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# BAIRE PROPERTIES OF (LF)-SPACES

P.P.NARAYANASWAMI (*)

**ABSTRACT.** *We relate the study of (LF)-spaces with some covering properties of locally convex spaces, which are variations of the theme of "Baire Space". All (LF)-spaces are partitioned into three classes, called $(LF)_1$, $(LF)_2$ and $(LF)_3$-spaces respectively. We then show that these classes are precisely the classes of (LF)-spaces that distinguish between the several Baire-type coverings we considered. The role of the sequence space $\varphi$ in this context is studied. The interaction between $(LF)_3$-spaces and the Separable Quotient Problem is also discussed.*

## 1 (LF)-SPACES

All spaces considered in this paper are locally convex (Hausdorff) topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$. Let $\{(E_n, \tau_n)\}_{n=1}^{\infty}$ be a sequence of locally convex spaces such that for each n, $E_n \subsetneq E_{n+1}$, and on $E_n$, $\tau_{n+1}$ induces a topology coarser than $\tau_n$. Such a sequence is an *inductive sequence*. If $E = \bigcup_{n=1}^{\infty} E_n$, and $\tau$ is the finest Hausdorff locally convex topology on E such that $\tau$ induces on each $E_n$, a topology coarser than $\tau_n$, then $(E,\tau)$ is said to be the *inductive limit* of the sequence $\{(E_n, \tau_n)\}$, and we write $(E, \tau) = \underset{n}{\mathrm{ind}}(E_n, \tau_n)$.

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The sequence $\{(E_n,\tau_n)\}$ is a *defining sequence* for the inductive limit. Note that we are only using a narrow definition of the notion of an inductive limit that befits our needs. If for each n, $\tau_{n+1}|_{E_n} = \tau_n$, then the inductive limit and the corresponding defining sequence are said to be *strict*. If each $(E_n,\tau_n)$ is a Fréchet space [Banach space], the inductive limit is called an (LF)-*space* [(LB)-*space*]. The terms *strict* (LF), *strict* (LB)-spaces have their obvious meanings. While referring to defining sequences for an (LF) or an (LB)-space, we shall always mean a defining sequence consisting of Fréchet spaces. Two inductive sequences $\{(E_n^{(1)},\tau_n^{(1)}\}$, $\{(E_n^{(2)},\tau_n^{(2)})\}$ on E (defining possibly two different Hausdorff topologies on E) are said to be *equivalent*, if for $i\in\{1,2\}$ and n arbitrary, there exists k such that $E_n^{(i)} \subset E_k^{(3-i)}$ and $\tau_k^{(3-i)}\Big|_{E_n^{(i)}} \le \tau_n^{(i)}$; i.e., each member of either sequence is continuously included in some member of the other. One readily sees that equivalent inductive sequences of Fréchet spaces define the same (LF)-space.

**THEOREM 1 (EQUIVALENCE THEOREM) [17]**

*Let* $(E,\tau^{(i)}) = \operatorname{ind}_n(E_n^{(i)},\tau_n^{(i)})$, $(i=1,2)$. *The following are equivalent statements:*

(a) $\{(E_n^{(1)},\tau_n^{(1)})\}$ *is equivalent to* $\{(E_n^{(2)},\tau_n^{(2)})\}$;

(b) $\tau^{(1)} = \tau^{(2)}$ ;

(c) *The infimum of* $\tau^{(1)}$ *and* $\tau^{(2)}$ *is Hausdorff.*

**COROLLARY.** If $(E,\tau)$ is a Hausdorff locally convex space, there is at most one topology on E finer than $\tau$, which makes E an (LF)-space.

It is quite possible for a strict (LF)-space to possess a non-strictt defining sequence. Also, for an (LB)-space, not every defining sequence need consist of Banach spaces only.

**EXAMPLE 1.** Let $\tau_n$ denote the product (Banach space) topology on

$$E_n = \underbrace{\ell_1 \times \ell_1 \times \ldots \times \ell_1}_{n \text{ factors}} \times \{0\} \times \{0\} \times \ldots$$

Clearly, $(E_n, \tau_n)$ is a strict defining sequence of Banach spaces, defining the strict (LB)-space $(E,\tau) = \operatorname{ind}_n (E_n, \tau_n)$. Consider

$$F_n = \underbrace{\ell_1 \times \ell_1 \times \ldots \times \ell_1}_{n \text{ factors}} \times s \times \{0\} \times \{0\} \ldots$$

with the product (non-Banach, Fréchet space) topology $\eta_n$, where s denotes the non-normable, nuclear Fréchet space of all rapidly decreasing sequences of scalars (s is continuously included in $\ell_1$). One sees that $\{(E_n, \tau_n)\}$ is equivalent to $\{(F_n, \eta_n)\}$, which is a non-strict defining sequence of non-Banach-spaces, defining the strict (LB)-space $(E,\tau)$.

Replacing $\ell_1$ by $\ell_2$, and s by $\ell_1$, we obtain a strict (LB)-space with a non-strict defining sequence of Banach spaces.

The space $\varphi$ of all scalar sequences with only a finite number of non-zero coordinates, equipped with the finest locally convex topology, can be recognized as the inductive limit of finite-dimensional spaces. It is the only strict (LB)-space for which every defining sequence is strict. The dual of $\varphi$ is the space $\omega$, the space of all scalar sequences, with the product (Fréchet space) topology.

We observe that no (LF)-space is both complete and metrizable. It is well-known ([19], p.225) that strict (LF)-spaces are complete, hence non-metrizable. Also, (LB)-spaces are never metrizable, even though some are incomplete. In [9], §31.6, there is a classical example of an incomplete (LB)-space, while the (LB)-space $\ell_{p^-} = \operatorname{ind}_n \left(\ell_{p - \frac{1}{N+n}}\right)$, where $p>1$ and N is chosen so that $p - \frac{1}{N+1} > 1$, is a complete (LB)-space. (Note that $\ell_{p^-}$ is independent of the choice of N). The (LF)-space $\omega \times \ell_{p^-} = \operatorname{ind}_n \left(\omega \times \ell_{p-\frac{1}{N+n}}\right)$ is a non-strict (LF), non-(LB), non-metrizable (LF)-space (see [16]). However, there do exist plenty of metrizable, as well as normable (LF)-spaces. For instance, see [17], [22] for constructions of such spaces. The following is a quick example.

**EXAMPLE 2.** Let $E_n = \underbrace{\omega \times \omega \times \ldots \times \omega}_{n \text{ factors}} \times \ell_p \times \ell_p \times \ldots$ with the product

(Fréchet) topology. Now $\ell_p (p \geq 1)$ is densely, and continuously included in $\omega$. So $\{E_n\}$ is a strictly increasing sequence of Fréchet spaces, with $E_n$ continuously included in $E_{n+1}$ for each n. It follows that $\text{ind } E_n = \bigcup_{n=1}^{\infty} E_n$ is a dense subspace of the Fréchet space $F = \omega \times \omega \times \ldots$, which, with the relative topology, is a metrizable (LF)-space. Since F is isomorphic to $\omega$, it follows that $\omega$ contains a dense, (metrizable) (LF)-subspace.

Normable (LF)-spaces are not easy to come by, but an example due to De Wilde is cited in [8], p.210.

At this point, a natural question arises. "When is an (LF)-space metrizable?" In the next section, we see that this leads to some covering properties of spaces. This is not unexpected, since in the definition of an (LF)-space $(E, \tau)$, E is "covered" by the defining sequence $\{E_n\}$.

## 2. BAIRE-TYPE COVERINGS

In [1], Amemiya-Kōmura observed that if E is barrelled and metrizable, then E is *not* the union of an increasing sequence of nowhere-dense, absolutely convex sets. The current terminology for this property is Baire-likeness, and a detailed study of Baire-like spaces can be found in [13]. White Baire-like spaces are always barrelled, it is shown in [13], as a generalization of the Amemiya-Kōmura result, that a barrelled space that does not contain (an isomorphic copy of) $\varphi$, is Baire-like. We note that $\varphi$ is not metrizable. As a consequence of these observations, we have the following result.

THEOREM 2 [16]

*An (LF)-space is metrizable, if and only if it is Baire-like, if and only if it does not contain a copy of* $\varphi$ .

We now consider several variants of the Baire-like covering property. A locally convex space E is

*Baire* if E is *not* the union of an increasing sequence of nowhere-dense sets;

*unordered Baire-like* [19] if E is *not* the union of a sequence of nowhere dense, absolutely convex sets, equivalently E has *property* (R-R) (Robertson and Robertson [11] Todd-Saxon [19]): if E is covered by a sequence of subspaces, at least one of the subspaces is both dense and barrelled.

a *(db)-space* [16], if E has *property* (R-T-Y) (Robertson, Tweddle and Yeomans [12]): if E is covered by an increasing sequence of subspaces, at least one of the subspaces is (hence almost all of them are) both dense and barrelled. (Valdivia [21] uses the terminology-superbarrelled space).

*quasi-Baire* if E *is* barrelled, and is *not* the union of an increasing sequence of nowhere-dense subspaces. Note that unordered Baire-like property is the same as "unordered" (db) property. In the definition of an unordered Baire-like space, if we demand that the absolutely convex sets are "increasing", we obtain a Baire-like space.

All these spaces (except Baire spaces) enjoy "reasonable" permanence properties. They are stable under the formation of arbitrary products, quotients and countable-codimensional subspaces. (See [10], [13], [16], [19]). The so-called Wilansky-Klee conjecture ([15],[19]) that "every dense one-codimensional subspace of a Banach space is Baire", was answered in the negative by Arias de Reyna [2], using Martin's Axiom. Using continuum hypothesis, he further showed in [3] that there exist two pre-Hilbertian spaces whose product is not Baire. Clearly,

$$\text{Baire} \Longrightarrow \text{unordered Baire-like} \Longrightarrow \text{(db)} \Longrightarrow$$
$$\Longrightarrow \text{Baire-like} \Longrightarrow \text{quasi-Baire} \Longrightarrow \text{barrelled.}$$

The Amemiya-Kōmura result, together with a result of De-Wilde and Houet [5] and/or Saxon [13], shows that in the class of metrizable spaces, Baire-likeness coincides with barrelledness, and even with a weaker property, namely property (S): the dual E' is $\sigma(E',E)$-sequentially complete. Valdivia [20] generalized the Amemiya-Kōmura result by showing that a Hausdorff barrelled space whose completion is Baire must be a Baire-like space. It then turns out that in the "smallest" variety [7], namely the variety of real Hausdorff spaces with their weak topology, the completion of any member is a product of reals, and hence a Baire space; so in the smallest variety, barrelledness is equivalent to Baire-likeness. In [13], it is shown that barrelled spaces are Baire-like in a wider class of spaces not containing $\varphi$ . Also in [10], we prove that in a still wider class of locally convex spaces not containing a complemented copy of $\varphi$ , barrelled spaces are quasi-Baire.

We want to show that none of the above implication arrows is reversable. Examples of unordered Baire-like spaces that are not Baire are plenty (see [6],[13],[14],[15]). The abundant existence of (db)-spaces that are not unordered Baire-like is demonstrated by the following theorem:

THEOREM 3 [16]

*Every infinite-dimensional Fréchet space has a dense subspace that is a (metrizable) (db)-space, but not unordered Baire-like.*

For the remaining three implications, we employ a classification of (LF)-spaces.

## 3. A CLASSIFICATION OF (LF)-SPACES

Since Fréchet spaces are barrelled, and inductive limits of barrelled spaces are again barrelled, it follows that (LF)-spaces are barrelled. On the other hand, no (LF)-space is a (db)-space. For, otherwise if $(E,\tau) = \mathrm{ind}(E_n,\tau_n)$, some $(E_k,\tau_k)$ is dense and barrelled in $(E,\tau)$. The identity map from $(E_k,\tau_k)$ onto $(E_k,\tau|E_k)$ is continuous from a Pták space onto a barrelled space, hence must be open, by Ptak's open mapping theorem. Thus, $E_k$ is closed in E, yielding $E_k=E$, a contradiction. A similar argument, using an increasing sequence of multiples of the unit balls in $E_n$'s shows that (LB)-spaces are never Baire-like. Also no strict (LF)-space is quasi-Baire, since in the definition, $E_n$ is a proper, closed subspace of E. These observations fit into the scheme (db) $\Longrightarrow$ Baire-like $\Longrightarrow$ quasi-Baire $\Longrightarrow$ barrelled

nicely, and enable us to classify all (LF)-spaces into three disjoint classes as follows.



DEFINITION [10]

An (LF)-space $(E, \tau)$ is an

$(LF)_1$-*space* if $(E, \tau)$ has a defining sequence *none* of whose members is dense in E.

$(LF)_2$-*space* if $(E, \tau)$ is non-metrizable, and has a defining sequence each of whose members is dense in E (equivalently, at least one of the members is dense in E);

$(LF)_3$-*space* if $(E, \tau)$ is metrizable.

These three classes are mutually disjoint - $(LF)_1 \cap (LF)_2 = \phi$ since two defining sequences must be equivalent; $(LF)_2$ is disjoint from $(LF)_3$ by definition. $(LF)_1$-spaces are never Baire-like, so by Theorem 2, are disjoint from the class of $(LF)_3$-spaces. Each of these classes is sufficiently rich. All strict (LF)-spaces are $(LF)_1$-spaces; $\varphi \times \ell_{p^-}$ is a non-strict $(LB)_1$-space. Some (LB)-spaces,namely those with a defining sequence of dense subspaces, for instance, the space $\ell_{p^-}$, is an $(LF)_2$-space - in fact an $(LB)_2$-space. (Since no (LB)-space is metrizable, $(LB)_3$-spaces do not exist). Every metrizable and every normable (LF)-space is an example of an $(LF)_3$-space. It is demonstrated in [17] that there exist plenty of $(LF)_3$-spaces.

The following theorem, which characterizes barrelled spaces that are not quasi-Baire is very useful, in this context.

**THEOREM 4** [10]

For a barrelled space E, the following are equivalent.

(a) E is *not* quasi-Baire;

(b) E contains a complemented copy of $\varphi$ ;

(c) E contains a closed $\aleph_0$-codimensional subspace ;

(d) $E \simeq E \times \varphi$;

(e) E is a strict inductive limit of a strictly increasing sequence of closed, barrelled subspaces of E.

As a consequence, all strict (LF)-spaces contain a complemented copy of $\varphi$ ; also, in the class of spaces *not* containing a complemented copy of $\varphi$ , the notions barrelled, and quasi-Baire coincide. Along with Theorem 2, these observations enable us to characterize $(LF)_i$-space (i=1,2,3) is terms of Baire-type notions, as well as in terms of the incidence of $\varphi$ . Explicitly, we have the following two characterization theorems.

**THEOREM 5** [10]

An (LF)-space $(E,\tau)$ is an

$(LF)_1$-space $\iff$ $(E,\tau)$ is *not* quasi-Baire;

$(LF)_2$-space $\iff$ $(E,\tau)$ is quasi-Baire, but not Baire-like;

$(LF)_3$-space $\iff$ $(E,\tau)$ is Baire-like.

Since (LF)-spaces are never (db)-spaces but always barrelled, we see that

$(LF)_1$-spaces are *precisely* the class of (LF)-spaces that distinguish between barrelled spaces and quasi-Baire spaces;

$(LF)_2$-spaces are precisely the class of (LF)-spaces that distinguish between quasi-Baire and Baire-like spaces;

$(LF)_3$-spaces are precisely the class of (LF)-spaces that distinguish between (db) and Baire-like spaces.

The next theorem characterizes $(LF)_i$-spaces (i=1,2,3) in terms of the space $\varphi$ .

**THEOREM 6** [10]

An (LF)-space E is an

$(LF)_1$-space $\Longleftrightarrow$ E contains a complemented copy of $\varphi$ ;

$(LF)_2$-space $\Longleftrightarrow$ E contains $\varphi$, but not a complemented copy of $\varphi$;

$(LF)_3$-space $\Longleftrightarrow$ E does not contain $\varphi$.

**REMARK** $(LF)_3$-spaces form incomplete quotients of complete spaces. (See [9], page 225).

## 4. STABILITY PROPERTIES OF $(LF)_i$-SPACES (i=1,2,3)

Various permanence properties of $(LF)_i$-spaces (i=1,2,3) are studied in [10] and [17]. A finite-codimensional subspace of an $(LF)_i$-space is an $(LF)_j$-space, $1 \leq i, j \leq 3$, if and only if i=j. A countable-codimensional subspace of an (LF)-space is an (LF)-space if and only if it is closed, and not contained in any member of a defining sequence. A Hausdorff inductive limit of an increasing sequence of (LF)-spaces is again an (LF)-space. An infinite product of (LF)-spaces is never an (LF)-space. But the cartesian product of an $(LF)_i$-space with an $(LF)_j$-space is an $(LF)_k$-space, where k = minimum of {i,j}, and $1 \leq i,j,k, \leq 3$. If M is a closed subspace of an $(LF)_i$-space, E, then the quotient E/M is either an $(LF)_j$-space, with $j \geq i$, $(1 \leq i, j \leq 3)$, or else, a Fréchet space (in case $E_n + M = E$ for some n). This result on quotients is fascinating, since it is possible for a Fréchet space to be the quotient of an (LF)-space. Since the index i cannot decrease while passing to quotients, we can regard the class of Fréchet spaces as $(LF)_4$-spaces, by agreeing to relax the requirement that the inductive sequence $\{E_n\}$ is *strictly* increasing, in our original definition of an (LF)-space. The class of $(LF)_3$-spaces are better behaved for quotients. Every $(LF)_3$-space admits a quotient, which is separable, infinite-dimensional Fréchet space. Such a result need not hold for $(LF)_1$ or $(LF)_2$-spaces. For example, no quotient of $\varphi$ (an $(LB)_1$-space) or $\ell_{p^-}$ (an $(LB)_2$-space) is a Fréchet space. On the other hand, if E is an $(LF)_i$-space, (i=1,2) and F, a Fréchet space, then the $(LF)_i$-space $E \times F$, (i=1,2) has the Fréchet space F as a quotient.

The classical Separable Quotient Problem (for Banach spaces) asks whether every infinite-dimensional Banach space admits a Hausdorff quotient, (by a closed subspace) which is separable and infinite-dimensional. While this problem is still open, we have an affirmative answer to the corresponding problem for the class of (LF)-spaces.

**THEOREM 7** [16]

Every (LF)-space admits an infinite-dimensional, separable quotient.

The proof in [16] actually constructs the separable quotient. For the class of Banach/Fréchet spaces, we have the following equivalent formulation.

**THEOREM 8** ([16], [17], [18])

The following are equivalent for a Banach/Fréchet space E.

(a) E has a separable quotient;

(b) E has a dense, non-barrelled subspace;

(c) E has a dense, non-(db)-subspace;

(d) E has a dense $S_\sigma$-subspace (i.e., a union of a strictly increasing sequence of closed subspaces);

(e) E has a dense subspace which, with a topology stronger than the relative topology is a normable/metrizable (LF)-space;

(f) E has a dense, proper subspace which, with a topology stronger than the relative topology is a Banach/Fréchet space (Bennett-Kalton [4]).

It is a classical result of Eidelheit, (see [9], p.432) that every non-normable, Fréchet space has a quotient, isomorphic to $\omega$. Hence, it is clear that all such spaces possess separable quotients, and properties (a) through (f) of the above theorem (for Fréchet spaces) hold for them. Furthermore, by Example 2 of Section 1, $\omega$ contains a dense $(LF)_3$-subspace. Hence it follows, as observed in [22] that every non-normable Fréchet space, in particular all nuclear Fréchet spaces contain dense $(LF)_3$-subspaces. The next two theorems also enable us to construct $(LF)_3$-spaces.

THEOREM 9 [17]

Let E be a Fréchet space, with a sequence $\{P_n\}$ of orthogonal prtojections such that each of the (necessarily closed) subspaces $P_n[F]$ has a separable, Hausdorff, infinite-dimensional quotient. Then E contains a dense $(LF)_3$-subspace.

THEOREM 10 [17]

Let $q : E \to F$ be a continuous linear surjection of a Fréchet space E onto a Fréchet space F. Then F has a dense subspace $F_0$, which, with the relative topology, is an $(LF)_3$-space, if and only if E has a dense subspace $E_0$, which, with the relative topology is an $(LF)_3$-space, containing $q^{-1}[0]$.

Yet another classical problem is the splitting problem. A Banach space E *splits infinitely often* if there exist sequences $\{M_n\}$, $\{N_n\}$ of subspace of E such that $E=M_1 \oplus N_1$, $M_1 = M_2 \oplus N_2$, $M_2=M_3 \oplus N_3, \ldots$ Equivalently, there exists a sequence of orthogonal projections with infinite-dimensional ranges. Theorems 9 and 10 essentially state that

A Fréchet space E has a dense $(LF)_3$-subspace if

either E splits infinitely often, and each of the parts has a separable quotient, or, E has a separable quotient, that splits infinitely often.

Since non-normable Fréchet space always has $(LF)_3$-subspace, the above discussion (Theorems 9 and 10) are needed only for Banach spaces i.e., every Banach space will be the completion of some (LF)-space, provided the separable quotient problem and the splitting problem have affirmative answers for Banach spaces.

Independently, one can easily construct $(LF)_3$-subspaces of standard Banach spaces. If a Banach space E has an unconditional basis $\{x_n\}$, partition the natural numbers $\mathbb{N}$ into infinite disjoint sets $\{S_n\}$, and define $P_n : E \to E$ by $P_n(x) = \sum_{i \in S_n} a_i x_i$, where $x = \sum_{i=1}^{\infty} a_i x_i$. Then $\{P_n\}$ is a sequence of orthogonal projections, and each of the infinite dimensional subspaces $P_n[E]$ admit a separable quotient by the trivial subspace $\{0\}$. For $\ell_\infty$, $P_n[\ell_\infty] \cong \ell_\infty$ which is known to have a separable quotient. For $C[0,1]$ (which has no unconditional basis), choose a sequence $\{[a_n,b_n]\}_{n=1}^{\infty}$ of disjoint, non-degenerate subintervals of $[0,1]$, and set $a_n < c_n < d_n < b_n$ for each n. Define projections $P_n : C[0,1] \to C[0,1]$ by

$$P_n(f)(t) = \begin{cases} f(t) & c_n \leq t \leq d_n \\ 0 & t_n \notin (a_n,b_n) \\ \text{linear in } [a_n,c_n] \text{ and } (d_n,b_n] & \end{cases}$$

Each $P_n(C[0,1])$ is isomorphic to $C[0,1]$, which is infinite-dimensional and separable, with $\|P_n\| = 1$. Theorem 9 applies.

## REFERENCES

[1] I.AMEMIYA and Y.KŌMURA: Über nicht-vollständige Montelräume, *Math. Ann.* 177(1968), 273-277.

[2] J.ARIAS de REYNA: Dense hyperplanes of first category, *Math.Ann.* 249(1980), 111-114.

[3] J.ARIAS de REYNA: Normed barely Baire spaces, *Israel J. Math.* 42(1982), 33-36.

[4] G.BENNETT and N.J.KALTON: Inclusion theorems for FK-spaces, *Canadian J.Math.* 25(1973), 511-524.

[5] M. De WILDE and C.HOUET, On increasing sequences of Absolutely Convex Sets in Locally Convex Spaces, *Math. Ann.* 192(1972), 257-261.

[6] P.DIEROLF, S.DIEROLF and L.DREWNOWSKI, Remarks and examples concerning unordered Baire-like and ultrabarrelled spaces, *Colloq. Math.* 39(1978), 109-116.

[7] J.DIESTEL,S.A.MORRIS and S.A.SAXON: Varieties of Linear Topological Spaces, *Trans. Amer. Math. Soc.,* 172(1972), 207-230.

[8] K.FLORET: *Some aspects of the theory of locally convex inductive limits,* Functional Analysis: Surveys and Recent Results II, North Holland Mathematics Studies, Vol.38, Amsterdam, (1980), 205-237.

[9] G.KÖTHE: *Topological Vector Spaces I,* Springer-Verlag, Berlin-Heidelberg New York, 1969.

[10] P.P.NARAYANASWAMI and S.A.SAXON: (LF)-spaces, Quasi-Baire spaces and the Strongest Locally Convex Topology, *Math.Ann.* 274(1986), 627-641.

[11] A.P.ROBERTSON and W.ROBERTSON: On the closed graph theorem, *Proc. Glasgow Math. Assoc.*, 3(1956), 9-12.

[12] W.J.ROBERTSON, I.TWEDDLE and F.E.YEOMANS: On the stability of barrelled topologies, III, *Bull.Austral.Math.Soc.* 22(1980), 99-112.

[13] S.A.SAXON: Nuclear and Product spaces, Baire-like spaces and the strongest locally convex topology, *Math.Ann.* 197(1972), 87-106.

[14] S.A.SAXON: Some normed barrelled spaces which are not Baire, *Math. Ann.* 209(1974), 153-160.

[15] S.A.SAXON: Two characterizations of linear Baire spaces, *Proc. Amer.Math.Soc.*, 45(1974), 204-208.

[16] S.A.SAXON and P.P. NARAYANASWAMI: Metrizable (LF)-spaces, (db)-spaces and the Separable Quotient Problem; *Bull.Austral. Math. Soc.* 23(1981), 65-81.

[17] S.A.SAXON and P.P.NARAYANASWAMI: Metrizable [Normable] (LF)-spaces and two classical problems in Fréchet [Banach] spaces. (Submitted).

[18] S.A.SAXON and A.WILANSKY: The equivalence of some Banach space problems, *Colloq. Math.* 37(1977), 217-226.

[19] A.TODD and S.A.SAXON: A property of locally convex Baire-spaces, *Math. Ann.* 206(1973), 23-34.

[20] M.VALDIVIA: Absolutely convex sets in barrelled spaces, *Ann. Inst. Fourier, Grenoble,* 21(1971), 3-13.

[21] M.VALDIVIA: *On superbarrelled spaces, Functional Analysis, Holomorphy and Approximation Theory,* Lecture Notes in Mathematics, Springer-Verlag, Vol. **843**(1981), 572-580.

[22] M.VALDIVIA and P.P.CARRERAS, On Metrizable (LF)-spaces. *Collect. Math.* 33(1982), 299-303.

**Department of Mathematics**
**Memorial University of Newfoundland,**
**St John's, Newfoundland,**
**CANADA A1C 5S7**