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NORMAL SETS AND THEIR ORDER-AUTOMORPHISMS

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ABSTRACT.- *For a finite set* X *and the prower set* P_X *of* X *, we define* normal *subsets of* P_X. *We then describe all order-automorphisms of a normal subset of* P_X*, and, in particular, characterize those normal subsets of* P_X *for which each order-automorphism is* induced, *i.e. determined by a permutation of* X .

Let $X = \{1,2,\dots,n\}$, $K \subseteq X$ and

$$A(K) = \{ A \subseteq X : |A| \in K \}.$$

Such a subset of the power set P_X of X is termed *normal*. The present paper describes all *order-autonorphisms* of $A(K)$, i.e. the permutations ϕ of $A(K)$ such that

$$A \subseteq B \text{ if and only if } \phi(A) \subseteq \phi(B),$$

for all $A,B \in A(K)$. Let $0\text{-Aut } A(K)$ denote the group of all order-automorphisms of $A(K)$, S_n be the symmetric group of degree n. Obviously, $0\text{-Aut } A(\emptyset) = \{\emptyset\}$, and $0\text{-Aut } A(K) \simeq 0\text{-Aut } A(K\setminus\{n\})$ for all K. Therefore, we shall always assume that $K \neq \emptyset$ and $n \notin K$. It is said that $\phi \in 0\text{-Aut } A(K)$ is *induced* by an $\alpha \in S_n$ if $\phi(A) = \alpha(A)$ $(= \{\alpha(a) : a \in A\})$, for every $A \in A(K)$. The next theorem is our main result.

THEOREM. *Let* $K \neq \emptyset$, $n \notin K$ *and* $A(K)$ *be the associated normal subset of* P_X*. Then*

1. *Every* $\phi \in$ O-Aut$A(K)$ *is induced if and only if either of the following three conditions holds:*

(i) $|K| > 1$;

(ii) $K = \{1\}$;

(iii) $K = \{n-1\}$.

In all these cases, O-Aut $A(K) \cong S_n$.

2. *If* $K = \{k\}$, *then* O-Aut $A(K) = S_{\binom{n}{k}}$.

While the second statement of the Theorem is trivial, the proof of the first statement is given in Lemmas 1-6 below with Lemma 1 proving the "only if" part.

LEMMA 1. *If* $K = \{k\}$ *with* $1<k<n-1$ *then* $A(K)$ *has non-induced order-automorphisms.*

Proof. Since every permutation of such $A(K)$ is an order-automorphism, the result follows from the observation that

$$|\text{O-Aut } A(K)| = |S_{\binom{n}{k}}| = \binom{n}{k}! \neq |S_n|,$$

for $1< k< n-1$.

For the "if" part of the first statement of the Theorem observe that when $A(K)$ consists of all singletons in P_X or all co-singletons in P_X (that is $K = \{1\}$ or $\{n-1\}$), then any order-automorphism ϕ of $A(K)$ determines a permutation α of X which induces ϕ. Hence in what follows we assume that $|K| > 1$, i.e., $A(K)$ contains subsets of X of distinct orders. The next lemma readily follows from the definition of an order-automorphism ϕ.

LEMMA 2. *For every* $A \in \mathcal{A}(K)$, $|A| = |\phi(A)|$.

Our aim now is to produce a permutation α of X associated with ϕ. Let m=max K.

LEMMA 3. *Let* $|A_1| = |A_2| = \dots = |A_\ell| = m$ *with* $|\cap\{A_i : i=1,2,\dots\ell\}| \geq k$ *for some* $k \in K$. *Then* $|\cap\{A_i : i=1,2,\dots\ell\}| = |\cap\{\phi(A_i) : i=1,2,\dots,\ell\}|$.

Proof. Let $C(A_1,A_2,\dots,A_\ell)=\{C \in \mathcal{A}(K) : |C|=k \text{ and } C \subseteq A_i, i=1,2,\dots\ell\}$. Then $|C(A_1,A_2,\dots,A_\ell)| = \binom{|\cap\{A_i : i=1,2,\dots\ell\}|}{k}$, and in view of Lemma 2, $\phi(C(A_1,A_2,\dots,A_\ell)) = C(\phi(A_1),\phi(A_2),\dots\phi(A_\ell))$. The result follows.

LEMMA 4. *Let* $|A| = |B| = m$. *Then* $|A\setminus B| = 1$ *iff* $|\phi(A) \setminus \phi(B)| = 1$.

Proof. With the aid of Lemma 3 we observe that
$|A\setminus B| = 1$ iff $|A \cap B| = m-1$ iff $|\phi(A) \cap \phi(B)| = m-1$ iff $|\phi(A) \setminus \phi(B)| = 1$.

LEMMA 5. *If* $|A| = |B| = |C| = |D| = m$ *with* $A\setminus B = C\setminus D = \{x\}$ *for some* $x \in X$, *then there exists* $y \in X$ *such that* $\phi(A) \setminus \phi(B) = \phi(C) \setminus \phi(D) = \{y\}$.

Proof. We start with the following observation. Given A_1,A_2,A_3 with $|A_1| = |A_2| = |A_3| = m$ and $|A_1\setminus A_2| = |A_2\setminus A_3| = |A_1\setminus A_3| = 1$, we have

(1) $$A_1\setminus A_2 = A_1\setminus A_3 \text{ iff } A_1\setminus A_2 \neq A_3\setminus A_2.$$

Indeed, if $A_1\setminus A_2 = A_1\setminus A_3 = \{x\}$, then $x \notin A_3$, so $A_1\setminus A_2 = \{x\} \neq A_3\setminus A_2$. Conversely, if $A_1\setminus A_2 = \{x\} \neq A_3\setminus A_2$, then $x \in A_1$ and $x \notin A_3$, so that $\{x\} = A_1\setminus A_3$.

Now let A,B,C and D be as in the statement of the Lemma. We consider 3 cases.

Case 1. $A = C$, $B \neq D$.

Letting in (1) $A_1 = A$, $A_2 = B$ and $A_3 = D$, we have

$$A\setminus B = A\setminus D \text{ iff } A\setminus B \neq D\setminus B.$$

Since $|A\cap B\cap D| = |A\cap D| - |(A\cap D)\setminus B| = |A\cap D| - |(A\setminus B)\cap(D\setminus B)|$ $= m-1-|(A\setminus B)\cap(D\setminus B)|$, $A\setminus B \neq D\setminus B$ is equivalent to $|A\cap B\cap D|=m-1$.

Thus

(2) $A\setminus B = A\setminus D$ iff $|A\cap B\cap D| = m-1$

iff $|\phi(A)\cap\phi(B)\cap\phi(D)| = m-1$, by Lemma 3

iff $\phi(A)\setminus\phi(B) = \phi(A)\setminus\phi(D)$, by (3) and Lemma 4,

replacing A by $\phi(A)$, B by $\phi(B)$ and D by $\phi(D)$.

Case 2. $A \neq C$, $B = D$.

Observe that (1) is equivalent to

(3) $A_1\setminus A_2 = A_3\setminus A_2$ iff $A_1\setminus A_2 \neq A_1\setminus A_3$.

In (3) let $A_1 = A$, $A_2 = B$, $A_3 = C$.

Then

$$A\setminus B = C\setminus B \quad \text{iff} \quad A\setminus B \neq A\setminus C,$$

and the result follows by applying the result of Case 1 to the last inequality.

Case 3. A,B,C and D are distinct.

We show that if $|A\setminus B| = |C\setminus D| = 1$ then $A\setminus B = C\setminus D$ iff there exists a finite sequence

$A_1\ (=A), A_2, \ldots, A_{2\ell-1}, A_{2\ell}(=C)$ of sets of order m such that

(4)
$$A \setminus B = A_1 \setminus A_2 = A_{2i+1} \setminus A_{2i} = A_{2i+1} \setminus A_{2i+2} = C \setminus D, \text{ for } i=1,\ldots,\ell-1.$$

Then by applying alternatively the results of Case 1 and 2 to the last chain of equalities we produce the desired result. To show the validity of (4) let $A \setminus C = \{x_1,\ldots,x_\ell\}$, $C \setminus A = \{y_1,\ldots y_\ell\}$, and define the required sets as follows:

$$A_1 = A, \text{ and for } i = 1,\ldots, \ell-1,$$
$$A_{2i} = (A \cap C\ \ \{x\})\ \dot{\cup}\ \{x_i,\ldots, x_\ell\}\ \dot{\cup}\ \{y_1,\ldots,y_i\},$$
$$A_{2i+1} = (A \cap C)\ \dot{\cup}\ \{x_{i+1},\ldots,x_\ell\}\dot{\cup}\{y_1,\ldots,y_i\},$$
$$A_{2\ell} = C.$$

Clearly, for each $i=1,\ldots,\ell-1$, $|A_{2i}| = |A_{2i+1}| = m$. Also it is easy to check that for each i

$$A_1 \setminus A_2 = A_{2i+1} \setminus A_{2i} = A_{2i+1} \setminus A_{2i+2} = \{x\},$$

so that $A_1, A_2, \ldots, A_{2\ell}$ is the sequence required in (4).

Now we are in a position to define a mapping $\alpha: X \to X$ via $\alpha(x)=y$ iff $\{y\} = \phi(A) \setminus \phi(B)$, for some $A,B \in \mathcal{A}(K)$ with $|A|=|B|=m$ and $A \setminus B=\{x\}$.

Lemma 5 ensures that α is well-defined. Also in view of Lemma 5 we define a mapping $\beta: X \to X$ associated with ϕ^{-1} by $\beta(x) = \phi^{-1}(A) \setminus \phi^{-1}(B)$, for some $A,B \in \mathcal{A}(K)$ with $|A| = |B| = m$ and $A \setminus B = \{x\}$. A straightforward computation shows that β is the inverse of α and so α is a permutation of X.

LEMMA 6. *ϕ is induced by α.*

Proof. Our aim is to show that for any $A \in \mathcal{A}(K)$, $\phi(A) = \alpha(A)$. The result is clear if $|A| = m$. If $|A| \neq m$, then $|A| < m(=\max K)$ and

$$\phi(A)=\cap\{\phi(B):\phi(A) \subseteq \phi(B),\ |\phi(B)|=m\} = \cap\{\alpha(B):\alpha(A) \subseteq \alpha(B),\ |\alpha(B)|=m\}$$

$$= \alpha(\cap\{B : A \subseteq B,\ |B| = m\}) = \alpha(A).$$

This, and the observation that if each $\phi \in$ O-Aut $\mathcal{A}(K)$ is induced by an $\alpha \in S_n$ then O-Aut $\mathcal{A}(K) \simeq S_n$, completes our proof of the Theorem.

REMARK 7. If X is an infinite set, a subset $\mathcal{A}$ of P_X is said to be normal if whenever $B \subseteq X$ such that for some $A \in \mathcal{A}$, $|B| = |A|$ and $|X \setminus B| = |X \setminus A|$ then $B \in \mathcal{A}$. This definition of a normal set is equivalent to the one stated at the beginning of the paper (note that if X is finite and A, $B \subseteq X$ with $|A|=|B|$ then $|X \setminus A|=|X \setminus B|$ automatically).

In [1] we gave a complete description of all order-automorphisms of an arbitrary normal subset $\mathcal{A}$ of the power set of an *infinite* set X. We showed that all order-automorphisms of $\mathcal{A}$ are induced precisely when one of the following holds.

(a) $(\mathcal{A}, \subseteq)$ is *non-trivial*, i.e., there are $A,B \in \mathcal{A}$ with $A \subsetneq B$;

(b) $\mathcal{A}$ consists of singletons;

(c) $\mathcal{A}$ consists of co-singletons, i.e. $\mathcal{A} = \{A \subseteq X : |X \setminus A| = 1\}$.

If $\mathcal{A}$ satisfies either of the above conditions then O-Aut $\mathcal{A} \simeq S_X$, the symmetric group on X. If $\mathcal{A}$ satisfies none of the above conditions then there is an $n \in \mathbb{N}$, $n > 1$ such that either $\mathcal{A} = \{A \subseteq X : |A| = n\}$ or $\mathcal{A} = \{A \subseteq X : |X-A|=n\}$. In this case every permutation of A is an order-automorphism, hence O-Aut $A = S_A$, the symmetric group on A.

REFERENCES

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