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On Turán's inequality for complex polynomials

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Abstract. In this paper, we shall derive a generalized form of Turan's inequality for a special class of polynomials with restricted zeros.

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1 Introduction and Statement of Results

Turán type polynomial inequalities are quite interesting and the developments in this regard are very well explained in [8] and [10]. We begin with a well-known theorem of Turán [11] on the complex polynomials having all its zeros in the unit disc can be stated as follows.

Theorem 1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.1}$$

The result is sharp and the equality holds in (1.1) if all zeros of $p(z)$ lie on the unit circle.

More generally, if the polynomial $p(z)$ has all its zeros in $|z| \leq K \leq 1$, then Malik [7] (also see Govil and Rahman [6]) proved the following inequality.

Theorem 2. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K} \max_{|z|=1} |p(z)|. \tag{1.2}$$

The result is sharp and the equality holds in (1.2) if $p(z) = z^n + K$.

Govil [5] considered the problem of determining the maximum modulus of the derivative of a complex polynomial whose zeros are in the disc $|z| \leq K$, $K \geq 1$.

Theorem 3. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K, \quad K \geq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |p(z)|. \tag{1.3}$$

The result is sharp and the equality holds in (1.3) if $p(z) = z^n + K^n$.

Govil [4] extended the inequality (1.3) to an inequality involving moduli of all the zeros of a complex polynomial rather than restricting only to the zero of maximum modulus. Generalizations of Turán's inequality have appeared in the literature quite significantly (See [10]). Recently Prasanna Kumar [9] proved a generalization of Turán's inequality for complex polynomials with restricted zeros. But that paper does consider all the zero coefficients of the polynomial also. This paper makes an attempt to overcome this to some extent and prove a new inequality in generalized form involving a special class of polynomials with restricted zeros. We state the theorem as follows.

Theorem 4. *Let $p(z) = a_n z^n + \sum_{k=0}^{t} a_k z^k = a_n \prod_{k=1}^{n} (z - z_k), \quad t \leq n-1$ with $a_0 \neq 0$, is a polynomial of degree $n > 3$ having all its zeros in $|z| \leq K, \quad K \geq 1$, and $s(z) = z^n p(\frac{1}{z})$. Then if $s'(z)$ has no zeros in the disk $|z| < \frac{1}{K}$ and $0 \leq r \leq R \leq K$, with $r \leq 1$, we have*

$$\begin{aligned}
\max_{|z|=1} |p'(z)| \geq{}& \frac{R^n}{BK^{2(n-1)}} \sum_{k=1}^{n} \frac{1}{K^2 + r|z_k|} \max_{|z|=K^2/R} |p(z)| \\
&+ \left(\frac{R^n - r^n}{BK^{n-2}}\right) \sum_{k=1}^{n} \frac{1}{K^2 + r|z_k|} \left(\frac{(1+K^{(n-t)})^{\frac{n}{(n-t)}-1}}{(r^{(n-t)} + K^{(n-t)})^{\frac{n}{(n-t)}}}\right) \min_{|z|=K} |p(z)| \\
&+ 2\frac{K^2}{B} |a_{n-1}| \sum_{k=1}^{n} \frac{1}{K^2 + r|z_k|} \left(\frac{R^n - r^n}{n(n+1)} - \frac{R - r}{n+1}\right) \\
&+ 2\frac{K^2}{B} |a_{n-2}| \sum_{k=1}^{n} \frac{1}{K^2 + r|z_k|} \left[\frac{R^n - r^n}{n(n-1)} - \frac{R^{n-2} - r^{n-2}}{(n-2)(n-3)} + \frac{2(R-r)}{(n-1)(n-3)}\right] \\
&+ \frac{2(K^{2(n-1)} - r^{n-1})}{K^{2n}(n+1)} |a_1| + 2|a_2| \left[\frac{K^{2(n-1)} - r^{n-1}}{K^{2n}(n-1)} - \frac{r^2(K^{2(n-3)} - r^{n-3})}{K^{2n}(n-3)}\right]
\end{aligned} \tag{1.4}$$

where

$$B = \left[1 + (R^n - r^n)\left(\frac{(1+K^{(n-t)})^{\frac{n}{(n-t)}-1}}{(r^{(n-t)} + K^{(n-t)})^{\frac{n}{(n-t)}}}\right)\right].$$

The result is sharp and the equality holds in (1.4) if $p(z) = z^n + K^n, \;\; r = 0$ *and* $R = K$.

2 Lemmas

The following lemmas are required to prove our result. First lemma is due to Girox, Rahman and Schmeisser [3].

Lemma 1. *If* $p(z) = a_n \prod_{k=1}^{n}(z - z_k)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq 1$, *then*

$$\max_{|z|=1} |p'(z)| \geq \sum_{k=1}^{n} \frac{1}{1 + |z_k|} \max_{|z|=1} |p(z)|. \tag{2.1}$$

Equality in (2.1) holds if every z_k *is positive real.*

Dewan et al. [2] derived the following lemma, which we will use in proving our results.

Lemma 2. *If* $p(z) = \sum_{k=0}^{n} a_k z^k$ *is a polynomial of degree* $n > 2$, *then for any* $R \geq 1$,

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - \frac{2(R^n - 1)}{n+2}|a_0| - |a_1| \left[\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right]. \tag{2.2}$$

Equality holds in (2.2) if $p(z) = z^n + 1$.

Aziz and Shah [1] proved an important inequality on lacunary polynomials, which is our next lemma.

Lemma 3. *If* $p(z) = a_0 + \sum_{k=t}^{n} a_k z^k, \;\; t \geq 1$, *is a polynomial of degree* n *having no zeros in* $|z| < K, \;\; K > 0$ *then for any* $0 \leq r \leq R \leq K$,

$$\max_{|z|=R} |p'(z)| \leq n \left(\frac{R^{t-1}(R^t + K^t)^{\frac{n}{t}-1}}{(r^t + K^t)^{\frac{n}{t}}}\right) \left(\max_{|z|=r} |p(z)| - \min_{|z|=K} |p(z)|\right). \tag{2.3}$$

There is an equality in (2.3) if $p(z) = (z^t + K^t)^{\frac{n}{t}}$. *where* n *is a multiple of* t.

Now we prove a lemma, which will play a crucial role in proving our theorem.

Lemma 4. *If* $p(z) = a_0 + \sum_{k=t}^{n} a_k z^k, \;\; t \geq 1$, *is a polynomial of degree* $n > 3$ *such that* $p(z)$ *and* $p'(z)$ *have no zeros in* $|z| < K$, *then for any* $0 \leq r \leq R \leq K$, *and* $r \leq 1, \;\; K \geq 1$,

$$\max_{|z|=R}|p(z)| \leq \left[1+(R^n-r^n)\left(\frac{(1+K^t)^{\frac{n}{t}-1}}{(r^t+K^t)^{\frac{n}{t}}}\right)\right]\max_{|z|=r}|p(z)|$$

$$-(R^n-r^n)\left(\frac{(1+K^t)^{\frac{n}{t}-1}}{(r^t+K^t)^{\frac{n}{t}}}\right)\min_{|z|=K}|p(z)|-2|a_1|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{R-r}{n+1}\right)$$

$$-2|a_2|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right]. \tag{2.4}$$

Proof. Let the polar representation of z be $z=re^{i\theta}$. Then for every θ with $0\leq\theta<2\pi$, we have

$$p(Re^{i\theta})-p(re^{i\theta})=\int_r^R e^{i\theta}p'(re^{i\theta})dr.$$

But then,

$$|p(Re^{i\theta})-p(re^{i\theta})|\leq\int_r^R|p'(re^{i\theta})|dr.$$

Now applying Lemma 2 and then Lemma 3 to the polynomial $p'(z)$ which is of degree greater than 2, with no zeros in $|z|<K$, we get

$$|p(Re^{i\theta})-p(re^{i\theta})|$$

$$\leq\int_r^R\left[t^{n-1}\max_{|z|=1}|p'(z)|-\frac{2(t^{n-1}-1)}{n+1}|a_1|-2|a_2|\left[\frac{t^{n-1}-1}{n-1}-\frac{t^{n-3}-1}{n-3}\right]\right]dt$$

$$=\left(\frac{R^n-r^n}{n}\right)\max_{|z|=1}|p'(z)|-2|a_1|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{(R-r)}{n+1}\right)$$

$$-2|a_2|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right]$$

$$\leq(R^n-r^n)\left(\frac{(1+K^t)^{\frac{n}{t}-1}}{(r^t+K^t)^{\frac{n}{t}}}\right)\left(\max_{|z|=r}|p(z)|-\min_{|z|=K}|p(z)|\right)$$

$$-2|a_1|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{(R-r)}{n+1}\right)-2|a_2|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right],$$

which further implies

$$
\begin{aligned}
&|p(Re^{i\theta})| - |p(re^{i\theta})| \\
&\qquad \le (R^n - r^n)\left(\frac{(1+K^t)^{\frac{n}{t}-1}}{(r^t+K^t)^{\frac{n}{t}}}\right)\left(\max_{|z|=r}|p(z)| - \min_{|z|=K}|p(z)|\right) \\
&\qquad\qquad -2|a_1|\left(\frac{(R^n-r^n)}{n(n+1)} - \frac{(R-r)}{n+1}\right) \\
&\qquad -2|a_2|\left[\frac{R^n-r^n}{n(n-1)} - \frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)} + \frac{2(R-r)}{(n-1)(n-3)}\right].
\end{aligned}
$$

Rearranging the terms will yield the inequality (2.4). Hence the proof is complete.

3 Proof of the Theorem 4

Since the zeros of $p(z)$ are $z_k (1 \le k \le n)$, the zeros of the polynomial $q(z) = p((K^2/r)z)$ are $\frac{rz_k}{K^2}(1 \le k \le n)$. Also observe that, the zeros of $q(z)$ all lie in $|z| \le 1$, since all the zeros of $p(z)$ lie in $|z| \le K$. Now by Lemma 1, we have

$$\max_{|z|=1}|q'(z)| \ge \sum_{k=1}^{n}\frac{1}{1+\frac{r|z_k|}{K^2}}\max_{|z|=1}|q(z)|$$

which gives

$$\max_{|z|=\frac{K^2}{r}}|p'(z)| \ge r\sum_{k=1}^{n}\frac{1}{K^2+r|z_k|}\max_{|z|=\frac{K^2}{r}}|p(z)|. \tag{3.1}$$

Since the degree of $p(z)$ is greater than 3, the degree of $p'(z)$ is greater than 2. Therefore applying Lemma 2 to $p'(z)$ with $\frac{K^2}{r} \ge 1$,

$$
\begin{aligned}
&\max_{|z|=\frac{K^2}{r}}|p'(z)| \le \frac{K^{2(n-1)}}{r^{n-1}}\max_{|z|=1}|p'(z)| \\
&-\frac{2(K^{2(n-1)}-r^{n-1})}{r^{n-1}(n+1)}|a_1| - 2|a_2|\left[\frac{K^{2(n-1)}-r^{n-1}}{r^{n-1}(n-1)} - \frac{K^{2(n-3)}-r^{n-3}}{r^{n-3}(n-3)}\right].
\end{aligned} \tag{3.2}
$$

From equations (3.1) and (3.2), it follows that

$$\sum_{k=1}^{n}\frac{1}{K^2+r|z_k|}\max_{|z|=\frac{K^2}{r}}|p(z)| \le \frac{K^{2(n-1)}}{r^n}\max_{|z|=1}|p'(z)|$$

$$-\frac{2(K^{2(n-1)}-r^{n-1})}{r^n(n+1)}|a_1|-2|a_2|\left[\frac{K^{2(n-1)}-r^{n-1}}{r^n(n-1)}-\frac{K^{2(n-3)}-r^{n-3}}{r^{(n-2)}(n-3)}\right].$$

Equivalently,

$$\max_{|z|=1}|p'(z)|\geq\frac{r^n}{K^{2(n-1)}}\sum_{k=1}^{n}\frac{1}{K^2+r|z_k|}\max_{|z|=\frac{K^2}{r}}|p(z)|$$

$$+\frac{2(K^{2(n-1)}-r^{n-1})}{K^{2(n-1)}(n+1)}|a_1|+2|a_2|\left[\frac{K^{2(n-1)}-r^{n-1}}{K^{2(n-1)}(n-1)}-\frac{r^2(K^{2(n-3)}-r^{n-3})}{K^{2(n-1)}(n-3)}\right]. \tag{3.3}$$

Let $r(z)=z^nP(\frac{1}{z})$. Since the polynomial $p(z)$ has all its zeros in $|z|\leq K, K\geq 1$, the polynomial $r(\frac{z}{K^2})$ has all its zeros in $|z|\geq K$. Also observe that,

$$\max_{|z|=R}|r\left(\frac{z}{K^2}\right)|=\frac{R^n}{K^{2n}}\max_{|z|=\frac{z}{K^2}}|p(z)| \quad \max_{|z|=r}|r\left(\frac{z}{K^2}\right)|=\frac{r^n}{K^{2n}}\max_{|z|=\frac{K^2}{r}}|p(z)|$$

and

$$\min_{|z|=K}|r\left(\frac{z}{K^2}\right)|=\frac{1}{K^n}\min_{|z|=K}|p(z)|.$$

Now applying Lemma 4 to the polynomial $r\left(\frac{z}{K^2}\right)$, we get

$$\max_{|z|=R}|r\left(\frac{z}{K^2}\right)|\leq\left[1+(R^n-r^n)(\frac{(1+K^{(n-t)})^{\frac{n}{(n-t)}-1}}{(r^{(n-t)}+K^{(n-t)})^{\frac{n}{(n-t)}}}\right](\max_{|z|=r}|r\left(\frac{z}{K^2}\right)|$$

$$-(R^n-r^n)\left(\frac{(1+K^{(n-t)})^{\frac{n}{(n-t)}-1}}{(r^{(n-t)}+K^{(n-t)})^{\frac{n}{(n-t)}}}\right)\min_{|z|=K}|r\left(\frac{z}{K^2}\right)|$$

$$-2|a_{n-1})|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{R-r}{n+1}\right)$$

$$-2|a_{n-2}|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right].$$

$\Rightarrow$

$$R^n\max_{|z|=\frac{K^2}{R}}|p(z)|\leq r^n\left[1+(R^n-r^n)\left(\frac{(1+K^{(n-t)})^{\frac{n}{(n-t)}-1}}{(r^{(n-t)}+K^{(n-t)})^{\frac{n}{(n-t)}}}\right)\right]\max_{|z|=\frac{K^2}{r}}|p(z)|$$

$$-K^n\left(R^n-r^n\right)\left(\frac{\left(1+K^{(n-t)}\right)^{\frac{n}{(n-t)}-1}}{\left(r^{(n-t)}+K^{(n-t)}\right)^{\frac{n}{(n-t)}}}\right)\min_{|z|=K}|p(z)|$$

$$-2K^{2n}|a_{n-1}|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{R-r}{n+1}\right)$$

$$-2K^{2n}|a_{n-2}|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right].$$

$\Rightarrow$

$$\left[1+\left(R^n-r^n\right)\left(\frac{\left(1+K^{(n-t)}\right)^{\frac{n}{(n-t)}-1}}{\left(r^{(n-t)}+K^{(n-t)}\right)^{\frac{n}{(n-t)}}}\right)\right]\max_{|z|=\frac{K^2}{r}}|p(z)|$$

$$\geq\frac{R^n}{r^n}\max_{|z|=\frac{K^2}{R}}|p(z)|+K^n\left(\frac{R^n-r^n}{r^n}\right)\left(\frac{\left(1+K^{(n-t)}\right)^{\frac{n}{(n-t)}-1}}{\left(r^{(n-t)}+K^{(n-t)}\right)^{\frac{n}{(n-t)}}}\right)\min_{|z|=K}|p(z)|$$

$$+2\frac{K^{2n}}{r^n}|a_{n-1}|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{R-r}{n+1}\right)$$

$$+2\frac{K^{2n}}{r^n}|a_{n-2}|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right].$$

$\Rightarrow$

$$\max_{|z|=\frac{K^2}{r}}|p(z)|\geq\frac{R^n}{B}\max_{|z|=\frac{K^2}{R}}|p(z)|$$

$$+K^n\left(\frac{R^n-r^n}{B}\right)\left(\frac{\left(1+K^{(n-t)}\right)^{\frac{n}{(n-t)}-1}}{\left(r^{(n-t)}+K^{(n-t)}\right)^{\frac{n}{(n-t)}}}\right)\min_{|z|=K}|p(z)|$$

$$+2\frac{K^{2n}}{B}|a_{(n-1}|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{R-r}{n+1}\right)$$

$$+2\frac{K^{2n}}{B}|a_{n-2}|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right],\tag{3.4}$$

where

$$B=\left[1+\left(R^n-r^n\right)\left(\frac{\left(1+K^{(n-t)}\right)^{\frac{n}{(n-t)}-1}}{\left(r^{(n-t)}+K^{(n-t)}\right)^{\frac{n}{(n-t)}}}\right)\right].$$

Combining the inequalities (3.3)and (3.4), we get the desired result. Thus the proof is complete.

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