposition 9 of [2]).

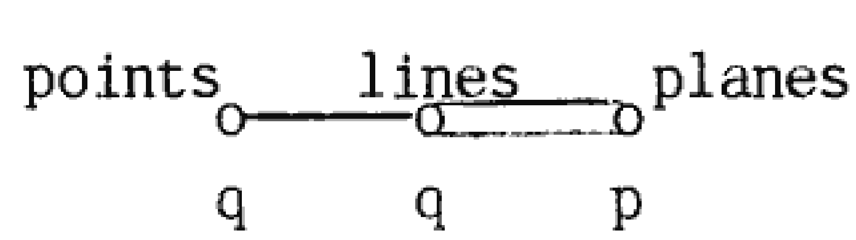
PROPOSITION C (Tits [17]).*The geometry* Γ *is a building if the properties* LL, LH *and* HH *hold in it.*
(This proposition is a specialization of Proposition 9 of [17]).

PROPOSITION D (Aschbacher [1]).*Let* Γ *be finite and thick and let us assume that all residues of* Γ *of type* A_2 *are desarguesian projective planes and all residues of* Γ *of type* C_2 *are classical generalized quadrangles. Then* Γ *is a building if it has a flag-transitive automorphism group.*
(This proposition is a specialization of Theorem 3 of [1]).

PROPOSITION E (Tits [17]).*The geometry* Γ *is 2-connected if it is a building.*
(This proposition is a specialization of Theorem 1 of [17]).

2. A CONJECTURE ON GEOMETRIES OF TYPE C_3.

In this section, except in Proposition G.1, G.2 and in their Corollary, the symbol Γ always denotes a geometry belonging to the diagram:



where q and p are positive integers and denote orders. The geometry is finite, by Corollary 2 of [9].

The following result is proved in [11].

LEMMA 1. *Given two distinct points of* Γ , *there are at most* pq+1 *lines incident with both them. Moreover, the following* *conditions are equivalent:*

(i) *Given any two collinear points, there are exactly* qp+1 *lines incident with both them.*

(ii) *Every point is incident with all planes.*

(see [11], Lemma 5.9 and 5.6).

If the (equivalent) conditions (i) and (ii) of Lemma 2 hold in Γ , then Γ is said to be *flat*. Trivially, if Γ is flat, then it has exactly (qp+1)(p+1) planes, $(qp+1)(q^2+q+1)$ lines and q^2+q+1 points and the collinearity graph of Γ is complete. Moreover:

LEMMA 2. *Let* Γ *be flat. Then* $q \leq p$, , *and we have* $q = p$ *if and only if the incidence relation of* Γ *induces on the set of planes and lines of* Γ *a geometry belonging to the diagram*

planes o—L—o lines

(see [11], Lemma 5.10).

Most of finite geometries of type C_3 admitting parameters are either buildings or flat. Indeed:

PROPOSITION F. (Ott [8], Rees & Scharlau [15], Rees [13], Leibler [6]).

Let $(q,p) = (q,q), (t^2,t^3)$ *(where* $t > 1$) *or* $(t-1,t+1)$ *(where* $t \geq 3$).
Then Γ *is either a building or flat.*

Let $(q,p) = (t,t^2), (t^2,t)$, $(t+1,t-1)$ *or* $(t,1)$ *(where* $t > 1$).
Then Γ *is a building.*

Moreover, there is not any geometry of type C_3 *with parameters*

$(q,p) = (t^3,t^2)$ $(t > 1)$.

The result on $(q,p) = (q,q)$ has been proved by Ott. He did not consider the case of $q=1$. At any rate the statement is trivial in this case.

The result by Ott has been generalized by Rees and Scharlau to all known 'thick' parameters except the cases of (t^2,t), $(t+1,t-1)$ and $(t-1,t+1)$. The case of $(t,1)$ is settled in [13]. In that paper Rees proves that Γ is covered by a building if it has parameters $(t,1)$. Then Γ must be a building by the result of [2] on C_n. All remaining cases have been settled by Liebler in [6].

We warn that the statement of Proposition F is false if $(q,p)=$ $= (1,t)$. A counterexample is implicitly given in [14]: set $r = s = t > 1$ and $n > 1$ in Example 2 of that paper.

The parameters listed above are all parameters for which we presently know examples of generalized quadrangles. So we shall call them *known* parameters. Then Proposition F can be restated as below:

PROPOSTION F. bis. *Let Γ have thick lines and known parameters. Then Γ is either a building or flat.*

The following lemma follows from Theorem 1 of [17] by the same argument used in the proof of lemma 10 of [9].

LEMMA 3. *Let the parameters q,p of Γ be such that every geometry with those parameters is either a building or flat. Then Γ is 2-connected.*

Then, by Proposition F. bis:

COROLLARY. *Let Γ have thick lines and known parameters. Then Γ is 2-connected.*

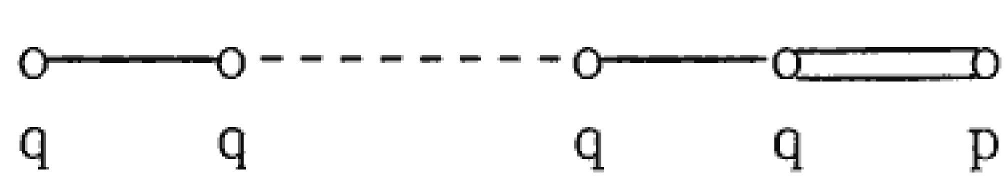
Proposition F. bis and the classification of finite geometries with thin lines given in [4] show that there is just one family of finite geometries with known parameters that are neither buildings nor flat, namely the class of geometries got by setting $r = s = t > 1$ and $n \geq 2$ in Example 2 of [14]. This fact suggests the following conjecture:

CONJECTURE. *If $q \neq 1$ then the following statement holds:*

(*) *The geometry Γ is either a building or flat.*

Let us mention two propositions related to this conjecture.

PROPOSITION G.1 (Pasini [10] and Liebler [6]). *Let Γ be a finite geometry belonging to the diagram*

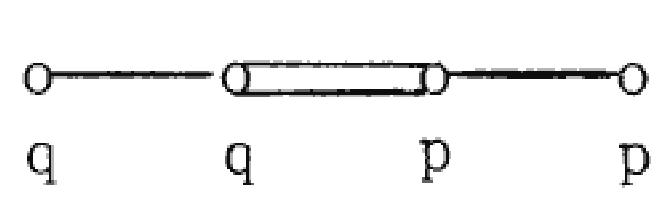


(n nodes , $n \geq 4$; $q \neq 1$)

Let us assume that the statement (*) holds in all residues of Γ of type C_3. Then Γ is a building.

Then, if the previous conjecture were true, all finite geometries of type C_n ($n \geq 4$) with thick lines and admitting parameters would be buildings.

PROPOSITION G.2 (Liebler [6]). *Let Γ be a finite geometry belonging to the diagram*



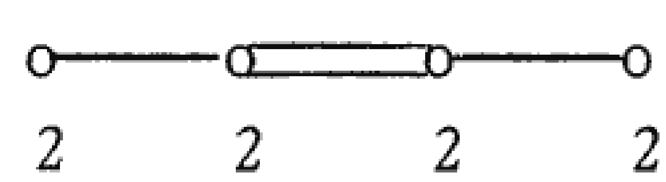
($q \geq p \neq 1$)

Let us assume that the statement (*) holds in all residues of Γ *of type* C_3*. Then either* Γ *is a building or* $qp^2 + 1$ *divides* $\prod_{i=1}^{4} (p^i - 1)$.

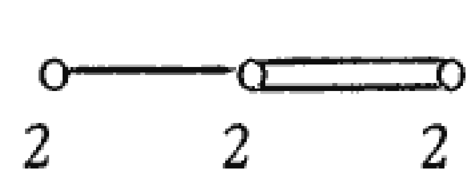
By Proposition G.1, G.2 and F.bis we have

COROLLARY. *Let* Γ *be a finite geometry of type* C_n $(n \geq 4)$ *with thick lines and known parameters. Then* Γ *is a building.*

Let Γ *be a finite thick geometry of type* F_4 *with known parameters. Then either* Γ *is a building or its parameters are as below*



Only one example is presently known of a finite thick non-building geometry belonging to a diagram of Lie type. First constructed by Neumanier in [7], it has many different descriptions. Here is the easiest one. Let us take a set S of 7 objects as set of points, all 3-subsets of S as lines and let π be a projective plane of order 2 over S. Let us take the orbit of π under the action of the alternating group A_7 on S as set of planes and define incidences by set-theoretic inclusion. We get a flat geometry in the diagram



This geometry is often known as the A_7-*geometry*. The reader can see [1],[11] and [12] for further characterizations of this geometry. The following result is due to Rees:

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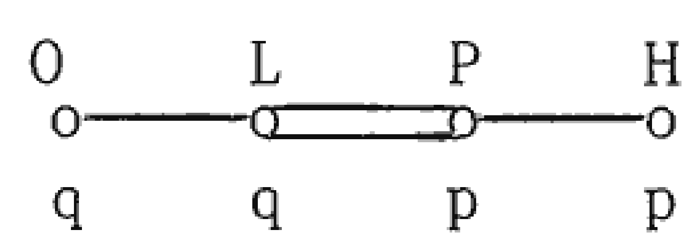
They form two conjugacy classes $\mathscr{C}$ and $\mathscr{C}^*$ in A_7. Each of these two conjugacy classes has size 15. Every outer automorphism of A_7 induced by inner automorphisms of S_7 exchange $\mathscr{C}$ with $\mathscr{C}^*$.

The A_7-geometry can be described as follows:

The subgroups of the first kind are points and those of the third kind are lines. The planes are the subgroups belonging to one of the two classes $\mathscr{C}$ and $\mathscr{C}^*$ of subgroups of the fourth kind. If we choose the class $\mathscr{C}$, then the twin geometry is got by choosing $\mathscr{C}^*$. Let, H,K be subgroups of A_7 taken as above. They are incident in the geometry if $H \cap K \neq 1$.

3. AN IMPROVEMENT OF A THEOREM BY LIEBLER.

In this section Γ is a geometry belonging to the diagram



where q,p are integers such that $q \geq p > 1$ and denote orders.

Moreover, we assume that the property (*) (see Conjecture of §2) holds in all residues of Γ of type C_3.

We use the following notations:

$F_0 = \{ a | a$ is a point and the residue Γ_a of a is flat $\}$

$B_0 = \{ b | b$ is a point and Γ_b is a building $\}$

$F_H = \{ u | u$ is a byperline and Γ_u is flat $\}$

$B_H = \{ v | v$ is a byperline and Γ_v is a building $\}$

By (*) the set $F_0 \cup B_0$ is the set of all points of Γ and the set $F_H \cup B_H$ is the set of all hyperlines of Γ .

Moreover we have $F_H = \emptyset$ if $q > p$, by Lemma 2.

LEMMA 4. *Let* a,b *be points of* Γ *such that there is not any hyperline of* Γ *incident with both them. Then there are hyperlines* u,v *and a point* c *such that* $a * u * c * v * b$. *And dually.*
(See Lemma 4 of [9]).

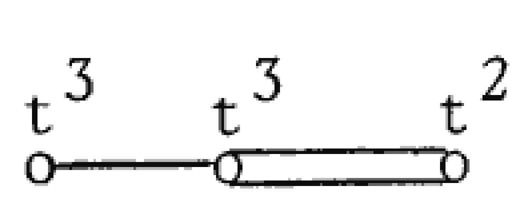
COROLLARY. *The geometry* Γ *is finite.*
(trivial, by the previous lemma and by Corollary 2 of [9]).

LEMMA 5. *Let* $a \in F_0$. *Then* $\sigma_H(a) \subseteq B_H$.

The statement is trivial if $q > p$ because in this case we have $F_H = \emptyset$. The case of q=p has been settled in [9], Lemma 12.

Now we can state the following theorem that improves Proposition G.2.

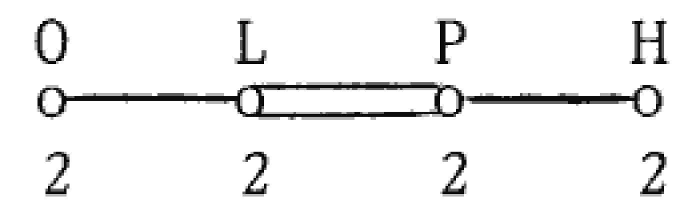
THEOREM 1. *Either* Γ *is a building or* $q = p = 2$.

The geometry Γ is finite by the Corollary of Lemma 4. By Lemma 5 at least one residue of Γ of type C_3 is a (finite thick) building. Then, by Theorem 4.11 of [18] we have either $q = p^2$ or $q = p$ (the case of $q = t^3$ and $p = t^2$ is excluded because, by Proposition F, there is not any geometry of type C_3 with parameters ). Then either Γ is a building or q=p=2, by the Corollary of Proposition G.2.

Q.E.D.

4. THE CASE OF $q = p = 2$.

In this section Γ is a geometry belonging to the diagram



Then every residue of Γ of type C_3 is eithor a building or flat, by Proposition F. The symbols F_0, B_0, F_H and B_H have the meaning stated in §3.

LEMMA 6. *The geometry* Γ *is a building iff* $F_0 = F_H = \emptyset$.

The "only if" part is trivial. The "if" part is an easy corollary of Proposition A and B.

LEMMA 7. *Let* a,b *be collinear points and let* $a \in F_0$. *Then there is exactly one line incident with both* a *and* b (see Lemma 11 of [9]).

LEMMA 8. *We have* $B_0 \neq \emptyset \neq B_H$.

This lemma can be proved by the same argument used in the proof of Theorem 2 of [9]. The reader is referred to that paper for all details.

LEMMA 9. *Let* a *be a point in* F_0 *and let* b,c *be distinct points in* $a^{\perp}$. *Let us assume that there is not any line incident with all of* a,b, *and* c. *Then there is just one plane incident with all of* a,b, *and* c.

Indeed let x,y be the lines through a and b and through a and c, respectively (the lines x,y are uniquely determined by Lemma 7). We have $x \neq y$ by our assumptions on a,b and c. There is just one plane incident with both x and y by Lemma 2. Moreover, every plane incident with all of a,b,c must be incident with x and y, by Lemma 7. We are done. Q.E.D.

COROLLARY. *Let* a *be a point in* F_0. *Then* $a^{\perp} \cup \{a\}$ *is a maximal clique in the collinearity graph of* Γ.

(trivial, by Lemma 9).

LEMMA 10. *The collinearity graph of* Γ *induces a discrete graph on the set* F_0.

(see Lemma 14 of [9]).

LEMMA 11. *Let* a *be a point in* F_0 *and let* b *be a point distinct from* a *such that* $\sigma_H(a) \cap \sigma_H(b)$ *contains at least two hyperlines. Let us assume that, for every point* c *collinear with* b, *there is just one line through* b *and* c. *Then we have* $b \in B_0$ *and* $a \perp b$.

Let u,u' be distinct hyperlines incident with both a and b. In Γ_a we find a plane w incident with both u and u', because $a \in F_0$. If w*b, then $a \perp b$ and $b \in B_0$ by Lemma 10. Let us assume that $a \not\perp b$, by contradiction. In Γ_u we find a line x and a plane v such that w*x*v*b (recall that Γ_u is a building by Lemma 5). Similarly, in $\Gamma_{u'}$ we find a line x' and a plane v' such that w*x'*v'*b. In Γ_w we find a point c incident with both x and x'. We have $c \in B_0$ by Lemma 10, because $c \perp a \in F_0$. In Γ_v and in $\Gamma_{v'}$ we find lines y,y' through c and b. We have y = y' by our hypothesis on b. Then we get y*w in the building Γ_c. Then b*w. We are done. Q.E.D.

COROLLARY. *Let* a,b *be distinct points in* F_0 . *Then there is at most one byperline incident with both them.*
(trivial, by Lemmas 7 and 11).

LEMMA 12. *Let us assume that the Property* LL *holds in* Γ . *Let* a,b *be distinct points in* F_0 . *Then* $\sigma_H(a) \cap \sigma_H(b) = \emptyset$.
Indeed let us assume that $\sigma_H(a) \cap \sigma_H(b) \neq \emptyset$, by contradiction, and let $u \in \sigma_H(a) \cap \sigma_H(b)$. Then $u \in B_H$ by Lemma 5. Then there are distinct points c,d in Γ_u such that both c and d are collinear with both a and b in Γ_u but c and d are not collinear in Γ_u. Let x_a, y_a, x_b, y_b be the lines in Γ_u incident with a and c, with a and d, with b and c and with b and d, respectively. Of course, we have $x_a \neq y_a$ and $x_b \neq y_b$. By Lemma 2 there is a plane w_a in Γ_a incident with both x_a and y_a and there is a plane w_b in Γ_b incident with both x_b and y_b. In Γ_{w_a} there is a line z_a indicent with both c and d. In Γ_{w_b} there is a line z_b incident with both c and d. We have $z_a = z_b$ by the property LL. Let us set $z = z_a$ (= z_b). In Γ_z there is a hyperline v incident with both w_a and w_b. In Γ_v the points c and d are collinear. Then $u \neq v$ because c and d are not collinear in Γ_u. We have a contradiction by the Corollary of Lemma 11.

Q.E.D.

The following two theorems deal with the properties LL,HH and LH.

THEOREM 2. *The following conditions are equivalent:*

(i) *The qeometry* Γ *is a building.*

(ii) *The Property* LH *holds in* Γ

(iii) *The Property* LL *and its dual hold in* Γ

The equivalence of (i) and (ii) has already been proved in [9] (see Theorem 4 of [9]). The equivalence of (i) and (iii) easily follows from Lemma 6.

THEOREM 3. *The geometry* Γ *is 2-connected if either the Property* LL *or the Property* HH *together with its dual hold in* Γ .

We have already proved in [9] (Theorem 6 of [9]) that Γ is 2-connected if LL holds in it.

Let us assume that the Property HH and its dual hold in Γ. If $F_0 \cup F_H = \emptyset$ then Γ is a building by Lemma 6. Then Γ is 2-connected by Proposition E. Let us assume that $F_0 \cup F_H \neq \emptyset$ Let $f : \bar{\Gamma} \to \Gamma$ be the universal 2-covering of Γ . Then f is a 3-covering by the Corollary of Lemma 3 and by Proposition F.

Let us set

$\bar{F}_0 = \{a \mid a$ is a point of $\bar{\Gamma}$ and $\bar{\Gamma}_a$ is flat$\}$.

$\bar{B}_0 = \{b \mid b$ is a point of $\bar{\Gamma}$ and $\bar{\Gamma}_b$ is a building$\}$.

We have $f(\bar{F}_0) = F_0$ and $f(\bar{B}_0) = B_0$ because f is a 3-covering. By Proposition 5.3 of [14] there is a subgroup A of Aut(Γ) acting regularly on each of its orbits in the chamber system and such that, for every chamber C of $\bar{\Gamma}$, the orbit A(C) of C under the action of A is the fibre $f^{-1}(f(C))$ of f over C. The stabilizer in A of an element of $\bar{\Gamma}$ is the trivial subgroup of A because f is a 3-covering and A is regular on each of its orbits. Let us prove that A stabilizes every element of $\bar{F}_0 \cup \bar{F}_H$. Then we shall have A=1 so that f is an isomorphism and the 2-connectedness of Γ will be proved.

Let us assume $F_0 \neq \emptyset$. Let $a \in \bar{F}_0$ and let us assume, by contradiction, that there is some $g \in A$ such that $g(a) \neq a$. There is not any hyperline incident with both a and g(a) because f is a 3-covering. By Lemma 1, there is a point b of $\bar{\Gamma}$ and there are hyperlines u,v of $\bar{\Gamma}$ such that $a * u * b * v * g(a)$. Of course, we have $u \neq v$ and $a \neq b \neq g(a)$. Then $f(b) \neq f(a)$ (= f(g(a)) and $f(u) \neq f(v)$ because f is a 3-covering. By the Property HH there is a line x of Γ incident with both f(a) and f(b). We have $x * f(u)$ and $x * f(v)$ because $f(a) \in F_0$. The line x lifts to a unique line $\bar{x} * v$ by the same reason. Then $a * \bar{x}$ in $\bar{\Gamma}_u$ and $g(a) * \bar{x}$ in $\bar{\Gamma}_v$ because f is a 3-covering. Then a = g(a) by the same reason. We have the contradiction. Then A fixes a. Then A = 1 and we are done. If we assume $F_H \neq \emptyset$ then we can use the argument dual of the previous one. Q.E.D.

In the forthcoming the symbol Aut(Γ) will denote the group of special automorphisms of Γ.

THEOREM 4. *The geometry* Γ *is a building if one of the following conditions holds:*

(i) *The group* Aut(Γ) *is transitive on the set of lines of* Γ .

(ii) *The group* Aut(Γ) *is transitive on the set of points of* Γ *and either the Property* LL *or its dual holds in* Γ .

(iii) *The group* Aut(Γ) *is transitive both on the set of points and on the set of hyperlines of* Γ .

Let us assume that Aut(Γ) is transitive on the set of lines of Γ.

Then $F_0 = \emptyset$. Indeed let $F_0 \neq \emptyset$, by contradiction. Then every line is incident with some point in F_0 because Aut(Γ) is transitive on the set of lines of Γ . But every plane is indicent with at most one point in F_0, by lemma 10. We have the contradiction. Then $F_0 = \emptyset$. let us assume that $F_H \neq \emptyset$. Every line is incident with exactly one huperline in F_H by Lemma 10 and because Aut(Γ) is transitive on the set of lines. Given a point a let N_a be the number of hyperlines in $\sigma_H(a) \cap F_H$. Then $N_a \cdot 15$ is the number of lines incident with a. Then $N_a=9$. Let P be the number of points and let N_F be the number of hyperlines in F_H. We have $9 \cdot P = 7 \cdot F_H$. Then there is a positive integer α such that $P=7\alpha$ and $N_F=9\alpha$. Let N_B be the number of hyperlines in B_H. If we count the number of incident point-hyperline pairs we get the equality $N_F+9 \cdot N_B=9 \cdot P$. Then $N_B=6\alpha$. Let us count the number of pairs (u,v) where $u \in F_H$, $v \in B_H$ and $u \top v$. By Lemmas 7 and 10 we have $N_F \cdot 30 \leq \sum_{v \in B_H} X_v$ where X_v is the number of hyperlines in F_H that are cocollinear with v. We easily get that $X_v = 45$ for every $v \in B_H$. Indeed $(v^{\top}) \cap F_H$ is an anticlique in the cocollinearity graph of Γ , by Lemma 10. For every $u \in (v^{\top}) \cap F_H$ let w_u be the plane through u and v (the plane w_u is uniquely determined by Lemma 7). Let us set $W_v = \{ w_u | u \in (v^{\top}) \cap F_H \}$. X_v is the number of planes in W_v, by Lemmas 10 and 7. But given any two planes w,w' in W_v there is not any line incident with both them, by Lemma 10. Then W_v contains at most 45 planes. Then $X_v = 45$ for every $v \in B_H$ because we have $N_F \cdot 30 \leq \sum_{v \in B_H} X_v$.

Given an element x of Γ we denote the stabilizer of x in Aut(Γ) by the symbol A_x. Given a line x let u be the hyperline in F_H incident with x. Of course A_x is a subgroup of A_u. Let L be the number of lines in Γ. We have $L = 35 \cdot N_F$. Then $L = 315\alpha$. The group A_x has index 315α in Aut(Γ) because Aut(Γ) is transitive on the set of lines of Γ. But Aut(Γ) is also transitive on F_H and F_H contains exactly 9α hyperlines. Then A_u has index 9α in Aut(Γ). Then A_x has index 35 in A_u. By Proposition H, the residue Γ_u of u is the A_7-geometry. Let G_u be the element-wise stabilizer of Γ_u. Then the group $\bar{A}_u = A_u/G_u$ is a subgroup of the alternating group A_7 in its action on the A_7-geometry. Moreover G_u is a normal subgroup of A_x, too. And the group $\bar{A}_x = A_x/G_u$ has index 35 in $\bar{A}_u$. But it is known that the alternating group A_7 has just one subgroup whose order is divisible by 35, namely A_7 itself (see page A.25 of [4]). Then $\bar{A}_u = A_7$. So A_u is transitive on the set of planes incident with u. Let y be such a plane. Then A_y is a subgroup of A_u (by Lemma 10). The group $\bar{A}_y = A_y/G_u$ is the stabilizer in A_7 of a plane of the A_7-geometry. Then $\bar{A}_y = L_3(2)$ and $\bar{A}_y$ has index 15 in $\bar{A}_u$ ($=A_7$). Then A_y has index 135α in Aut(Γ) because A_u has index 9α in Aut(Γ). Let v,v' be the two hyperlines in B_H incident with y. We have two possibilities.

Case 1. The group A_y permutes v and v'. Then Aut(Γ) is transitive on B_H because it is transitive on F_H and, for every $u' \in F_H$, the group A_u is transitive on the set of planes incident with u, and every $v \in B_H$ is cocollinear with some $u' \in F_H$. Trivially, the stabilizer $A_{y,v}$ of y and v has index 2 in A_y. But we have seen that $\bar{A}_y = L_3(2)$ and $L_3(2)$ is simple. Then $A_{y,v}$ acts as A_y on the residue of {y,u}. Then there is an element g of A_y that permutes v and v' and such that g acts as the identity mapping on the residue of $\{y,u\}$. Then g fixes every point in Γ_u, because Γ_u is flat. Then g fixes everything in Γ_u because Γ_u is the A_7-geometry. Let us take a point a in Γ_u. Let us consider the action of g in the building Γ_a. The building Γ_a is the polar space of the symplectic form

$$\phi = x_1y_2+x_2y_1+x_2y_3+x_3y_2+x_3y_4+x_4y_3+x_5y_5+x_5y_4+x_5y_4+x_5y_6+x_6y_5$$

over the 6-dimensional vector space K^6 over the Galois field $K = GF(2)$.

By the information that we have collected about g we can prove by long but easy computations that g acts in K^6 as a matrix of form

$$\begin{pmatrix} 1 & r & 0 & s & 0 & s \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & r+s & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (r,s = 0 \text{ or } 1)$$

if u corresponds to the vector $(1,0,0,0,0,0)$ in K^6. We know that g permutes v and v'. Then s=1, otherwise h should fix any hyperline cocollinear with u in Γ_a. Given g hyperline w in Γ_a, let $(x_1,x_2,\dots,x_6)$ be its coordinates in K^6. We have u $\mathcal{T}$ w in Γ_a iff $x_2=1$. We know that Γ_a contains 8 hyperlines in F_H different from u and they are not cocollinear with u or with each other. Moreover g fixes $F_H \cap (\sigma_H(a)-\{u\})$. Let $w \in F_H \cap (\sigma_H(a)-\{u\})$. Then $g(w) \neq w$.

Indeed, if otherwise, w is cocollinear with both v and v' in Γ_a, because it is cocollinear with some hyperline incident with y but it is not cocollinear with u, and g permutes v and v'. Then w is cocollinear with u because it is cocollinear with both v and v' and u is incident with the plane y that is incident with both v and v'. We have the contradiction. Then $g(w) \neq w$. Then, by the matrix representation of g, we get that, if $(x_1,x_2,\ldots,x_6)$ are the coordinates of w in K^6, we have ($x_2=1$ because $w \not\mathcal{T} u$ and) r=0 or r=1 and $x_4=x_6$. But we have $w \,\mathcal{T}\, g(w)$ because $g(w) \in F_H$. Then, by the matrix respresentation of g we get $x_6 + x_4 r = 0$.

But $F_H \cap (\sigma_H(a) - \{u\})$ contains 8 hyperlines. Then there is another hyperline w' in that set such that $w \neq w' \neq g(w)$. Let $(y_1,y_2,\ldots,y_6)$ be the coordinates of w' in K^6. We have $w \not\mathcal{T} w' \not\mathcal{T} g(w)$. By this conditions, by easy computations, we easily get that the equality $(x_4 + y_4)(1+r) = 1$ holds. Then $r = 0$. Then we get $x_6 = 0$ by the relation $x_6 + x_4 r = 0$ proved above. The stabilizer $A_{a,u}$ of a and u is transitive on the set of planes incident with u because $\bar{A}_u = A_7$. Then acting by elements of $A_{a,u}$ we can interchange the roles of the vectors (0,0,1,0,0,0), (0,0,0,1,0,0),(0,0,0,1,0) and (0,0,0,0,0,1) of K^6. But $A_{a,u}$ fixes the set $F_H \cap (\sigma_H(a) - \{u\})$. Then the equation $x_6 = 0$ leads to other 3 linearly independent equations. All these four equations must hold on the set of vectors representing hyperlines in $F_H \cap (\sigma_H(a) - \{u\})$.

Then this set cannot contain 8 hyperlines. Indeed we have collected in this may 5 linearly independent equations: $x_2 = 1$, $x_6 = 0$ and the further three equations got by $x_6 = 0$.

We have the contradiction.

Case 2. The group A_y fixes both v and v . Then $A_y \leq A_v$. Let m be the index of A_y in A_v. Every hyperline in B_H is cocollinear with 45 hyperlines in F_H. We have proved that Aut(Γ) is transitive on the set of planes incident with hyperlines in F_H. Then Aut(Γ) has either one or two orbits on B_H. If Aut(Γ) has just one orbit on B_H then 6 m is the index of A_y in Aut(Γ). Then $6 \cdot m \cdot \alpha = 135\alpha$. We have a contradiction. Then Aut(Γ) has two orbits on B_H. If X is the size of one of them we get $135\alpha = mX\alpha$. Then the two orbits B_1, B_2 of Aut(Γ) on B_H have the same size. Then $X = 3\alpha$ because B_H contains 6α hyperlines. Then $m = 45$ and A_v is transitive on $F_H \cap (v^T)$. Let us assume that v is incident with x. The stabilizer $A_{x,v}$ of x and v is the stabilizer of x and y. Then it has index 105 in A_u.

So $A_{x,y}$, has index 945α in Aut(Γ). Then it has index 315 in A_v. Then A_v is transitive on the set of lines of Γ_v. The group $A_{x,y}$ either fixes all planes incident with x and v or permutes two of them and fixes the third one. Let W_F be the set of planes of Γ that are not incident with any hyperline in F_H. The set W_F is an orbit of Aut(Γ) and Aut(Γ) has at most four orbits on W_B. Every line is incident with exactly 3 hyperlines in each of the orbits B_1,B_2 of Aut(Γ) on B_H and every plane in W_F is incident with exactly one hyperline in each of B_1 and B_2. Every plane in W_B is incident with at least two hyperlines in the same set B_i. Then Aut(Γ) has at least two orbits on W_B. Let us assume that Aut(Γ) has more than two orbits $W_1,W_2,W_3,\ldots$ on W_B. Let us assume, by contradiction, that for every i=1,2,3 there is a plane y_i in W_i such that $\sigma_H(y_i)\cap B_1\neq\emptyset$. Then there is some $v\in B_1$ such that v is incident with some plane in each of the sets W_1,W_2 and W_3. But this is not possible because we have already proved that A_v has at most two orbits on the set of planes $\sigma_P(v)\cap W_B$. We have the contradiction. Then we can assume that $\sigma_H(z)\cap B_1=\emptyset$ for every $z\in W_3$. Then $\sigma_H(z)\subseteq B_2$ for every $z\in W_3$. Then if x is a line there is just one plane in $\sigma_P(x)-(W_1\cup W_2)$. Then Aut($\Gamma$) has just three orbits on W_B because it is transitive on the set of lines of Γ. Moreover x must be incident with just one plane in W_1 (or in W_2) and with exactly two planes in W_2 (or in W_1, respectively). Then W_1 and W_2 have different sizes: one of them has size double of the other one. But the stabilizer of a pair (x,v), where $v\in B_1$ and $x\in\sigma_L(v)$, fixes all planes incident with both x and v. Then W_1 and W_2 have the same size. We have a contradiction. Then Aut(Γ) has just two orbits W_1,W_2 on W_B. We can assume that W_1 is the set of planes incident with at least two hyperlines in B_1 and W_2 is the set of planes incident with at least 2 hyperlines in B_2. For every line x we have 4 planes in $\sigma_P(x) \cap W_B$ and $\sigma_P(x) \cap W_1 \neq \emptyset \neq \sigma_P(x) \cap W_2$. We have two essentially distinct possibilities:

1) Both the sets $\sigma_P(x) \cap W_1$ and $\sigma_P(x) \cap W_2$ contain 2 planes. Then $\sigma_H(z) \cap B_i \neq \emptyset$ for every $z \in \sigma_P(x) \cap (W_1 \cup W_2)$ and for every i=1,2. A straightforward verification shows that it is not possible to select one point u in a projective plane of order 2 and to give wedges 1 or 2 to the other 6 points so that every line through u has one point of wedge 1 and one point of wedge 2 and two of the other four lines have two points of wedge 1 and one point of wedge 2 and the remaining two lines have two points of wedge 2 and one point of wedge 1.

Then this case is not possible.

2) The set $\sigma_P(x) \cap W_1$ contains just one plane z and the set $\sigma_P(x) \cap W_2$ contains three planes z_1, z_2, z_3. A straightforward verification shows that the only possibility is the following one: $\sigma_H(z) \subseteq B_1$ and $\sigma_H(z_i) \cap B_2$ contains two hyperlines and $\sigma_H(z_i) \cap B_1$ contains one hyperline, for every i=1,2,3.

Let u be the hyperline in $\sigma_H(x) \cap F_H$. Let v be a hyperline in $\sigma_H(x) \cap B_2$. The stabilizer $A_{x,v}$ of x and v coincides with the stabilizer in A_u of the pair (x,y) where y is the plane incident with v and u. By the action of A_7 on the A_7-geometry we get that $A_{x,v}$ is transitive on $\sigma_0(x)$ and permutes the two planes in $\sigma_P(x) \cap W_2$ whereas it fixes y. Let G_v be the elementwise stabilizer of Γ_v and let us set $\bar{A}_v = A_v/G_v$. The group $\bar{A}_v$ acts on Γ_v as a subgroup of the symplectic group $S_6(2)$ in its natural action on the building Γ_v and it is transitive on the set of incident point-line pairs of Γ_v. Then its order divides 2^9 3. Moreover the stabilizer $\bar{A}_{y,v}$ of y in $\bar{A}_v$ acts flag-transitively on the plane y. Then $\bar{A}_{y,v} = G \cdot L_3(2)$ or $G \cdot Frob(21)$ where G is a 2-group of order 2^m where $m \leq 6$. Moreover the orbit of y under the action of $\bar{A}_v$ has size 45. There are 135 planes in Γ_v. Then $\bar{A}_v$ has index $2^{6-m} \cdot 3$ or $2^{9-m} \cdot 3$ in $S_6(2)$ according to whether $\bar{A}_{y,v} = G \cdot L_3(2)$ or $G \cdot Frob(21)$. Let a be a point incident with y. Let $\bar{A}_{a,v}$ be the stabilizer of a in $\bar{A}_v$ and let $\bar{A}_{a,y,v}$ be the stabilizer of the pair (a,y) in $\bar{A}_v$. The group $\bar{A}_{y,v}$ is flag transitive on the plane y. Then $\bar{A}_{a,y,v}$ has index 7 in $\bar{A}_{y,v}$. Moreover $\bar{A}_{a,v}$ has index 63 in $\bar{A}_v$ because $\bar{A}_v$ is transitive on the set of points of Γ_v. Then $\bar{A}_{a,y,v}$ has index 5 in $\bar{A}_{a,v}$. That is, the orbit of y under the action of $\bar{A}_{a,v}$ has size 5. The group $\bar{A}_{a,v}$ has order $2^h \cdot 15$ where $h = m+3$ or m according to whether $\bar{A}_v$ has index $2^{6-m} \cdot 3$ or $2^{9-m} \cdot 3$ in $S_6(2)$. Then $\bar{A}_{a,v}$ has index $2^{9-h} \cdot 3$ in the stabilizer $K \cdot S_4(2)$ of a in $S_6(2)$ (K is a group of order 2^5). Let A^* be the action of $\bar{A}_{a,v}$ on the residue of the flag (a,v). The group A^* has index $2^k \cdot 3$ in $S_4(2)$ for some nonnegative integer $k \leq 4$. Moreover A^* has an orbit of size 5 on the set of planes of the residue $\Gamma_{a,v}$ of the flag (a,v). The group $S_4(2)$ has one normal subgroup A_6 of index 2 in $S_4(2)$ and acting flag-transitively on $\Gamma_{a,v}$. Then $A^* \cap A_6$ has index $2^k \cdot 3$ or $2^{k-1} \cdot 3$ in A_6 and has an orbit of size 5 on the set of planes of $\Gamma_{a,v}$. But there is not any subgroup of A_6 having these properties (see [4], page A.25). We have the contradiction. Then $F_H = \emptyset$.

Then Γ is a building by Lemma 6.

We have proved that Γ is a building if Aut(Γ) is transitive on the set of lines of Γ.

Let us assume that Aut(Γ) is transitive on the set of points of Γ. Then $F_0 = \emptyset$ by Lemma 8. If either the Property LL holds or Aut(Γ) is transitive on the set of hyperlines of Γ, we have $F_H=\emptyset$ and Γ is a building by Lemma 6. Let us assume that the dual of LL holds in Γ and let us assume by contradiction that $F_H \neq \emptyset$. Then every point is incident with exactly one hyperline in F_H by Lemma 12 and because Aut(Γ) is transitive on the set of points of Γ. Trivially, the group Aut(Γ) is transitive on F_H. Let N_F be the number of hyperlines in F_H, let N_B be the number of hyperlines in B_H and let N_0 be the number of points. By easy computations we have $N_0 = 7 \cdot N_F$ and $N_F + 9 \cdot N_B = 9 \cdot N_0$. Then there is a positive integer α such that $N_F = 9\alpha$, $N_B = 62\alpha$ and $N_0 = 63\alpha$. Given an element x of Γ, we denote the stabilizer of x in Aut(Γ) by the symbol A_x. The symbol G_x will denote the elementwise stabilizer of the residue Γ_x of x and we set $\bar{A}_x = A_x/G_x$. Let a be a point and let u be the hyperline in $\sigma_H(a) \cap F_H$. We have $A_a \leq A_u$. Moreover we have $|A_a| 63\alpha = |A_u| 9\alpha$ because Aut(Γ) is transitive on F_H and on the set of points of Γ. Then A_a has index 7 in A_u. Then A_u is transitive on the set of points of Γ_u. But Γ_u is the A_7-geometry by Proposition H. Then $\bar{A}_u$ is a subgroup of A_7 in its natural action on the A_7-geometry. But $\bar{A}_u$ is transitive on the set of points of Γ_u. Then one of the following conditions holds:

1) We have $\bar{A}_u = A_7$.

2) We have $\bar{A}_u = L_3(2)$ and $\bar{A}_u$ is either the stabilizer of a plane of Γ_u or the stabilizer of a plane in the geometry twin of Γ_u.

3) We have $\bar{A}_u = \mathrm{Frob}(21)$.

4) The group $\bar{A}_u$ is cyclic of order 7.

Every hyperline v in B_H is cocollinear with at most 5 hyperlines in F_H. Indeed given $u \in F_H \cap (v^\top)$, let w(u) be the plane through u and v and let us set $W_v = \{ w(u) / u \in F_H \cap (v^\top) \}$. Two planes in W_v cannot have points in common by the dual of Lemma 7 and because every point is incident with just one hyperline in F_H. Then W_v contains at most 5 planes. But W_v and $F_H \cap (v^\top)$ contains the same number of elements by the duals of Lemmas 7 and 10. Then $F_H \; (v^\top)$ contains at most 5 elements. For every i=0,1,...,5 let B_i be the set of hyperlines v in B_H such that $F_H \cap (v^\top)$ contains i hyperlines and let $B_{i,1},\ldots,B_{i,n_i}$ be the orbits of Aut(Γ) on B_i (if $B_i = \emptyset$ we set $n_i = 0$). Let $|B_i|$ be the size of B_i. We have $62\alpha = \sum_{i=0}^{5} |B_i|$ and, if we compute the number of pairs (u,v) such that $u \in F_H$, $v \in B_H$ and $u \top v$, we get $9\alpha \cdot 30 = \sum_{i=0}^{5} i|B_i|$. Then $B_i \neq \emptyset \neq B_j$ for at least two distinct values of i,j=0,1,...,5, because 62 does not divide $9 \cdot 30$. For every i=0,1,...,5 and for every $k=1,\ldots,n_i$, let $n_{i,k}$ be the number of hyperlines in $\sigma_H(a) \cap B_{i,k}$ for some point a (the number $n_{i,k}$ does not depend on the choice of a because Aut(Γ) is transitive on the set of points of Γ). An easy computation shows that $|B_{i,k}| = n_{i,k}\alpha$. Moreover, if $i \neq 0$ and $B_i \neq \emptyset$, let $C_{i,k}$ be the number of hyperlines in $(u^\top) \cap B_{i,k}$ where $u \in F_H$ (the number $C_{i,k}$ does not depend on the choice of u because Aut(Γ) is transitive on F_H). Of course we have $1 \leq C_{i,k} \leq 30$. An easy computation shows that $|B_{i,k}| \cdot i = 9\alpha \cdot C_{i,k}$. Then $n_{i,k} = (9 \cdot C_{i,k})/i$. Let $v \in (u^\top) \cap B_H$ and let $A_{u,v}$ be the stabilizer of the pair (u,v). We have $|A_{u,v}| \cdot [A_u : A_{u,v}] \cdot 9\alpha = |A_{u,v}| \cdot [A_v : A_{u,v}] \cdot n_{i,k}\alpha$. Then $[A_u : A_{u,v}] = [A_v : A_{u,v}] \cdot C_{i,k}/i$. Let us write $h_{u,v}$ instead of $|A_v : A_{u,v}|$. We have $h_{u,v} \leq i$, $i \cdot [A_u : A_{u,v}]/h_{u,v} = C_{i,k}$ and $n_{i,k} = 9 \cdot [A_u : A_{u,v}]/h_{u,v}$. But $\sum_{i,k} n_{i,k}\alpha \leq 62\alpha$. Then an easy computation shows that we have $h_{u,v} = 5$ for some choice of u and v. Then i=5 for that value of $h_{u,v}$ and $C_{i,k} = [A_u : A_{u,v}]$ must be divisible by 5. This condition is not satisfied if $\bar{A}_u = L_3(2)$, Frob(21) of Z_7. Indeed if $\bar{A}_u = L_3(2)$ and is the stabilizer of a plane of the geometry twin of Γ_u, then it has orbits of size 7 and 8 on the set of planes of Γ_u. If $\bar{A}_u = L_3(2)$ is the stabilizer of a plane of Γ_u or if $\bar{A}_u$ = Frob(21) or Z_7 then its orbits on the set of planes of Γ_u have sizes 1 and 14 or 1,7,7 or 1,7,7 respectively.

Then $\bar{A}_u = A_7$ and it is transitive on the set of planes of Γ_u. We have $C_{i,k}$ = 15 or 30. if $C_{i,k} = 30$ then i=5 and $B_H - B_0$ is an orbit of Aut(Γ). It has size 54α. So B_0 contains 8α hyperlines. If $C_{i,k} = 15$ then i=3 or 5 because i divides $9 \cdot 15$. If i=3 then $n_{i,k} = 45$. So one orbit of Aut(Γ) on B_H has size 45α. Let $C_{j,h}$ be the other orbit of Aut(Γ) on $B_H - B_0$. Then $C_{j,h}$ has size 27α or 45α according to whether j=5 or 3. In both cases B_H contains more that 62α hyperlines. We have a contradiction. Then $i=j=5$ and both the orbits of $Aut(\Gamma)$ on B_H-B_0 have size 27α. Then B_H-B_0 contains 54α hyperlines and B_0 contains 8α hyperlines. Every point is incident with 8 hyperlines in B_0 and 54 hyperlines in B_H-B_0. Let $a\in\sigma_0(u)$. The group A_a is a subgroup of A_u and acts as A_6 on Γ_u. Then $\bar{A}_a$ is a subgroup of the stabilizer $K\cdot S_4(2)$ of u in the symplectic group $S_6(2)$ (in its natural action on the building Γ_a) where K is a group of order 2^5. Moreover $\bar{A}_a$ has the form $G\cdot A_6$ where G is a group of order 2^m $(m\leq 5)$. The stabilizer in $S_6(2)$ of u and v, where v is any hyperline in Γ_a not cocollinear with u, is the group $S_4(2)$. Indeed there are 32 hyperlines in Γ_a that are not collinear with a, whereas the stabilizer in $S_6(2)$ of u and v is transitive on the pairs $(C,\mathcal{A})$ where $\mathcal{A}$ is an apartment of the generalized quadrangle $u^{\top}\cap v^{\top}$ in Γ_a and C is a chamber of $\mathcal{A}$. Then there is some hyperline v in Γ_a, not cocollinear with u, such that, if $G_{u,v}$ $(=S_4(2))$ is the stabilizer in $S_6(2)$ of the ordered pair (u,v), we have that $G_{u,v}\cap\bar{A}_a$ is the alternating group A_6 and $[\bar{A}_a:G_{u,v}\cap\bar{A}_a]=2^m$. Moreover, $\bar{A}_a$ fixes the set $B_0\cap\sigma_H(a)$ that contains 8 hyperlines. Let $w\in\sigma_H(a)-(B_0)\cup u^{\perp})$. Let $A_{w,a}$ be the stabilizer of the flag $\{a,w\}$. We have $|A_{w,a}|\cdot[a_a:A_{w,a}]\cdot 63\alpha = |A_{w,a}|\cdot[A_w:A_{w,a}]\cdot Y$ where $Y=27\alpha$ or $Y=54\alpha$. Then $[A_a:A_{w,a}]\cdot 7 = [A_w:A_{w,a}]\cdot Z$ where $Z=3$ or 6. Then 3 divides $[A_a:A_{w,a}]$. But we have $[a_a:A_{y,a}] = [\bar{A}_a : \bar{A}_a\cap G_{u,v}] = 2^m$. Then $v\in B_0$. Therefore m=0,1,2 or 3 because $B_0\cap\sigma_H(a)$ contains 8 hyperlines. The set $\sigma_H(a) -((v^{\top})\cup(u^{\top})\cup B_0)$ contains 9 hyperlines. Their orbits under the action of $\bar{A}_a$ have sizes divisible by 3, Then the stabilizer in $\bar{A}_a$ of any of those hyperlines gives a subgroup M of $\bar{A}_a$ of index divisible by 3 in $\bar{A}_a$. Moreover, each of these subgroups has index ≤ 9 in $\bar{A}_a$. But A_5 is the only subgroup $\bar{M}$ of A_6 such that $9 \geq [A_6:\bar{M}]$ and $[A_6:\bar{M}] \equiv 0$ (mod.3). We have $[A_6:A_5] = 6$. Then we are compelled to split a set of 9 hyperlines in disjoint orbits each of one has size $2^t 6$ for some nonnegative integer t. This is not possible. We have the final contradiction.

Then $F_H = \emptyset$ and Γ is a building by Lemma 6.

Q.E.D.

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