Note di Matematica Vol. IX - n. 1, 95-143 (1989)

FINITE GROUPS ADMITTING A FIXED-POINT-FREE AUTOMORPHISM OF ORDER rst

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1. INTRODUCTION

Here we present a proof of the following

Theorem. ***Lei G be a finite group admitting a jìxed-point-free coprirne automorphism of order rst, where r, s and t are distinct primes and rst is a non-Fermat number. Then G is soluble.***

(A non-Fermat number is a positive integer which is not divisible by an integer of the form $2^m + 1$ ($m \geq 1$) ; note that there are infinitely many non-Fermat numbers which are the product of three distinct primes).

The above result appears in the author's thesis [4]. The condition that rst be a non-Fermat number was removed in subsequent work giving rise to the 'four-headed' hydra [5]-[8], and as a consequence [4] remained unpublished. Unfortunately, the minutia and the proliferazion of subcases in [5]-[8] somewhat obscures the direction of the proof. To have an account which better illustrates the development of these ideas, and also to serve as a guide for those wishing to traverse [5]-[8], is what prompted the present revised version of [4].

The proof of the above theorem proceeds by considering a counterexample G of minima1 order (let α denote the accompanying fixed-point-free automorphism) and endeavouring to show that certain α -invariant Hall subgroups of G permute with one another. The inconclusive information obtained in this direction, as evidenced by results in section 3, forces us to widen our horizons in the shape of linking theorems presented in section 4. Armed with the linking theorems we are able, in section 6, to show that G factorizes (in two possible ways) as a product of two α -invariant soluble Hall subgroups. In the final section these factorizations are analysed and shown to be untenable, which completes the proof of the theorem.

Now a few words on the rôle of the various intermediate results (for notation refer to section 2). Lemma 4.1, the quintessential linking result, is used frequently. While Theorem 4.3's only purpose is to help in showing that at least two of L_1, L_2 and L_3 permute (Lemma 6.2). The linking results Theorem 4.4 and Lemma 4.5 are used in conjunction with Theorem 5.1 to produce factorizations of G given in Theorem 6.3, and Theorem 4.4 is used again in Lemma 7.5.

Further discussion of ideas and strategies relevant to this work may be found in sections 1 and 2 of [5].

2. NOTATION

We use [5] as our basis reference and results ($y.x$) , Theorem y.x or Lemma y.x of [5] will, for brevity, all be referred to by $I(y.x)$. Below we review a little of the notation from [5]. For further relevant notation and concepts we refer the reader to [5] and for details conceming the Thompson subgroup to [Chapter 8, 3].

For the remainder of this paper G denotes a counterexample of minimal order to the theorem. Thus G admits a fixed-point-freecoprime automorphism, say α, of order rst where rst is a non-Fermat number. So all proper a-invariant subgroups of G are soluble and, by I(2.1) (i), G possesses no non-trivial proper α-invariant normal subgroups. Hence, appealing to [2], we see that (G, α) satisfies Hypothesis III of [section 2, 5].

We let ρ, cr, τ denote (respectively), α^{st}, α^{rt}, α^{rs}. Sometimes we choose to write $\rho = \alpha_1$, $\sigma = \alpha_2$ and $\tau = \alpha_3$. Let $\Lambda = \{1,2,3\} \supseteq$ A andlet P bean α-invariant Sylow p-subgroup of G. We say P is of type A if $P_{\alpha_i} \neq 1$ for $i \in$ A and $P_{\alpha_i} = 1$ for $i \notin$ A (where $P_{\alpha_i} = C_P(\alpha_i)$) . For $i \in \Lambda$ (respectively $\{i, j\} \subseteq \Lambda$), L_i (respectively L_{ij}) denotes the subgroup of G genemted by the α-invariant Sylow subgroups of type $\Lambda \setminus \{i\}$ (respectively $\Lambda \setminus \{i, j\}$). Set $\mathscr{L}_1 = L$, $L_{12} L_{13}$, $\mathscr{L}_2 = L_2 L_{12} L_{23}$ and $\mathscr{L}_3 = L_3 L_{13} L_{23}$. By I(3.13) $\mathscr{L}_1$, L_i and L_{ij} are all nilpotent Hall subgroups of G. Thus we have:

$$L_{1_\rho} = 1, L_{1_\sigma} \neq 1 \neq L_{1_\tau} \qquad \text{(if } L_1 \neq 1)$$

$$L_{2_\sigma} = 1, L_{2_\rho} \neq 1 \neq L_{2_\tau} \qquad \text{(if } L_2 \neq 1)$$

$$L_{3_\tau} = 1, L_{3_\rho} \neq 1 \neq L_{3_\sigma} \qquad \text{(if } L_3 \neq 1)$$

$$L_{12_\tau} \neq 1, L_{12_\rho} = 1 = L_{12_\sigma} \qquad \text{(if } L_{12} \neq 1)$$

$$L_{13_\sigma} \neq 1, L_{13_\rho} = 1 = L_{13_\tau} \qquad \text{(if } L_{13} \neq 1)$$

$$L_{23_\rho} \neq 1, L_{23_\sigma} = 1 = L_{23_\tau} \qquad \text{(if } L_{23} \neq 1)$$

We use L (instead of L_0 in [5]) to denote the subgroup of G generated by the α-invariant Sylow subgroups of type Λ . For $H \geq$ G, H^G denotes the normal closure of H in G.

In this work, since rst is a non-Fermat number, we see that I(5.3), I(5.7) and I(5.8) hold without the condition excluding the prime 2. However, a word of caution: I(5.5) differs from the above in its reliance upon I(2.23).

Suppose H is a proper α-invariant subgroup of G, and let X (respectively Y) be α-invariant λ -(respectively μ-) subgroups of H. Then (X, Y) $\leq H_{\lambda \cup \mu}$ (H_π , where π is a set of primes, denotes the unique α -invariant Hall π -subgroup of H). This observation, together with those in I(2.21), will be used without further mention.

3. THE STRUCTURE OF CERTAIN MAXIMAL α -INVARIANT SUBGROUPS

By I(2.22), if L and M are (respectively) α -invariant Hall λ - and μ -subgroups of G which do not permute, and $\lambda \cap \mu = \phi$, then $|\mathcal{M}(\lambda, \mu)| = 2$. The purpose of this section is to analyse the structure of the subgroups in $\mathcal{M}(\lambda, \mu)$ for various choices of λ and μ.

Lemma3.1. *Let $\Lambda = \{i, j, k\}$. If $L_iL_j \neq L_jL_i$, then $\mathcal{M}(\pi_i, \pi_j) = \{L_iN_{L_j}(L_i), L_jN_{L_i}(L_j)\}$ and either $L_{i_{\alpha_k}} \leq N_{L_i}(L_j)$ or $L_{j_{\alpha_k}} \leq N_{L_j}(L_i)$. Moreover, $[N_{L_i}(L_j), \alpha_j] \leq C_{L_i}(L_j)$ and $[N_{L_j}(L_i), \alpha_i] \leq C_{L_j}(L_i)$.*

Proof. By I(2.22) $\mathcal{M}(\pi_i, \pi_j) = \{L_i\mathcal{P}_{L_j}(L_i), L_j\mathcal{P}_{L_i}(L_j)\}$. Applying I(5.7) twice gives $\mathcal{P}_{L_i}(L_j) = N_{L_i}(L_j)$ and $\mathcal{P}_{L_j}(L_i) = N_{L_j}(L_i)$. Since $L_{i_{\alpha_k}} L_{j_{\alpha_k}} = (G_{\alpha_k})_{\pi_i \cup \pi_j}$ is an α-invariant $\{\pi_1 \cup \pi_2\}$-subgroup, it is clear that either $L_{i_{\alpha_k}} \leq N_{L_i}(L_j)$ or $L_{j_{\alpha_k}} \leq N_{L_j}(L_i)$. The remainder of the lemma follows using I(2.11).

Lemma 3.2. *Let P be an α-invariant Sylow p-subgroup* ***of*** *G* ***of*** *type Λ, and let $\Lambda = \{i, j, k\}$. If $PL_{ij} \neq L_{ij}P$, then $\mathcal{M}(p, \pi_{ij}) = \{P, L_{ij}N_P(L_{ij})\}$, and $1 \neq P_{\alpha_k} \leq C_P(L_{ij})$ and $[N_P(L_{ij}), \alpha_i\alpha_j] \leq C_P(L_{ij})$. (Hence $Z(P) = Z(P)_{\alpha_i\alpha_j} \leq N_P(L_{ij})$.).*

Proof. From I(3.13) (iii) $1 \neq P_{\alpha_k} \leq C_P(L_{ij})$. *Thus* $\mathcal{P}_{L_{ij}}(P) = 1$ by I(5.3) whence $\mathcal{P}_P(L_{ij}) = N_P(L_{ij})$ by I(2.20). By I(2.21) (iv) and I(5.1) (b) we have $Z(P) = Z(P)_{\alpha_i\alpha_j} \leq N_P(L_{ij})$, and $[N_P(L_{ij}), \alpha_i\alpha_j] \leq C_P(L_{ij})$ by I(2.11).

Lemma 3.3. *Suppose* ***$PL_1 \neq L_1 P$ where P is*** *an α-invariant* ***Sylow p-subgroup of type Λ***, *and set $\mathcal{M}(p, \pi_1) = \{PY, XL_1\}$. Then*

(i) Neither $P_\sigma \leq X$ and $L_{1_\tau} \leq Y$ nor $P_\tau \leq X$ and $L_{1_\sigma} \leq Y$ can hold.

(ii) Either $P_\sigma, P_\tau \leq X$ or $L_1^* \leq Y$.

Proof. (i) Suppose $P_\sigma \leq X$ and $L_{1_\tau} \leq Y$ holds. By I(5.7) $X = N_P(L_1)$. Because $Y \neq 1, O_p(XL_1) = 1$ by I(5.3) and so, using I(2.11) $P_\sigma \leq X \leq P_\rho$. Since X normalizes Y, I(2.14) (i) implies $L_1 = YC_{L_1}([X, \tau])$. Clearly $[X, \tau] \neq 1$ andso $PL_1 \neq L_1P$ forces $C_P([X, \tau]) \leq X$, whence $P = P_\rho$ by I(2.3) (v). But then $Y \trianglelefteq PY$ by I(2.3) (xi) and then (see I(2.21) (v)) $PL_1 = L_1 P$, a contradiction. So $P_\sigma \leq X$ and $L_{1_\tau} \leq Y$ cannot hold, and a similar argument rules out $P_\tau \leq X$ and $L_{1_\tau} \leq Y$.

(ii) This follows directly from (i).

Lemma 3.4. *Suppose $PL_1 \neq L_1 P$ where P is an α-invariant Sylow p-subgroup* ***of type*** *Λ, and set $\mathcal{M}(p, \pi_1) = \{PY, XL_1\}$.*

(i) If, furthermore, $L_1^* \leq Y$, then

(a) $\mathcal{M}(p, \pi_1) = \{PN_{L_1}(P), L_1\}$;

(b) $P_{\sigma\tau} = 1$;

(c) either $L_{1_\sigma} = L_{1_\tau}$ or $Z(L_1) \leq N_{L_1}(P)$;

(d) if $Z(L_1) \leq N_{L_1}(P)$, then $Z(L_1) = Z(L_1)_{\sigma\tau}$;

(e) $P_{\rho\sigma} \neq 1 \neq P_{\rho\tau}$; and

(f) P is not equal to $P_\rho P_\sigma$ or P_τ.

(ii) If, furthermore, $P_\sigma, P_\tau \leq X$, then

(a) $X = N_P(L_1)$ and $Y = N_{L_1}(P)$;

(b) $X = X_\rho C_P(L_1)$ and $[X, \rho] \leq C_P(L_1)$;

(c) if $C_P(L_1) \neq 1$, then $\mathcal{M}(p, \pi_1) = \{N_P(L_1)L_1, P\}$ and $Z(P) = Z(P)_\rho \leq X$;

(d) if $[X, \rho] \neq 1$, then $N_P(X)^* \leq X$;

(e) if P is star-covered, then $P = P_\rho$;

(f) is $C_P(L_1) = 1$, then $P^* = P_\rho \geq X, P_{\sigma\tau} = 1$ and $Y \leq L_{1_{\sigma\tau}}$; and

(g) if $L_1 = L_1^*$ and $X \leq P_\rho$, then $P = P_\rho$.

Proof. (i) (a). By I(2.21) (vi) and I(5.1)(d) X = 1 , and then $P \trianglelefteq PY$ by I(2.20). Thus $\mathcal{M}(p, \pi_1) = \{PN_{L_1}(P), L_1\}$.

(b) Since [L, , $P_{\sigma\tau}$] = 1, clearly $P_{\sigma\tau} \leq X = 1$.

(c) If $L_{1_\sigma} \neq L_{1_\tau}$, then we have, say $L_{1_\sigma} \nleq L_{1_\tau}$. Hence $0_{\pi_1}(P_\sigma L_{1\sigma}) \neq 1$ by I(4.5). Since $N_G(0_{\pi_1}(P_\sigma L_{1\sigma})) \geq Z(P)$, L_{1_σ} and X = 1, this forces $Z(P) \leq Y = N_{L_1}(P)$, as required.

(d) Since $Z(L_1) \leq N_{L_1}(P), Z(L_1)^* = Z(L_1)$ by I(5.1) (e). So if $Z(L_1) \neq Z(L_1)_{\sigma\tau}$, then, say $Z(L_1)_\sigma \nleq Z(L_1)_\tau$ which implies $Z(L_1) \cap 0_{\pi_1}(P_\sigma L_{1\sigma}) \neq 1$, contradicting X = 1. Therefore $Z(L_,) = Z(L_1)_{\sigma\tau}$.

(e) Suppose $P_{\rho\sigma} = 1$. Then $[P_\sigma, L_{1_\sigma}] = 1$ by (b) and I(2.8). Hence $Z(L_1) \leq N_{L_1}(P)$ by the shape of $\mathcal{M}(p, \pi_1)$. But then $Z(L_,) \leq L_{1_{\sigma\tau}}$ by (d) forces $P_\sigma \leq$ X = 1. Therefore $P_{\rho\sigma} \neq 1$ and, similarly, $P_{\rho\tau} \neq 1$.

(f) Clearly $P\# P_\sigma$ and $P \neq P_\tau$ since $P_{\sigma\tau} = 1$. While $P = P_\rho$ would imply Y $\trianglelefteq PY$, by I(2.3) (ix), contradicting $PL, \neq L_1P$. So $P \neq P_\rho$.

(ii) If $O_p(XL_1) = 1$, then $L_1 \trianglelefteq L_1X$ and X $\leq P_\rho$ by I(2.13). Hence Y centralizes $O_p(PX)$, and $O_p(PX)$, . Now X $\neq 1$, I(5.3) and I(2.11) yield Y $\leq L_{1_{\sigma\tau}}$. From X $\leq P_\rho$ and Y $\leq L_{1_{\sigma\tau}}$ we obtain $[X, \mathrm{Y}] = 1$ and thus $P \trianglelefteq PY$ by I(2.20). Whilst, if $O_p(XL_1) \neq 1$, then Y = 1 by I(5.3) whence $L_1 \trianglelefteq$ X L_1 by I(2.20). These remarks establish (a), (c) and (f). Part (b) follows from I(2.13), and (b) and I(2.3) (viii) yield (d).

(e) By (d) [X, p] $\neq 1$ is not possible. Therefore $P = P^* = P_\rho$, as required.

(g) Suppose $P \neq P_\rho$ and argue for a contradiction. Put $\overline{L}_1 = L_1/\phi(L_1)$. By I(3.3) (vi) q = $\overline{L}_1^* = \overline{L}_{1_\sigma} \overline{L}_{1_\tau}$. Because $P_\sigma \leq N_P(L_1)$ by (a), P_σ acts upon $\overline{L}_1$ and $\overline{L}_{1_\sigma}$. Applying I(2.3) (x) to $P_\sigma(\overline{L}_1/\overline{L}_{1_\sigma})$ gives, as $P_\sigma \leq X \leq P_\rho, \overline{L}_1 = \overline{L}_{1_\sigma} C_{\overline{L}_1}(P_\sigma)$. From $P \neq P_\rho$ and I(2.3) (v) $C_P(P_\rho) \nleq$ X and thus $C_{L_1}(P_\sigma) \leq$ Y $\leq L_{1_{\sigma\tau}}$ by (c) and (f). Therefore, as $C_{\overline{L}_1}(P_\sigma) = \overline{C_{L_1}(P_\sigma)}$, we deduce $\overline{L}_1 = \overline{L}_{1_\sigma}$. Hence $L_1 = L_{1_\sigma}$ by [Theorem 5.14; 3] and by a similar argument $L_1 = L_{1_\tau}$. 'Now I(2.3) (xi) gives [L, , X] = 1, a contradiction. Therefore $L_1 = L_1^*$ and $X \leq P_\rho$ implythat $P = P_\rho$.

Remark. Clearly there are results analagous to Lemmas 3.3 and 3.4 for L_2 and L_3.

We now examine the behaviour between α-invariant Sylow subgroups of typc Λ.

Lemma 3.5. ***Let P and Q be cu-invariant Sylow p- and q-subgroups of G of type Λ which do not permute, and let $\mathscr{M}(p, q) = \{PY, QX\}$. Then, with possible interchanging of p and q and rearrangement of p, σ and τ, one of the following occurs:***

(i) $P^* \leq X$, andfurthermore

(a) $\mathscr{M}(p,q) = \{P, N_P(Q)Q\}$;

(b) $Z(P) \leq N_P(Q)$;

(c) Z(P) is contained in one of $P_{\sigma\tau}, P_{\rho\sigma}$ or $P_{\rho\tau}$;

(d) (suppose. in (c), that $Z(P) \leq P_{\sigma\tau}$) $Q_{\sigma\tau} = 1$ and $Q_{\rho\sigma} \neq 1 \neq Q_{\rho\tau}$;

(e) Q is not equal to Q_ρ, Q_σ or Q_τ; or

(ii) $P_\rho \leq X$ and $Q_\sigma, Q_\tau \leq Y$, and furthermore

(a) p = 2;

(b) $Y < Q_\rho = Q^* \neq Q$ (and so Q is not star-covered);

(c) $Q_{\sigma\tau} = 1$ and $Q_{\rho\sigma} \neq 1 \neq Q_{\rho\tau}$;

(d) for all non-trivial cu-invariant subgroups R of P_ρ, $N_P(R) \leq X$;

(e) $Z(P) \leq X_{\sigma\tau}$;

(f) $1 \neq [X,\sigma] \leq P_\rho, 1 \neq [X,\tau] \leq P_\rho$ and $[X,\rho] \leq X_{\sigma\tau}$;

(g) $X = N_P(Q)$;

(h) $N_P(X)^* \leq X$ (and so P is not star-covered); and

(i) either P is contained in a unique maximal cu-invariant subgroup of G or $J(P)_\rho = 1$.

Proof. Clearly, up to relabelling, either $P^* \leq$ X or $P_\rho \leq$ X and $Q_\rho, Q_\tau \leq$ Y. We now prove the statements in (i). So assume $P^* \leq$ X . By I(2.21) (vi) and I(5.1) (d) Y = 1, whence Q $\trianglelefteq$ QX by I(2.20). Hence (a) holds, Combining (a) with I(3.14) gives (b).

We now prove (c). If $Z(P)_\rho \neq Z(P)^*_{\rho\langle\sigma\tau\rangle}$, then Z(P) $\cap O_p(P_\rho Q_\rho) \neq 1$ by (I(4.5). Hence, as $Q_\rho \trianglelefteq P_\rho Q_\rho$, we obtain $Q_\rho \leq$ Y, contradicting (a). Therefore $Z(P)_\rho = Z(P)^*_{\rho\langle\sigma\tau\rangle}$ and, similarly, Z(P), = $Z(P)^*_{\sigma\langle\rho\tau\rangle}$ and Z(P), = $Z(P)^*_{\tau\langle\rho\sigma\rangle}$. We claim that at least two of $Z(P)_{\rho\sigma}$, $Z(P)_{\rho\tau}$ and Z(P),, aretrivial. For suppose, say, that Z(P),, $\neq 1 \neq Z(P)_{\rho\tau}$. Then, as $G_{\rho\sigma}$ and $G_{\rho\tau}$ are nilpotent and Y = 1, $Q_{\rho\sigma} = 1 = Q_{\rho\tau}$. Hence $[P_\rho, Q_\rho] = 1$ by I(2.8) which then yields $Q_\rho \leq C_Q(Z(P),) \leq$ Y , a contradiction. So, without loss of generality, we may assume $Z(P)_{\rho\tau} = 1 = Z(P)_{\rho\sigma}$. This then implies $Z(P)_\rho = 1$ and Z(P), = Z(P), , and so Z(P)* = Z(P),,. Since Z(P) $\leq X$ by (b), I(5.1) (e) gives Z(P) = Z(P)* = $Z(P)_{\sigma\tau}$, which proves (c).

Because Z(P) $\leq P_{\sigma\tau}$ and Y = 1, clearly $Q_{\sigma\tau} = 1$. If, say, $Q_{\rho\sigma} = 1$, then $[P_\sigma, Q_\sigma] = 1$ by I(2.8), which is at variance with Y = 1 . Therefore $Q_{\rho\sigma} \neq 1$ and, likewise, $Q_{\rho\tau} \neq 1$. Next we consider (d). Since $Q_{\sigma\tau} = 1$ clearly $Q_\sigma \neq Q \neq$ Q,. Suppose Q = Q_ρ were to hold. Then, by I(2.3) (ix), Z(P) = [Z(P), pl $\leq [X$, pl $\leq O_p(XQ)$, which contradicts $PQ \neq$ QP. So we also have Q $\neq Q_\rho$, and this finishes (i).

Now we suppose $P_\rho \leq$ X and P_σ, $P_\tau \leq$ Y. If $p \neq 2$, then, since Y $\neq 1$, a double application of I(5.5) gives $P_\rho \leq X \leq P_{\sigma\tau}$ wich is not possible. Therefore p = 2, and we have (ii) (a). Using I(5.5) again, as $q \neq 2$ and $X \neq 1$, yields Y $\leq Q_\rho$. Thus $Q^* =$ Q,. Next we prove that Q $\neq Q_\rho$. Suppose that Q = Q_ρ and argue for a contradiction. Because Y $\neq 1$, O_p(QX) = 1 by I(5.3). Hence QX $\leq G_\rho$ by I(2.3) (ix). Consequently, as $Q^*_{\langle\sigma\tau\rangle} \leq Y$, I(5.1) (d) yields Q = $YC_Q(X)$, whence Q = Y. From this contradiction we deduce that Q $\neq Q_\rho$. Clearly, by (i) (a) and X $\neq 1$, Y $< Q^*$ and so we have verified (b). Evidently (b) implies (c).

Combining I(2.14) (ii) and I(4.5) we obtain

$$Q = O_q(QX)Q^* = O_q(QX)Q_\rho = C_Q(P_\rho)Q_\rho.$$

Since $Q \neq Q_\rho$ by (ii) (b), $C_Q(P_\rho) \not\leq$ Y , from which (d) follows. From (d) we clearly have (e).

Before proceeding further we show

(3.1) $$X = X_{\sigma\tau}P_\rho, [X_{\sigma\tau}, Y] = 1 \quad \text{and} \quad P_\rho = P_{\rho\sigma}P_{\rho\tau}$$

Since O_p(PY) $\cap X$ centralizes 0,,(QX) $\cap Y \geq O_q$(QX),, O_q(QX), , I(2.14) (i) and I(5.3) yield that $O_p(PY) \cap X \leq P_{\sigma\tau}$. Hence $O_p(XY) \leq P_{\sigma\tau}$ by I(2.21) (ii). From Y $\leq Q_\rho$ and I(2.3) (ix) we obtain X = $X_{\sigma\tau}P_\rho$ and $O_p(XY)$ = X,,. So $[X_{\sigma\tau}, Y] = 1$ by I(2.3) (xi). Also we see that O_p((XY),) = 1 . Hence $P_\rho = P_{\rho\sigma}P_{\rho\tau}$ by I(2.10) (iii), and so (3.1) holds.

If X $\leq P_\sigma$ were to hold, then (d) and I(2.3) (v) imply $P = P_\sigma$ whence, since $O_q(PY)$ = 1, Y $\leq Q_\sigma$ by I(2.3) (ix). But then $Q_\tau \leq Y \leq Q_{\rho\sigma}$, a contradiction. Therefore $[X, \sigma] \neq 1$ and, similarly, [X, $\tau] \neq 1$. The remainder of (f) follows from (3.1).

Using (3.1) and I(2.10) (ii) gives $Q \trianglelefteq QX$ and then $X = N_P(Q)$. Combining (d), (f) and I(2.3) (viii) yields (h). Finally we prove (i). Suppose $J(P)_\rho \neq 1$. Then $R = Z(J(P)) \leq X$ by part (d). From (3.1) we see that $R_\rho = R_{\rho\sigma}R_{\rho\tau}$, $R_\sigma = R_{\rho\sigma}R_{\sigma\tau}$ and $R_\tau = R_{\rho\tau}R_{\sigma\tau}$. This together with (h) and I(2.6), I(4.7) and I(6.4), yields that P is contained in a unique maximal α-invariant subgroup of G, so proving (i).

4. LINKING THEOREMS

In this section we use the results of the previous section to analyse configumtions involving three or more α-invariant nilpotent Hall subgroups.

Lemma 4.1. *Let P be an α-invariant Sylow p-subgroup **of** G **of** type Λ and let $i, j \in \Lambda$ with $i \neq j$. Then at least two **of** P, L_i and L_j permute.*

Proof. Suppose the lemma is false and, without loss of generality, that $i = 1$ and $j = 2$. Thus we are supposing

$$L_1L_2 \neq L_2L_1, \quad PL_1 \neq L_1P \quad \text{and} \quad PL_2 \neq L_2P.$$

The proof is broken up into cases depending on the form of $\mathcal{M}(p, \pi_1)$ and $\mathcal{M}(p, \pi_2)$. Let $\mathcal{M}(p, \pi_k) = \{PY_k, L_kX_k\}$ for $k = 1, 2$; by Lemma 3.4 $Y_k = N_{L_k}(P)$ and $X_k = N_P(L_k)$.

Case 1. $P_\sigma, P_\tau \leq N_P(L_1)$ and $P_,,, P_\tau \leq N_P(L,)$.

First we consider the possibility $C_P(L,) = 1 = C_P(L_2)$. Applying Lemma 3.4 (ii) (f) to both L_1X_1 and L_2X_2 gives $P_\rho = P^* = P_\sigma$. But then $1 \neq P_\tau = C_P(\alpha)$ contradicts α acting fixed-point-freely upon G. Thus, at least one of $C_P(L,)$ and $C_P(L_2)$ must be non-trivial. Without loss of generality we may assume $C_P(L_1) \neq 1$. Hence $Z(P) = Z(P), \leq N_P(L_1)$ by Lemma 3.4 (ii) (c). Therefore $Z(P) \leq P_\rho \leq N_P(L_2)$ and consequently, by I(5.1) (b), $Z(P) = Z(P)_\sigma$. Thus $Z(P)$, $= 1$ and $Z(P) \leq N_P(L_1) \cap N_P(L_2)$. Clearly $Z(P)$ normalizes both $N_{L_1}(L,)$ and $N_{L_2}(L,)$. Since $L_1L_2 \neq L_2L_1$, either $L_{1_\tau} \leq N_{L_1}(L_2)$ or $L_{2_\tau} \leq N_{L_2}(L,)$ by Lemma 3.1. Suppose (say) that $L_{1_\tau} \leq N_{L_1}(L_2)$ holds. Then, since $Z(P)$, $= 1$, I(2.14) (i) applied to $Z(P)$ normalizing L_1 and $N_{L_1}(L_2)$ gives $L_1 = N_{L_1}(L_2)C_{L_1}(Z(P))$. *Now* $C_{L_1}(Z(P)) \leq N_{L_1}(P) \leq L_{1_{\sigma\tau}}$ by Lemma 3.4 (ii) (c) and (f) and so

$$L_1 = N_{L_1}(L_2)C_{L_1}(Z(P)) = N_{L_1}(L_2)L_{1_{\sigma\tau}} = N_{L_1}(L_2).$$

This contradicts $L, L_2 \neq L_2L_1$, and so disposes of case 1.

Case 2. $P_\sigma, P_\tau \leq N_P(L_1)$ and $L_{2_\rho}, L_{2_\tau} \leq N_{L_2}(P)$.

Since $L_1 L_2 \neq L_2 L_1$, either $L_{1_\tau} \leq N_{L_1}(L_2)$ or $L_{2_\tau} \leq N_{L_2}(L_1)$ holds. Suppose for the moment that $L_{2_\tau} < N_{L_2}(L_1)$ pertains. Then $L_{2_\tau} \leq N_{L_2}(L_1) \cap N_{L_2}(P)$ and so L_{2_τ} normalizes $N_P(L_1)$. Using T(2.14) (i) yields, since $P_\sigma \leq N_P(L_1)$, that $P = N_P(L_1) C_P(L_{2_\tau})$. Now, appealing to Lemma 3.4(i) (c) and (d), gives that either $L_{2_\tau} = L_{2_\rho} = L_2^*$ or $Z(L_2) = Z(L_2)_{\rho\tau}$. In either case (using I(3.6) (iii) for the former) we deduce that $P = N_P(L_1) C_P(L_{2_\tau}) = N_P(L_1)$, which is not possible. Thus $L_{2_\tau} \leq N_{L_2}(L_1)$ is untenable and so we have $L_{1_\tau} \leq N_{L_1}(L_2)$. Inparticular, $N_{L_1}(L_2) \neq 1$. From Lemma 3.4 (i) (c)and (d) applied to P and L_2 wehave that either $Z(L_2) = Z(L_2)_{\rho\tau}$ or $L_{2_\rho} = L_{2_\tau}$. Suppose $Z(L_2) = Z(L_2)_{\rho\tau}$ holds. Then 1(2.3)(x) applied to $N_{L_1}(L_2) Z(L_2)$ gives $[N_{L_1}(L_2), Z(L_2)] = 1$, and hence, since $N_{L_1}(L_2) \neq 1$, $Z(L_2) \leq C_{L_2}(N_{L_1}(L_2)) \leq N_{L_2}(L_1)$. Therefore

$$Z(L_2) \leq N_{L_2}(L_1) \cap L_{2_{\rho\tau}} \leq N_{L_2}(L_1) \cap N_{L_2}(P),$$

and so $Z(L_2)$ normalizes $N_P(L_1) \geq P_\sigma$. Hence $P = N_P(L_1) C_P(Z(L_2)) = N_P(L_1)$, since $C_P(Z(L_2)) = 1$ by Lemma 3.4(i) (a). Thus $Z(L_2) = Z(L_2)_{\rho\tau}$ cannot hold. Now $L_{2_\rho} = L_{2_\tau}$ yields, using I(6.4), that $N_{L_1}(L_2) \trianglelefteq N_{L_1}(L_2) L_2$ whence, since $N_{L_1}(L_2) \neq 1$, I(2.21) (v) implies that $L_1 L_2 = L_2 L_1$. *Thus* $L_{2_\rho} = L_{2_\tau}$ is also untenable, and this deals with case 2.

Case3. $L_1^* \leq N_{L_1}(P)$ and $L_2^* \leq N_{L_2}(P)$.

A double application of Lemma 3.4(i)(b) and (e) yields $P_{\sigma\tau} = 1$, $P_{\rho\sigma} \neq 1 \neq P_{\rho\tau}$ and $P_{\rho\tau} = 1$, $P_{\sigma\tau} \neq 1 \neq P_{\rho\sigma}$. Clearly this situation is impossible.

As the possibility $L_{1_\sigma}, L_{1_\tau} \leq N_{L_1}(P)$ and $P_\rho, P_\tau \leq N_P(L_1)$ may be dealt with as in case 2 we see that all the altematives for $\mathcal{M}(p, \pi_1)$ and $\mathcal{M}(p, \pi_2)$, as given by Lemma 3.3, yield a contradiction, as required.

The next result will be required in the proof of Theorem 4.3. Lemma 4.22 is a special case of I(5.10) (b), however we give a proof here.

Lemma *4.2. Suppose $L_i L_j \neq L_j L_i$ and $L_j L_k \neq L_k L_j$ where $\{i, j, k\} = \Lambda$. If J is a non-trivial α-invariant subgroup of $N_{L_k}(L_i) \cap N_{L_k}(L_j)$ and $L_{j\alpha_k} \leq N_{L_j}(L_i)$, then $C_{L_j}(J) \not\leq N_{L_j}(L_k)$.*

Proof. Without loss of generality we set $i = 1$, $j = 2$, and $k = 3$. *So we* have $L_1 L_2 \neq L_2 L_1$, $L_2 L_3 \neq L_3 L_2$, $J \leq N_{L_3}(L_1) \cap N_{L_3}(L_2)$ and $L_{2_\tau} \leq N_{L_2}(L_1)$. Suppose $C_{L_2}(J) \leq N_{L_2}(L_3)$, and argue for a contradiction.

Since J normalizes L_1 and L_2, J must normalize $N_{L_2}(L_1)$. Hence, as $L_{2_\tau} \leq N_{L_2}(L_1)$, $J_\tau = 1$ and J normalizes L_2, I(2.14) (i) gives

$$L_2 = C_{L_2}(J)N_{L_2}(L_1) = N_{L_2}(L_3)N_{L_2}(L_1).$$

Since $L_1 L_2 \neq L_2 L_1$, clearly $N_{L_2}(L_3) \not\leq N_{L_2}(L_1)$. Therefore $N_{L_2}(L_3) \not\leq L_{2_\tau}$. Hence $0_\pi(L_1 N_{L_2}(L_3)) \neq 1$ by X(2.33). But then $\mathscr{P}_{L_3}(L_2) = N_{L_3}(L_2) = 1$ by I(5.3), contrary to $J \neq 1$. Then we conclude that $C_{L_2}(J) \not\leq N_{L_2}(L_1)$, as desired.

Theorem 4.3. *Assume that $L_iL_j \neq L_jL_i$ for all $i, j \in \Lambda$ with $i \neq j$. Then one of the following holds:*

(i) $L_{1_\sigma} = L_1, L_{2_\tau} = L_2, L_{3_\rho} = L_3$.

(ii) $L_{1_\tau} = L_1, L_{2_\rho} = L_2, L_{3_\sigma} = L_3$.

Proof. By Lemma 3.1 we have that $\mathscr{M}(\pi_i, \pi_j) = \{L_iN_{L_j}(L_i), L_jN_{L_i}(L_j)\}$.

First we establish

$$\langle L_{2_\tau}, L_{3_\sigma}\rangle \not\leq N_G(L_1). \tag{4.1}$$

Supposing $\langle L_{2_\tau}, L_{3_\sigma}\rangle \leq N_G(L_1)$ we seek a contradiction. Without loss of generality we may assume that $\{N_G(L_1)\}_{\pi_2,\pi_3} \leq L_2 N_{L_3}(L_2)$. So

$$L_{3_\sigma} \leq N_G(L_1) \cap L_3 = N_{L_3}(L_1) \leq N_{L_3}(L_2).$$

Applying Lemma 4.2 with i = 1, j = 2, k = 3 and $J = L_{3_\sigma}$ yields

$$C_{L_2}(L_{3_\sigma}) \not\leq N_{L_2}(L_3). \tag{4.2}$$

From (4.2) we deduce that $Z(L_3)_\sigma = 1$ and that $Z(L_3) \leq N_{L_3}(L_2)$. Hence, as $L_{2_\sigma} = 1$, σ acts fixed-point-freely upon $Z(L_3)L_2$, and so $[Z(L_3), L_2] = 1$ by I(2.2) (i). But then $\langle L_3, L_2\rangle \leq C_G(Z(L_3))$, contrary to $L_2L_3 \neq L_3L_2$. This is the desired contradiction, and so we have proved (4.1).

The arguments used to prove (4.1) also yield

$$\langle L_{1_\tau}, L_{3_\rho}\rangle \not\leq N_G(L_2) \quad \text{and} \quad \langle L_{1_\sigma}, L_{2_\rho}\rangle \not\leq N_G(L_3). \tag{4.3}$$

The form of $\mathscr{M}(\pi_i, \pi_j)$ together with (4.1) and (4.3) imply that one of the following must hold:

(4.4) $$L_{1_\tau} \le N_{L_1}(L_2), L_{2_\rho} \le N_{L_2}(L_3) \quad \text{and} \quad L_{3_\sigma} \le N_{L_3}(L_1);$$

or

(4.5) $$L_{1_\sigma} \le N_{L_1}(L_3), L_{2_\tau} \le N_{L_2}(L_1) \quad \text{and} \quad L_{3_\rho} \le N_{L_3}(L_2).$$

Since the ensuing arguments apply equally to (4.4) and (4.5) we shall suppose, without loss of generaly, that case (4.4) holds.

(4.6) If $L_1^* = L_{1_\sigma}$, then $L_1 = L_{1_\sigma}$.

Since $L_{3_\sigma} \le N_{L_3}(L_1)$ (by (4.4)), L_{3_σ} normalizes $L_{1_\sigma} = L_1^*$ and so $L_1 = L_{1_\sigma} C_{L_1}(L_{3_\sigma})$ by I(2.14) (ii). Supposing $L_1 \ne L_{1_\sigma}$ we argue for a contradiction. Clearly we must have $C_{L_1}(L_{3_\sigma}) \not\le L_{1_\sigma} = L_1^*$. If $C_{L_1}(L_{3_\sigma}) \le N_{L_1}(L_3)$, then I(4.5) forces $O_{\pi_1}(L_3 N_{L_1}(L_3)) \ne 1$. But then $N_{L_3}(L_1) = 1$ by I(5.3) whereas $1 \ne L_{3_\sigma} \le N_{L_3}(L_1)$. Thus we conclude that

(4.7) $$C_{L_1}(L_{3_\sigma}) \not\le N_{L_1}(L_3).$$

Hence $Z(L_3) \le N_{L_3}(L_1)$ and $Z(L_3)_\sigma = 1$ by (4.7). Thus σ acts fixed-point-freely upon $Z(L_3)N_{L_2}(L_3)$ andso $[Z(L_3), N_{L_2}(L_3)] = 1$ by I(2.2) (i). Since $N_{L_2}(L_3) \ne 1$ by (4.4), this implies that $Z(L_3) \le N_{L_3}(L_2)$.

Therefore we have

(4.8) $$Z(L_3) \le N_{L_3}(L_1) \cap N_{L_3}(L_2) \quad \text{and} \quad L_{1_\tau} \le N_{L_1}(L_2).$$

However (4.8) is at variance with Lemma 4.2 (taking $J = Z(L_3)$, $i = 2$, $j = 1$ and $k = 3$). This is the desired contradiction, and so we have (4.6).

Clearly the arguments used in proving (4.6) will also yield

(4.9) If $L_2^* = L_{2_\tau}$ (respectively $L_3^* = L_{3_\rho}$), then $L_2 = L_{2_\tau}$ (respectively $L_3 = L_{3_\rho}$).

We now show that

(4.10) $$L_1 = L_{1_\sigma}$$

Assuming $L_1 \neq L_{1_\sigma}$ we seek a contradiction. Thus, by (4.6), $L_1^* \neq L_{1_\sigma}$ and consequently, as $L_{1_\tau} \leq N_{L_1}(L_2)$, we have $N_{L_1}(L_2) \not\leq L_{1_\sigma}$. Therefore, using I(2.13) (i), we obtain

(4 .11) $$O_{\pi_1}(L_2 N_{L_1}(L_2)) \neq 1.$$

I(5.3) and (4.11) imply

(4.12) $$N_{L_2}(L_1) = 1.$$

Also from (4.11) we infer that

(4.13) $$Z(L_1) = Z(L_1)_\sigma \leq N_{L_1}(L_2).$$

Lemma4.2, together with (4.13) and $L_{2_\rho} \leq N_{L_2}(L_3)$ (taking $J = Z(L_1)$), forces

(4.14) $$Z(L_1) \not\leq N_{L_1}(L_3).$$

We now turn our attention to L_3 and prove that

(4.15) $$L_{3_\sigma} = N_{L_3}(L_1)$$

Suppose (4.15) were false. Then $[N_{L_3}(L_1), \sigma] \neq 1$. From I(2.3) (x) and (4.13) we have $[Z(L_1), [N_{L_3}(L_1), \sigma]] = 1$, and then (4.14) dictates that

$$C_{L_3}([N_{L_3}(L_1), \sigma]) \leq N_{L_3}(L_1).$$

In particular, $Z(L_3) \leq N_{L_3}(L_1)$,andso $[Z(L_3), \sigma] \leq [N_{L_3}(L_1), \sigma]$. Hence $[Z(L_3), \sigma] \neq 1$ would imply $Z(L_1) \leq N_{L_1}(L_3)$, contradicting (4.14). So $Z(L_3) = Z(L_3)_\sigma$. By considering $Z(L_3) N_{L_2}(L_3)$,1(2.3)(x) yields that $[Z(L_3), N_{L_2}(L_3)] = 1$.

Therefore, since $N_{L_2}(L_1) \neq 1$, we see that $Z(L_3) \leq N_{L_3}(L_2)$. So we have $Z(L_3) \leq N_{L_3}(L_2) \cap N_{L_3}(L_1)$ and $L_{1_\tau} \leq N_{L_1}(L_2)$ which is against Lemma 4.2. With this contradiction we have established (4.15).

If $O_{\pi_3}(L_1 N_{L_3}(L_1)) \neq 1$, then $C_{L_3}(N_{L_3}(L_1)) \leq N_{L_3}(L_1)$ and thence, by (4.15) and 1(2.3)(v), $L_3 = L_{3_\sigma} = N_{L_3}(L_1)$, which contradicts $L_1 L_3 \neq L_3 L_1$. Hence $O_{\pi_3}(L_1 N_{L_3}(L_1)) = 1$, and so $N_{L_3}(L_1) \leq L_{3_\rho}$ by 1(2.13)(i). Therefore $L_3^* = L_{3_\rho}$ and then $L_3 = L_{3_\rho}$ by (4.9). We claim that $O_{\pi_2}(L_3 N_{L_2}(L_3)) = 1$. For $O_{\pi_2}(L_3 N_{L_2}(L_3)) \neq 1$ gives $Z(L_2) \leq N_{L_2}(L_3)$, and then $L_3 L_2 \neq L_2 L_3$, $L_3 = L_{3_\rho}$ and I(2.3) (x) imply that $Z(L_2) = Z(L_2)_\rho$. Applying 1(2.3)(x) to $Z(L_2) N_{L_1}(L,)$ yields $[Z(L_2), N_{L_1}(L_2)] = 1$. Since $N_{L_1}(L,) \neq 1$ we then obtain $Z(L_2) \leq N_{L_2}(L,)$. But $N_{L_2}(L,) = 1$ by (4.12) and so we see that $O_{\pi_2}(L, N_{L_2}(L_3)) \neq 1$ is untenable, so verifying the claim.

From $O_{\pi_2}(L_3 N_{L_2}(L_3)) = 1$, 1(2.13)(i) gives $N_{L_2}(L_3) \leq L_{2_\tau}$ and so $L_2^* = L_{2_\tau}$. By (4.9) $L_2 = L_{2_\tau}$. *Now* $Z(L_1) \leq N_{L_1}(L_2)$ by (4.13), and so $L_1 L_2 \neq L_2 L_1$ and I(2.3)(x) give $Z(L_1) = Z(L_1)_\tau$. Applying I(2.13) (x) to $Z(L_1) N_{L_3}(L_1)$ gives $[Z(L_1), N_{L_3}(L_1] = 1$ whence, as $N_{L_3}(L_1) \neq 1, Z(L_1) \leq N_{L_1}(L_3)$. But by (4.14) $Z(L_1) \not\leq N_{L_1}(L_3)$. This is the desired contradiction, and so we have verified (4.10).

A similar argument will establish that $L_2 = L_{2_\tau}$ and $L_3 = L_{3_\rho}$ so giving case (i) of the theorem. We observe that (4.5) will give rise to case (ii), and so the proof of Theorem 4.3 is complete.

Theorem 4.4. *Let P and Q be (respectively)* α*-invariant Sylow p- and q -subgroups* **of** *type* Λ, $p \neq q$, *and let* $i, j \in \Lambda, i \neq j$. *If* $PQ = QP, PL_j = L_j P$ *and* $QL_i = L_i Q$, *then at least one of* $PL_i = L_i P$ *and* $QL_j = L_j Q$ *holds.*

Proof. Suppose the theorem is false, and, without loss of generality, that $i = 1$ and $j = 2$. So the following is assumed to hold:

(4.16) $\quad PQ = QP,\ PL, = L_2 P, QL_1 = L, Q, PL, \neq L_1 P$ and $QL, \neq L_2 Q$.

We derive a contradiction in the following series of statements.

(4.17) $\quad L_1^* \leq N_{L_1}(P)$ and $L_2^* \leq N_{L_2}(Q)$ cannot both hold at the same time.

Suppose $L_1^* \leq N_{L_1}(P)$ and $L_2^* \leq N_{L_2}(Q)$ hold. By Lemma 3.4 (i)(a) and (b) $\mathcal{M}(p, \pi_1) = \{PN_{L_1}(P), L_1\}, \mathcal{M}(q, \pi_2) = \{QN_{L_2}(Q), L_2\}$ and $P_{\sigma\tau} = 1 = Q_{\rho\tau}$. So $\sigma\tau$ and $\rho\tau$ act (respectively) fixed-point-freely upon PL_2 and QL_1. Consequently, as $L^*_{1\langle\rho\tau\rangle} = L_{1_\tau}, L^*_{2\langle\sigma\tau\rangle} = L_{2_\tau}$ and $(PL,)$, and $(QL,)$, are nilpotent, I(3.7) gives

$$[P_\tau, L_2] = 1 = [Q_\tau, L_1].$$

Because $P_\tau Q_\tau$ is soluble, without loss of generality, we must have $O_p(P_\tau Q_\tau) \neq 1$. Hence

$$Q_\tau, L_1 \leq N_G(O_p(P_\tau Q_\tau))$$

whence $Q_\tau \leq \mathscr{P}_Q(L_1) = 1$, which is not possible. This verifies (4.17).

Before proceeding further we investigate the interaction between L_1 and L_2.

(4.18) $$L_1 L_2 \neq L_2 L_1.$$

Suppose $L_1 L_2 = L_2 L_1$ holds. Because of (4.17) and Lemma 3.3 it may be assumed that (say) $Q_\rho, Q_\tau \leq N_Q(L_2)$. Employing I(5.8) (f) with $\gamma = \rho$, $L = L_1$, $M = Q$ and $N = L_2$ (note that $G \neq L_2(L_1 Q)$ since $P \neq 1$) yields $O_{\pi_1}(L_1 L_2) = 1$, whence $L_1 = L_{1_\sigma}$ by I(2.13) (i). Consequently, by Lemma 3.3, we must have $P_\sigma, P_\tau \leq N_P(L_1)$. A further application of I(5.8)(f) with $\gamma = \sigma$, $L = L_2$, $M = P$ and $N = L_1$ gives $O_{\pi_2}(L_1 L_2) = 1$. But then $F(L_1 L_2) = 1$, which contradicts a well-known property of soluble groups. Hence we must have $L_1 L_2 \neq L_2 L_1$.

Our next two assertions prepare the ground for our later work.

(4.19) If P (respectively Q) is not star covered, then $Q_\rho, Q_\tau \leq N_Q(L_2)$ (respectively $P_\sigma, P_\tau \leq N_P(L_1)$).

Suppose $L_2^* \leq N_{L_2}(Q)$ were to hold. Then applying I(5.8) (f) with $\gamma = \alpha$, $L = P$, $M = L_2$ and $N = Q$ gives that $O_p(PQ) = 1$. Hence P is star-covered by I(4.4), contrary to the hypothesis of (4.19). Thus $L_2^* \not\leq N_{L_2}(Q)$ and so, by Lemma 3.3, $Q_\rho, Q_\tau \leq N_Q(L_2)$, as required.

From Lemma 3.4(ii) (e) we have

(4.20) If $P_\sigma, P_\tau \leq N_P(L_1)$ (respectively $Q_\rho, Q_\tau \leq N_Q(L_2)$) and P (respectively Q) is star covered, then $P = P_\rho$ (respectively $Q = Q_\sigma$).

We have reached a stage in the proof where it is necessary to subdivide into the following cases:

Case 1: Both P and Q are not star-covered;

Case 2: Both P and Q are star-covered; and

Case 3: P is not star-covered and Q is star-covered.

Case 1: Both P and Q are not star-covered.

A double application of (4.19) immediately gives

(4.21) $$P_\sigma, P_\tau \leq N_P(L_1) \text{ and } Q_\rho, Q_\tau \leq N_Q(L_2).$$

Weassert that $N_P(L_1) \not\leq P_\rho$. For suppose $N_P(L_1) \leq P_\rho$ did hold. Then $P^* = P_\rho$ by (4.21). Since Q is assumed to not be star-covered, $R = O_q(PQ) \cap O_q(QL_1) \neq 1$ by I(4.7). By considering $C_G(R)$ we infer that either $O_p(PQ) \leq N_P(L_1)$ or $O_{\pi_1}(QL_1) \leq N_{L_1}(P)$. The former possibility, using I(4.7), implies that

$$P = P^*O_p(PQ) = P^*N_P(L_1) = P_\rho,$$

contrary to P being not star-covered. Thus $O_{\pi_1}(QL_1) \leq N_{L_1}(P)$ holds, and so $O_{\pi_1}(QL_1) \leq L_{1_{\sigma\tau}}$ by Lemma 3.4(ii) (c) and (f).

Consequently $L_1 = L_1^*$ by I(4.4) and then Lemma 3.4(ii)(g) gives that $P = P_\rho$, which again contradicts P being not star-covered. Therefore $N_P(L_1) \not\leq P_\rho$ as asserted. Likewise wemayestablishthat $N_Q(L_2) \not\leq Q_\sigma$. Thus $[N_P(L_1), \rho] \neq 1 \neq [N_Q(L_2), \sigma]$ andhence Lemma 3.4(ii) (c) and (d) yield

(4.22) $\mathcal{M}(p, \pi_1) = \{P, N_P(L_1)L_1\}$ and $\mathcal{M}(q, \pi_2) = \{Q, N_Q(L_2)L_2\}$

(4.23) $N_P(N_P(L_1))^* \leq N_P(L_1)$ and $N_Q(N_Q(L_2))^* \leq N_Q(L_2)$.

Since $L_1L_2 \neq L_2L_1$ by (4.18) and our situation is symmetric with respect to P and Q, we may suppose that $L_{1_\tau} \leq N_{L_1}(L_2)$. In particular $F = N_{L_1}(L_2) \neq 1$. Recalling that $Q_\rho \leq N_Q(L_2)$ (by (4.21)), I(2.14) (i) and I(2.13) (i) yield

(4.24) $Q = N_Q(L_2)O_q(QL_1) = N_Q(L_2)C_Q(F)$.

We claim that

(4.25) $C_Q(F)$ is star covered

For, if this were not the case, I(4.5) implies $C_Q(F) \cap O_q(PQ) \neq 1$. Since $F \neq 1$, (4.22) then yields $O_p(PQ) \leq N_p(L_1)$. But then (4.23) and I(4.6) together force $P = N_P(L_1)$, a contradiction. Therefore (4.25) holds.

Put $C = C_Q(F)$. From (4.24)

$$N_Q(N_Q(L_2)) = N_Q(L_2)N_C(N_Q(L_2)).$$

Combining (4.23) and (4.25) we obtain

$$N_C(N_Q(L_2)) = N_C(N_Q(L_2))^* \leq N_Q(N_Q(L_2))^* \leq N_Q(L_2),$$

which then implies that $N_Q(N_Q(L_2)) = N_Q(L_2)$. Hence $N_Q(L_2) = Q$, contrary to $L_2 Q \neq Q L_2$. This contradiction disposes of case 1.

Case 2. Both P and Q are star-covered.

Suppose, for the moment, that $P_\sigma, P_\tau \le N_P(L_1)$ and $Q_\rho, Q_\tau \le N_Q(L_2)$ hold. Then $P = P_\rho$ and $Q = Q_\sigma$ by (4.20). By I(2.3)(ix) and I(2.21)(v) $\mathscr{P}_{L_1}(P) = 1$. Also, by I(2.3)(ix) and I(2.13) (i)

$$[Q,\rho] \trianglelefteq PQ \quad \text{and} \quad [Q,\rho] \le O_q(QL_1).$$

Since, $1 \neq Q_\tau = Q_{\sigma\tau} \le [Q,\rho]$, we deduce that $O_{\pi_1}(QL_1) \le \mathscr{P}_{L_1}(P) = 1$. Consequently, by I(4.4), $L_1 = L_{1_\tau}$ because $\rho\tau$ acts fixed-point-freely upon QL_1 and $L^*_{1\langle\rho\tau\rangle} = L_{1_\tau}$. Further. $Q = Q_\sigma$ and I(2.3) (ix) gives $L_1 = L_{1_\sigma}$. So $L_1 = L_{1_{\sigma\tau}}$ and therefore $N_P(L_1) \trianglelefteq L_1 N_P(L_1)$ by I(6.4). Then $PL_1 = L_1 P$ by I(2.21)(v). So we see that $P_\sigma, P_\tau \le N_P(L_1)$ and Q_ρ, $Q_\tau \le N_Q(L_2)$ cannot both hold.

In view of (4.17) and the symmetric conditions on P and Q we may assume, without loss of generality, that $P_\sigma, P_\tau \le N_P(L_1)$ and $L_2^* \le N_{L_2}(Q)$ pertains. From Lemma 3.4 (i) (b) $Q_{\rho\tau} = 1$ and so $[Q,\rho] \neq 1$. Since $P = P_\rho$ by (4.20) we may argue as in the previous paragraph to obtain

(4.26) $$O_{\pi_1}(QL_1) = 1 \text{ and } L_1 = L_{1_\tau}.$$

By I(2.10)(i) QL_1 has Fitting length at most two, and so (4.26) gives $Q \trianglelefteq QL_1$. Hence

(4.27) $$L_2^* \le N_{L_2}(L_1).$$

Our aim now is to show that $L_2 \le N_{L_2}(L_1)$.

If $[N_{L_2}(L_1),\rho] \neq 1$, then, as $P = P_\rho$ gives $[L_2,\rho] \le O_{\pi_2}(PL_2)$, we obtain

$$O_p(PL_2), L_1 \le C_G([N_{L_2}(L_1),\rho]).$$

Hence $O_p(PL_2) \le N_P(L_1)$. But then I(2.13) (i) forces $P = O_p(PL_2)P_\sigma \le N_P(L_1)$, a contradiction. Therefore $N_{L_2}(L_1) \le L_{2_\rho}$, and so using (4.27) we have

(4.28) $$L_{2_\tau} \le N_{L_2}(L_1) = L_{2_\rho}.$$

Now $O_{\pi_2}(L_1 N_{L_2}(L_1)) \neq 1$ would imply, by I(2.3) (x) and I(2.21) (iv), that $L_2 = L_{2_\rho}$, contrary to $L_1 L_2 \neq L_2 L_1$. Hence $O_{\pi_2}(L_1 N_{L_2}(L_1)) = 1$, and then $L_1 = L_{1_\tau}$ and I(2.3)

(ix) yield $L_{2_\tau} = N_{L_2}(L_1)$. Therefore $L_{2_\tau} = L_{2_\rho}$, and so an application of I(6.4) to PL_2 yields $P \trianglelefteq PL_2$. In particular, $[P, O_{\pi_2}(PL_2)] = 1$. Now $P = P_\rho$ and $L_1 = L_{1_\tau}$ imply $1 \neq P_\sigma \leq [N_P(L_1), \tau] \leq C_P(L_1)$ and so

$$O_{\pi_2}(PL_2), L_1 \leq C_G(P_\sigma),$$

which gives $O_{\pi_2}(PL_2) \leq N_{L_2}(L_1)$. Combining this with (4.27) and I(4.5) gives

$$L_2 = O_{\pi_2}(PL_2)L_2^* \leq N_{L_2}(L_1).$$

This is the desired contradiction which completes case 2.

We now move onto the final case, which, unfortunately, is somewhat lengthy.

Case 3. *P* is not star-covered and Q is star-covered.

Since *P* is not star-covered, (4.19) implies that $Q_\rho, Q_\tau \leq N_Q(L_2)$. Consequently Q = Q_σ by (4.20), and so I(2.3) (ix) gives

(4.29) $$\mathscr{M}(q, \pi_2) = \{Q, N_Q(L_2)L_2\}.$$

Furthermore, we may deduce that

(4.30)
(i) $O_{\pi_2}(PL_2) = 1$.
(ii) L_2 is star covered.

From Q = Q_σ and I(2.3) (ix) we have $[P, \sigma] \trianglelefteq PQ$. Now $[P, \sigma] \leq O_p(PL_2)$ by I(2.13)(i), and $[P, \sigma] \neq 1$ since *P* is not star-covered. Then $N_G([P, \sigma])$ and (4.29) imply (4.30) (i). Part (ii) follows from (i) and I(4.4).

Suppose $L_{2_\tau} \leq N_{L_2}(L_1)$ holds. Then L_2 being star-covered implies, by I(2.3) (viii), that $[N_{L_2}(L_1), \rho] \neq 1$ is impossible. Consequently we obtain $L_{2_\rho} = L_2^* = L_2$. Hence, recalling that Q = Q_σ, I(2.3) (x) gives

$$Q_\tau \leq [N_Q(L_2), \rho] \leq C_Q(L_2).$$

Also, $Q_{\rho\tau} = 1$, and so $\rho\tau$ acts fixed-point-freely on QL_1. Because $L^*_{1\langle\rho\tau\rangle} = L_{1_\tau}$, I(3.7) yields $[Q_\tau, L_1] = 1$. But then

$$L_1, L_2 \leq C_G(Q_\tau),$$

contradicting (4.18). Thus we conclude that

(4.31) $$L_{1_\tau} \leq N_{L_1}(L_2).$$

We next show that

(4.32) $$L_2 = L_{2_\tau}$$

Since $Q_\rho \leq N_Q(L_2), L_{1_\rho} = 1$ and, by (4.31), $L_{1_\tau} \leq N_{L_1}(L_2)$, Q = $N_Q(L_2)C_Q(L_{1_\tau})$ by 1(2.13)(i) and 1(2.14)(i). Put $\overline{L}_2 = L_2/\phi(L_2)$. From (4.30) (ii) $\overline{L}_2 = \bar{L}_{2_\rho}\bar{L}_{2_\tau}$. Clearly -CZ, $\trianglelefteq \overline{L}_2 L_{1_\tau}$ and hence, because $L_{1_\rho} = 1$, I(2.3) (x) yields $\overline{L}_2 = \overline{L}_{2_\tau} C_{\overline{L}_2}(L_{1_\tau})$.

If $C_{L_2}(L_{1_\tau}) \neq 1$, then (4.29) forces $C_Q(L_{1_\tau}) \leq N_Q(L_2)$, whence $Q = N_Q(L_2)C_Q(L_{1_\tau}) = N_Q(L_2)$, against $QL_2 \neq L_2Q$. So $C_{L_2}(L_{1_\tau}) = 1$. Hence $C_{\overline{L}_2}(L_{1_\tau}) = 1$, and therefore $\overline{L}_2 = \overline{L}_{2_\tau}$. By a well-known property of the Frattini subgroup, we obtain $L_2 = L_{2_\tau}$, as desired.

Since $Q = Q_\sigma$, $1 \neq Q_\rho \leq [N_Q(L_2), \tau] \leq C_Q(L_2)$ by (4.32) and I(2.3)(x). So $Z(Q) \leq N_Q(L_2)$, and, since $QL_2 \neq L_2Q$, $Z(Q) \leq$ Q,. Recallingthat $[Q_\tau, L_1] = 1$ (as $(QL_1)_{\rho\tau} =$ 1) we obtain

(4.33) $$[Z(Q), L_1] = 1.$$

We claim that $O_p(PQ) = 1$. Suppose this were false. Then Z(Q) $\cap O_q(PQ) \neq 1$, which, together with (4.33), gives $O_p(PQ) \leq \mathscr{P}_P(L_1)$. Because ***P*** is not star-covered, $O_p(PQ) \neq 1$ and so, by Lemma 3.4(i)(a), $L_1^* \not\leq N_{L_1}(P)$. Thus $O_p(PQ), P_\sigma, P_\tau \leq N_P(L_1)$. If $N_P(L_1) \leq P_\rho$ holds, then, by I(4.5), $P = O_p(PQ)P^* = P_\rho$, contrary to P not being star-covered. Whilst $[N_P(L_1), \rho] \neq 1$ implies, by Lemma 3.4(ii) (d), that $N_P(N_P((L_1))^* \leq N_P(L_1)$, and then I(4.6) gives the untenable $P = N_P(L_1)$. This establishes the claim. Using I(2.6) we now deduce that

(4.34) $$Q = N_Q(J(P))C_Q(Z(P)).$$

If the Fitting length of PL_2 were at most two, then (4.30) (i) would give $P \trianglelefteq PL_2$. Then Z(P) $\trianglelefteq PL_2$ and ***J(P)*** $\trianglelefteq PL_2$, and hence (4.34) forces $QL_2 = L_2Q$, a contradiction. Thus we conclude, using I(2.10) (i), that

(4.35) $$P_{\sigma\tau} \neq 1.$$

We shall show that (4.35) gives rise to a contradiction. One observation we shall use is that

(4.36) $$Z(J(P)) \nleq P_\rho.$$

Suppose $Z(J(P)) \leq P_\rho$ were to hold. Then we may apply I(2.3) (x) to both $Z(J(P)) N_Q(J(P))$ and $Z(J(P))N_{L_2}(J(P))$. Since $Z(P) \leq Z(J(P))$ and $O_p(PQ) = 1 = O_{\pi_2}(PL_2)$, I(2.6) yields

$$Q = C_Q(Z(P))Q_\rho \quad \text{and} \quad L_2 = C_{L_2}(Z(P))L_{2_\rho}$$

Recall, from (4.29), that $Q_\rho \leq N_Q(L_2)$, and hence $C_Q(Z(P)) \nleq N_Q(L_2)$. Therefore $C_{L_2}(Z(P)) \leq \mathscr{P}_{L_2}(Q) = 1$ by (4.29), which then gives $L_2 = L_{2_\rho}$. Hence, using (4.32), $L_2 = L_{2_{\rho\tau}}$. Combining I(6.4) and I(2.21)(v) gives $L_2Q = QL_2$, a contradiction. Thus we have established that $Z(J(P)) \nleq P_\rho$.

From (4.35) and I(3.13)(iii) $1 \neq P_{\sigma\tau} \leq C_P(L_1)$ and so

(4.37) $$\mathscr{M}(p, \pi_1) = \{P, N_P(L_1)L_1\}$$

by Lemma 3.4(ii)(c). We assert that

(4.38) $$L_1^* = L_{1_\sigma} \neq L_1$$

First we verify that $L_1^* = L_{1_\sigma}$. Supposing $L_1^* \neq L_{1_\sigma}$ we seek a contradiction. So $1 \neq [N_{L_1}(L_2), \sigma] \leq C_{L_1}(L_2)$ by (4.31) and I(2.13)(i). Hence $Z(L_1) \leq N_{L_1}(L_2)$. Because $L_1L_2 \neq L_2L_1$ and, by (4.32), $L_2 = L_{2_\tau}$, I(2.3) (x) and I(2.13)(i) force $Z(L_1) \leq L_{1_{\sigma\tau}}$. But then $[Z(L_1), N_P(L_1)] = 1$ by I(6.4), which, as $N_P(L_1) \neq 1$, yields $Z(L_1) \leq \mathscr{P}_{L_1}(P)$, against (4.37). So we have proved that $L_1^* = L_{1_\sigma}$.

Observe that $P_{\rho\tau} \neq 1$. For $P_{\rho\tau} = 1$ would imply, as $Q = Q_\rho$, that $\rho\tau$ acts fixed-point-freely upon PQ. Recalling that $O_p(PQ) = 1$, I(2.10) (i) gives $P \trianglelefteq PQ$. Since $O_{\pi_2}(PL_2) = 1$ by (4.30) (i) I(2.6) implies $L_2 = N_{L_2}(J(P))C_{L_2}(Z(P))$, whence $QL_2 = L_2Q$, which is not possible.

Now suppose $L_1 = L_{1_\sigma}$. Then I(2.3)(x), I(2.13)(i) and (4.37) give $[L_1, P_\tau] = 1$. Now $[L_1, P_{\rho\tau}] = 1$ by I(3.13) (iii) and so $L_1, L_2 \leq C_G(P_{\rho\tau})$. Since $P_{\rho\tau} \neq 1$, we obtain the untenable $L_1L_2 = L_2L_1$. Therefore $L_1 \neq L_{1_\sigma}$, and we have (4.38).

Since $P_\sigma \leq N_P(L_1)$, (4.38) and I(2.14) (ii) imply that $L_1 = C_{L_1}(P_\sigma)L_{1_\sigma}$. Further, $C_{L_1}(P_\sigma) \neq 1$ by (4.38). Therefore the shape of $\mathcal{M}(p, \pi_1)$ gives Z(P) $\leq N_P(L_1)$ and $Z(P)_\sigma = 1$. Now $[L_1, P_\sigma] \leq L_{1_\sigma}$ and, since [Z(P), P_σ] = 1, $Z(P)$ normalizes $[L_1, P_\sigma]$. ApplyingI(2.3) (x) to Z(P) $[L_1, P_\sigma]$ wededuce that [Z(P), $[L_1, P_\sigma]$] = 1. Then theshape of $\mathcal{M}(p, \pi_1)$ forces $[L_1, P_\sigma] = 1$.

If $J(P)_\sigma \neq 1$,then $P_\sigma \leq C_P(L_1)$ yields $Z(J(P)) \leq N_P(L_1)$. By I(2.13) (i) and (4.36)

$$1 \neq [Z(J(P)), \rho] \leq C_P(L_1).$$

Then, using I(2.3) (viii), we infer that $P_\rho \leq N_P($ L,) , and hence $P^* \leq N_P(l_1)$. Employing 1(5.8)(f) (with L = Q, $M = P, N =$ L, and $\gamma = \alpha$) yields $O_p(QL_1) = 1$. However, by (4.33), [Z(Q), L_1] = 1, and so we see that J(P), = 1. Consequently (since Q = Q,),

$$J(P) \leq [P, \sigma] \leq O_p(PQ) \cap O_p(PL_2)$$

Then, by [Lemma 8.22(ii); 31, $J(O_p(PQ))$ = ***J(P)*** = $J(O_p(PL_2))$ and hence $Q, L_2 \leq N_G(J($ ***P)***) , a contradiction! This is the long sought contradiction and finishes the work on case 3.

The proof of Theorem 4.4 is complete.

The next linking result is of a similar nature to Theorem 4.4 though its proof is much shorter.

Lemma ***4.5.*** ***Let P and Q be (respectively) α -invariant Sylow p- and q -subgroups of type Λ whichpermute, $p \neq q$, and set Λ = {i, j, k}. If $PL_{jk} = L_{jk}P$ and $QL_i = L_iQ$, then at least one of $PL_i = L_iP$ and $QL_{jk} = L_{jk}Q$ must hold.***

Proof. Suppose the lemma is false and argue fora contradiction. Without loss of generality we assume $i = 1, j = 2$ and $k = 3$. So we have

(4.39)
$$PQ = QP, PL_{23} = L_{23}P, QL_1 = L_1Q,$$
$$PL_1 \neq L_1P \quad \text{and} \quad QL_{23} \neq L_{23}Q$$

From Lemma 3.2, Z(Q) $\leq Q_{\sigma\tau}$ and so [Z(Q) , ***L,***] = 1 by I(3.13) (iii). Also note that $L_{23}^* = L_{23_\rho} \neq L_{23}$ by I(2.8) and I(6.1).

Now suppose Q is not star-covered. Then $O_q($ ***PQ)*** $\neq 1$ by I(4.4). Hence, by I(5.8) (f), $L_1^* \not\leq N_{L_1}(P)$ and so $P_\sigma, P_\tau \leq N_P(L_1)$. Moreover $O_q(PQ) \cap Z(Q) \neq 1$ and $[Z(Q), L_1] =$

1 yield $O_p(PQ) \leq N_P(L_1)$ whence $P = P_\rho$ by Lemma 3.4(ii) (d) and I(4.6). Therefore $\mathscr{M}(p,\pi_1) = \{P, N_P(L_1)L_1\}$, $[Q,\rho] \trianglelefteq PQ$ by I(2.3) (ix), and $P_{\sigma\tau} = 1$. Since $1 \neq [Q,\rho] \leq O_q(QL_1)$, we obtain $O_{\pi_1}(QL_1) \leq \mathscr{P}_{L_1}(P) = 1$. Thus

(4.40) L_1 is star covered.

Clearly PL_{23} admits $\sigma\tau$ fixed-point-freely and so $[P, L_{23}] = 1$ by I(2.8). If $L_1 L_{23} = L_{23} L_1$, then $L_{23}^* \neq L_{23}$, I(4.4) and $N_G(O_{\pi_{23}}(L_1 L_{23})) \geq P, L_1$ yields a contradiction to (4.39). Thus $L_1 L_{23} \neq L_{23} L_1$.

Since $P_{\sigma\tau} = 1$, $P_\sigma, P_\tau \leq N_P(L_1)$ and, by (4.4), $L_1 = L_1^*$ it follows (see Lemma 3.4(ii) (g)) that for at least one of P_σ and P_τ, say P_σ, $C_{L_1}(P_\sigma) \neq 1$ and $C_{L_1}(P_\sigma) \not\leq L_{1_{\sigma\tau}}$. Clearly $C_G(P_\sigma) \geq L_{23}$, $C_{L_1}(P_\sigma)$ and hence $O_{\pi_1}(L_{23}\mathscr{P}_{L_1}(L_{23})) \neq 1$ by I(2.13) (i) and $C_{L_1}(P_\sigma) \not\leq L_{1_{\sigma\tau}}$. Hence $Z(L_1) \leq L_{1_{\sigma\tau}}$ as $L_1 L_{23} \neq L_{23} L_1$. But then $[N_p(L_1), Z(L_1)] = 1$ by I(2.3) (xi) whence $Z(L_1) \leq \mathscr{P}_{L_1}(P)$ contrary to the shape of $\mathscr{M}(p,\pi_1)$.

Hence we conclude that Q must be star-covered. Then by Lemma 3.2 and I(2.3) (viii) either $N_Q(L_{23}) \leq Q_\sigma$ or $N_Q(L_{23}) \leq Q_\tau$. Suppose $N_Q(L_{23}) \leq Q_\sigma$. Hence as $C_Q(L_{23}) \neq 1$, $Q = Q_\sigma$ by I(2.21) (iv) and I(2.3) (v). So $[P, \sigma] \trianglelefteq PQ$. If $[P,\sigma] \neq 1$, then $N_G([P,\sigma]) \geq Q$, $O_{\pi_{23}}(PL_{23})$ implies $1 \neq O_{\pi_{23}}(PL_{23}) \leq \mathscr{P}_{L_{23}}(Q)$, which contradicts Lemma 3.2. Thus $P = P_\sigma$ and so $1 \neq P_\tau = P_{\sigma\tau}$. Hence $P_\sigma, P_\tau \leq N_P(L_1)$ by Lemma 3.4 (i) (b). But then $P \leq N_P(L_1)$, a contradiction.

This completes the proof of Lemma 4.5.

We close this section with two results, the first of which will be used in Lemmas 6.1 and 7.4 whilst the second is specifically designed for one application in Theorem 7.6.

Lemma 4.6. *Let P be an α-invariant Sylow p-subgroup of type Λ, $p \in \pi(G)$, for which $PL_2 \neq L_2 P$ and $PL_3 \neq L_3 P$. Then*

(i) $P_\rho, P_\tau \leq N_P(L_2)$ and $P_\rho, P_\sigma \leq N_P(L_3)$;

(ii) $Z(P) = Z(P)_{\sigma\tau} \leq N_P(L_2) \cap N_P(L_3)$;

(iii) P is not star-covered; and

(iv) either $N_G(Z(J(P))) = PC_G(Z(J(P)))$ or J(P) is contained in at least one of $N_P(L_2)$ and $N_P(L_3)$.

Proof. (i) From Lemma 4.1, $L_2 L_3 = L_3 L_2$. Suppose that $P_\rho, P_\sigma \not\leq N_P(L_3)$. Then $L_3^* \leq N_{L_3}(P)$ and by Lemma 3.4 (i),

$$P_{\rho\sigma} = 1, P_{\sigma\tau} \neq 1 \neq P_{\rho\tau} \quad \text{and} \quad \mathscr{M}(p,\pi_3) = \{L_3, N_{L_3}(P)P\}.$$

Since $P_{\rho\tau} \neq 1$, we must have $P_\rho, P_\tau \leq N_P(L_2)$ by Lemma 3.4 (i) (b). From $P_{\rho\sigma} = 1$ we see that $[N_P(L_2), \sigma] \neq 1$ whence $C_P(L_2) \neq 1$ and $Z(P) \leq N_P(L_2)$. The shape of $\mathcal{M}(p, \pi_3)$ forces $O_\pi(L_2L_3) = 1$, which then, by 1(2.13)(i), gives $L_2 = L_{2_\tau}$. Hence $Z(P) \leq P_\tau$ by 1(2.3)(x). But then $[Z(P), N_{L_3}(P)] = 1$ which gives the untenable $Z(P) \leq \mathscr{P}_P(L_3) = 1$. Thus we conclude that $P_\rho, P_\sigma \leq N_P(L_3)$ and, likewise, that $P_\rho, P_\tau \leq N_P(L_2)$.

(ii) Because $P_\rho \neq 1$, one of $[N_P(L_2), \sigma]$ and $[N_P(L_3), \tau]$ must be non-trivial. Hence we have, say, $C_P(L_2) \neq 1$ and so $Z(P) = Z(P)_\sigma \leq N_P(L_2)$. But then $Z(P) \leq P_\sigma \leq N_P(L_3)$, so proving (ii).

(iii) Since $P \neq P_{\sigma\tau}$, P cannot be star-covered by Lemma 3.4 (ii) (e).

(iv) Put $R = Z(J(P))$. If, say, $R_\rho \neq R_{\rho\sigma}R_{\rho\tau}$, then $O_p(R_\rho L_{3_\rho}) \neq 1$ by I(4.5). Since $L_{3_\rho} \not\leq \mathscr{P}_{L_3}(P)$ by (i) and Lemma 3.3(ii), this implies that $J(P) \leq N_P(L_3)$. So either J(P) is contained in at least one of $N_P(L_2)$ and $N_P(L_3)$ or

$$R_\rho = R_{\rho\sigma}R_{\rho\tau},\ R_\sigma = R_{\rho\sigma}R_{\sigma\tau} \quad \text{and} \quad R_\tau = R_{\rho\tau}R_{\sigma\tau}. \tag{4.41}$$

If (4.41) pertains, then applying I(6.4) to $RN_G(R)_p$, yields $N_G(R) = PC_G(R)$. This proves (iv).

Lemma 4.7. ***Suppose P is an α-invariant Sylow p-subgroup of G of type Λ which is not star-covered, and let $\Lambda = \{i, j, k\}$. Also suppose***

(i) P permutes with L_i and L_j but not with L_k;

(ii) $L_iL_j \neq L_jL_i$; and

(iii) $Z(J(P)) \not\leq N_P(L_k)$.

Then $P_{\alpha_i\alpha_j} \neq 1$.

Proof. Without loss of generality, we take $i = 1$, $j = 2$ and $k = 3$. So we have $PL_1 = L_1P$, $PL_2 = L_2P$, $PL_3 \neq L_3P$ and $L_1L_2 \neq L_2L_1$. Recall that $\mathscr{P}_P(L_3) = N_P(L_3)$.

First we show that either $L_1 = L_{1_\sigma}$ or $L_2 = L_{2_\rho}$ holds. Since $[P_{\rho\sigma}, L_3] = 1$, (iii) implies $J(P)_\tau = 1$. Applying I(4.5) to $J(P)N_{L_1}(J(P))$ and $J(P)N_{L_2}(J(P))$ yields

$$L_1 = C_{L_1}(D)L_{1_\sigma} \quad \text{and} \quad L_2 = C_{L_2}(D)L_{2_\rho}$$

where $D = O_p(PL_1) \cap O_p(PL_2) \cap Z(P)$. From I(4.7) $D \neq 1$ and so either $C_{L_1}(D) \leq N_{L_1}(L_2)$ or $C_{L_2}(D) \leq N_{L_2}(L_1)$ holds.

Assume, say, that $C_{L_1}(D) \leq N_{L_1}(L_2)$. Note that this implies $O_{\pi_1}(PL_1) \leq N_{L_1}(L_2)$. If $[N_{L_1}(L_2), \sigma] \neq 1$, then $C_{L_1}(L_2) \neq 1$ whence $N_{L_2}(L_1) = 1$ by I(5.7), and so $L_{1_\tau} \leq$

$N_{L_1}(L_2)$. Therefore, by I(2.3) (viii), $N_{L_1}(N_{L_1}(L_2))^* \leq N_{L_1}(L_2)$. But then $N_{L_1}(L_2) = L_1$ by I(4.6) which contradicts (ii). Thus we must have $C_{L_1}(D) \leq N_{L_1}(L_2) \leq L_{1_\sigma}$. Consequently

$$L_1 = C_{L_1}(D) L_{1_\sigma} = L_{1_\sigma}$$

If $C_{L_2}(D) \leq N_{L_2}(L_1)$, then we would obtain $L_2 = L_{2_\rho}$.

Without loss of generality we may assume that $L_1 = L_{1_\sigma}$. As a consequence, $\mathscr{M}(\pi_1, \pi_2) = \{L, , L_2 N_{L_1}(L_2)\}$. Moreover, because P is not star-covered and $[P, \sigma] \trianglelefteq PL,$, *we* have $O_{\pi_2}(PL_2) \leq N_{L_2}(L_1) = 1$. Also since, $L_{1_\rho} = 1$, we have $[P, \rho\sigma] \leq O_p(PL_1)$ and therefore $J(P) \leq O_p(PL_1)$. Thus $J(P) = J(O_p(PL_1)) \trianglelefteq PL_1$.

Now, if $P_{\rho\sigma} = 1$, then *PL,* would have Fitting length at most two which gives $P \trianglelefteq PL,$. But then $L_1, L_2 \leq N_G(J(P))$, contradicting (ii). Hence we have $P_{\rho\sigma} \neq 1$, which established the lemma.

5. SOLUBILITY OF *L*

The purpose of this section is to demonstrate that

Theorem 5.1. *L is a soluble Hall subgroup of G.*

Suppose Theorem 5.1. is false, Then $PQ \neq QP$ where P and Q are cr-invariant Sylow subgroups of G of type Λ. By Lemma 3.5 we may suppose our notation chosen so that

$$Z(P) = Z(P)_{\sigma\tau} \leq N_P(Q) \text{ and } \quad Q_{\sigma\tau} = 1,$$

where, if $P^* \not\leq N,(Q)$, *we* have $P_\rho \leq N,(Q)$ and $Q_\sigma, Q_\tau \leq \mathscr{P}_Q(P)$. If possible we chose P and Q so that $p \neq 2$.

In the following series of lemmas we deduce an appropriate contradiction. Our aim is to produce a factorization of G which then forces G to contain a non-trivial proper cr-invariant norma1 subgroup. Lemmas 5.2 to 5.7 serve as preparation for the task of constructing the factorization.

Let A (respectively *B)* denote the subgroup of G generated by the α-invariant Sylow subgroups of type Λ which permute with P (respectively, do not permute with *P).* Note thaat $P \leq A$ and $Q \leq B$.

(5.1) Let H be a soluble α -invariant subgroup of G .

(i) If $P \leq H$, then $O_p(H) \not\leq N_P(Q)$

(ii) Suppose $P_\rho \leq N_P(Q), Q_\sigma, Q_\tau \leq \mathscr{P}_Q(P)$ and $Q \leq H$. Then $O_q(H) \not\leq \mathscr{P}_Q(P)$.

From Lemma 3.5 either $P^* \leq N_P(Q)$ or $N_P(N_P(Q))^* \leq N_P(Q)$. Hence by either I(4.5) or I(4.6) $O_p(H) \not\leq N_P(Q)$, so proving (i). Similar considerations also yield (5.1) (ii).

Clearly we also have that

(5.2) P is not star-covered.

(5.3) Suppose $P^* \leq N_P(Q)$ and let N be an α-invariant Hall $\{p, q\}'$-subgroup of G which permutes with both ***P*** and Q. If $G \neq (PN)Q$, then (i) $P = N_P(Q)C_P(N)$; and (ii) $N_Q(N_1) = 1$ for all non-trivial α-invariant subgroups N_1 of N .

Using 1(58)(e)(i) and (ii) and Lemma 3.5(i)(a) immediately yields (5.3).

Lemma 5.2. (i) $L_{12} = L_{13} = 1$,

(ii) $PL_1 = L_1P$ ***with*** $[Z(P), L_1] = 1$.

(iii) If p = 2, then the set of α***-invariant Sylow*** w***-subgroups of type*** Λ ***with*** $w \neq 2$ ***generate a soluble Hall subgroup of*** G.

(iv) A and B are soluble Hall subgroups of G.

(v) L_{23} ***B is a soluble Hall subgroup of*** G.

(vi) If $L_{23} \neq 1$, ***then*** $PL_{23} \neq L_{23}P$ ***and*** $N_P(Q) = N_P(L_{23})$.

Proof. Since Z(P) $\leq P_{\sigma\tau}$ and $[\mathscr{L}_1, P_{\sigma\tau}] = 1$, P must permute with $\mathscr{L}_1$ and we have (ii). We now prove that $L_{12} = 1$. Suppose $L_{12} \neq 1$. Then $L_{12} \neq L_{12}^*$ by I(2.8) and I(6.1). Now $[L_{12}, Q_\tau] = 1$ and so, since $O_{\pi_{12}}(PL_{12}) \neq 1$ by I(4.5), Lemma 3.5(i)(a) implies that $P^* \not\leq N_P(Q)$. So $P_\rho \leq N_P(Q)$ and $Q_\sigma, Q_\tau \leq \mathscr{P}_Q(P)$. Suppose $L_{12}Q \neq QL_{12}$. Then $\mathscr{M}(q, \pi_{12}) = \{Q, L_{12}N_Q(L_{12})\}$ with $N_Q(L_{12}) = C_Q(L_{12})(N_Q(L_{12})_{\rho\sigma}$. Because $O_p(PL_{12}) \neq 1$ we obtain, using Lemma 3.5(ii)(b), $C_Q(L_{12}) \leq \mathscr{P}_Q(P) \leq$ **Q_τ**, whence $N_Q(L_{12}) \leq Q_\tau$. But then Q = Q_ρ by 1(2.3)(v), contrary to Lemma 3.5(ii) (b). Therefore $L_{12}Q = QL_{12}$. So L_{12} permutes with both P and Q and hence, since $L_{12} \neq L_{12}^*$, using I(4.7) gives either $O_p(PL_{12}) \leq N_P(Q)$ or $O_q(QL_{12}) \leq \mathscr{P}_Q(P)$, contradicting (5.1). Therefore we conclude that $L_{12} = 1$. A similar argument shows that $L_{13} = 1$, and we have proved (i).

(iii) This follows from the choice of *(P, Q)* .

(iv) If p = 2 , then (iii) implies (iv). So we may suppose $p \neq 2$. Let U and ***V*** be, respectively, α-invariant Sylow u - and v -subgroups of G which do not permute with P .

Because ***Z(P)*** $\leq P_{\sigma\tau}$, neither $U^* \leq N_,(P)$ nor $V^* \leq N_,(P)$ is possible by I(2.3) (xi) and Lemma 3.5(i) (a). While $P^* \leq N_P(V)$ and $P^* \leq N_P(V)$ yields, using Lemma 3.5 (i) (d), $U_{\sigma\tau} = V_{\sigma\tau} = 1$. But then Lemma 3.5 (i) (c), (d) and (ii) (c) (e) imply that ***UV*** $\neq VU$

is impossible. Since $p \neq 2$, by Lemma 3.5 (ii) (a), without loss of generality it only remains to consider the situation

$$P^* \leq N_P(U), v = 2, \text{ and } V_{\alpha_i} \leq N_V(P), P_{\alpha_j}, P_{\alpha_k} \leq \mathscr{P}_P(V) \text{ where } \{i, j, k\} = \Lambda.$$

Because $P_{\sigma\tau} \neq 1$, by Lemma 3.5(ii)(c) we may suppose

$$V_\tau \leq N_V(P) \quad \text{and} \quad P_\rho, P_\tau \leq \mathscr{P}_P(V).$$

Therefore Z(V) $\leq V_{\rho\sigma}$ by Lemma 3.5(ii) (e). Hence $U^* \leq$ N,(V) is not possible. If $V^* \leq N_V(U)$ were to hold, then Z(V) $\leq V_{\rho\sigma}$ and the shape of $\mathscr{M}(u, v)$ forces $U_{\rho\sigma} = 1$. But $U_{\rho\sigma} \neq 1$ by Lemma 3.4(i) (d) (applied to ***P*** and U). So $U^* \not\leq$ N,(V) and $V^* \not\leq$ ***N,(U).*** Now $P^* \leq N_P(U)$ implies $U_{\sigma\tau} = 1$ and therefore, as $v = 2$, Lemma 3.5 (ii) (c) shows that

$$V_\sigma \leq N_V(U) \quad \text{and} \quad U_\sigma, U_\tau \leq \mathscr{P}_U(V),$$

and thus Z(V) $\leq V_{\sigma\tau}$ by Lemma 3.5(ii)(e). But then Z(V) $\leq V_{\sigma\tau} \cap V_{\rho\sigma}$, which is not possible. Therefore we conclude that B is a soluble Hall subgroup of G.

Now let U and ***V*** denote α-invariant Sylow subgroups of G which permute with ***P.*** Suppose $UV \neq$ ***VU.*** If $V^* \not\leq N_V(U)$ and $U^* \leq$ N,(V) pertains, then, as ***P*** is not star-covered, I(4.7) force either $O_u($ ***PU)*** $\leq \mathscr{P}_U($ ***V)*** or $O_v($ ***PV)*** $\leq \mathscr{P}_V(U)$ which is not possible by Lemma 3.5(ii)(b) (h). So we must have, say, $V^* \leq N_V(U)$. But then, as $O_p($ ***PU)*** $\neq 1$, this situation contradicts I(5.8) (f) (with ***L*** = ***P,*** M = ***V*** and N = U). Thus ***UV = VU*** must hold whence ***A*** is a soluble Hall subgroup of G .

(v) Let ***V*** be an α-invariant Sylow subgroup of B. Since $V_{\sigma\tau} = 1$ I(2.8) yields $\mathscr{P}_V($ ***L,)*** $\trianglelefteq L_{23}\mathscr{P}(L_{23})$ and so L_{23} ***V*** = ***VL,,*** , which proves (v).

(vi) This is straightforward and so is omitted.

Lemma ***53. Suppose that P**** $\not\leq$ ***N,,(Q) and that*** $PL_i = L_iP$ ***where*** i ***= 2 or 3. Then***

(i) $[Z(O_p(PL_i)), L_i] = 1$ *, and*

(ii) $QL_i = L_iQ$.

Proof. Without loss of generality we take $i = 2$, and set $Z = Z(O_p($ ***PL,))*** .

(i) Now $[P, \sigma] \leq O_p($ ***PL,)*** and from Lemma 3.5(ii) (f) 1 $\neq [N_P(Q)$, al $\leq P_\rho$ and hence $O_p(PL_2)_\rho \neq 1$. Therefore $Z \leq N_P(Q)$ by Lemma 3.5(ii) (d). Because of (5.1) and Lemma 3.5(ii) (d), we must have $Z_\rho = 1$, and hence, by Lemma 3.5 (ii) (f),

$$Z \leq [N_P(Q), \rho] \leq P_{\sigma\tau}.$$

This immediately yields (i).

(ii) Suppose QL, $\neq L_2$ Q . Because $Q_{\rho\tau} \neq 1$ by Lemma 3.5 (ii) (c) we see that $C_Q(L,) \neq 1$ and so $Q_\rho, Q_\tau \leq N_Q(L_2)$ must hold. From part (i) we have $[Z, L_2] = 1$ and $Z \leq N_P(Q)$. Thus

$$Z \leq N_P(Q) \cap N_P(L_2).$$

Consequently, as Q* = Q_ρ by Lemma 3.5 (ii) (b), I(2.14) (ii) gives Q = $N_Q(L_2)C_Q(Z)$. Using Lemma 3.5(ii)(b) we then obtain

$$Q = N_Q(L_2)C_Q(Z) = N_Q(L_2)Q_\rho = N_Q(L_2),$$

contrary to $QL_2 \neq L_2$ Q . This proves (ii).

Lemma 5.4. *If PL, = L_iP where i = 2 or 3 , then $L_1 L_i = L_iL_1$.*

Proof. The case when $P^* \not\leq N_P(Q)$ is easily resolved by Lemmas 5.2 (ii) and 5.3 (i) since Z(P) $\cap$ $Z(O_p(PL,)) \neq 1$. So for the remainder of the lemma's proof we may assume $P^* \leq N_P(Q)$. Without loss of generality we take $i = 2$.

Since $C_Q(Z(P)) = 1$ and Z(P) $\leq P_{\sigma\tau}$, we observe that

(5.4) $$Q^* \neq Q_\rho \text{ and } Z(Q) \not\leq Q_\rho.$$

From the shape of $\mathcal{M}(p,q)$ and [L,, Q,] = 1 we have $O_{\pi_2}(PL,) = 1$, and so

(5.5) L_2 is star-covered.

From $[L_2$, Q,] = 1 and (5.3) we deduce that QL, $\neq L_2Q$. Moreover, using (5.4), we note

(5.6) $$N_Q(N_Q(L_2))^* \leq N_Q(L_2) \text{ and } Z(Q), Q_\rho, Q_\tau \leq N_Q(L_2).$$

We now suppose $L_1 L_2 \neq L_2 L,$, and seek a contradiction beginning with

(5.7) $$L_{1_\tau} \leq N_{L_1}(L_2).$$

If (5.7) is false, then $L_{2_\tau} \leq N_{L_2}(L_1)$ holds, which, by (5.5), implies that $L_2 = L_{2_\rho}$. Hence, by 1(2.3)(x) and $QL, \neq L_2$ Q , Z(Q) $\leq Q_\rho$, contrary to (5.4). This proves (5.7).

We claim that QL, $= L_1Q$. For suppose QL, $\neq L_1Q$. Then (5.4) immediately gives $L_1^* \leq N_{L_1}(Q)$. So $L_{1_\tau} \leq N_{L_1}(Q) \cap N_{L_1}(L_2)$ by (5.7). Since $Q_\rho \leq N_Q(L_2)$ by (5.6), I(2.14) (i) yields Q = $N_Q(L_2)C_Q(L_{1_\tau})$. But from Lemma 3.4 (i) (a), (c) and (d) we have $C_Q(L_{1_\tau})$ = 1, which contradicts $QL_2 \neq L_2Q$. Therefore $QL_1 = L_1Q$, as claimed. Again using $Q_\rho, L_{1_\tau} \leq N_G(L_2)$ and I(2.14) (i) we obtain

$$O_q(QL_1) = (O_q(QL_1) \cap N_Q(L_2))C_Q(L_{1_\tau}),$$

which, appealing to (5.3), then yields $O_q(QL_1) \leq N_Q(L_2)$. However, (5.6) and I(4.6) then imply $N_Q(L_2)$ = Q , a contradiction. This completes the proof of the lemma.

The next result is required in the proof of Lemma 5.6.

Lemma 5.5. ***Suppose*** $PL_i = L_iP$ ***where*** $i \in \Lambda$***, and lei W be an cu-invariant Sylow w - subgroup of A . If*** $L_iW \neq WL_i$***, then*** $W \leq G_{\alpha_i}$.

Proof. Without loss of generality we may assume i = 2. Since $PW = WP$ and P is not star-covered, 1(5.8)(f) rules out the possibility $L_2^* \leq N_{L_2}(W)$. So $W_\rho, W_\tau \leq N_W(L_2)$. From Lemma 3.3 we have $L_{2_\rho} \not\leq \mathscr{P}_{L_2}(W)$. Now, because P is not star-covered, I(3.3) (vii) and I(4.4) imply that $[O_p(PL_2), \rho] \neq 1$, and thus $O_w(PW) \leq N_W(L_2)$ by I(5.8) (c). Then Lemma 3.4 (ii) (d), I(4.6) and I(4.5) yield $W = W_\sigma$, so proving the lemma.

Lemma 5.6. *If* $L_iP = PL_i$ ***where*** $i \in \Lambda$***, then*** L_iA ***is a*** *soluble* ***Hall subgroup of*** G.

Proof. Suppose the lemma is false and argue for a contradiction. Thus $L_iW \neq WL_i$ for some cu-invariant Sylow w-subgroup of A, and hence $W \leq G_{\alpha_i}$ by Lemma 5.5. Clearly $\mathscr{M}(\pi_i, w) = \{W, L_iN_W(L_i)\}$. Observe that $W \leq G_{\alpha_i}$ and Lemma 3.5 (i) (e), (ii)(b) and (h) imply that $QW = WQ$. We now divide our proof into two cases: i = 1 and $i \neq 1$.

Case 1. i = 1.

Since $W \leq$ G,, WQ admits $\sigma\tau$ fixed-point-freely. If $P^* \leq N_P(Q)$, then (5.3) (ii) clearly gives $O_w(WQ)$ = 1. Hence $W = W_\sigma W_\tau$ by I(2.10) (ii). Consequently, as $W \neq 1$, I(2.10) (ii) and I(6.1) yield the contradiction $G \neq O^w(G)$. Now we consider the possibility $P^* \not\leq N_P(Q)$.

Because $L_1W \neq WL_1$, Lemma 3.5(ii) (i) shows that $J(P)_\rho$ = 1. From $W = W_\rho$, I(2.3) (i) gives $[P, \rho] \leq O_p(PW)$. Hence

$$J(P) \leq [P,\rho] \leq O_p(PW) \cap O_p(PL_1).$$

A well-known property of the Thompson subgroup yields $J(O_p(PW)) = J(P) = J(O_p(PL_1))$ and consequently $L_1, W \leq N_G(J(P))$, a contradiction. This settles case 1.

Case 2. $i \neq 1$.

Without loss of generality we shall suppose $i = 2$. Suppose to begin with that $P^* \leq N_P(Q)$. Then, because $[L_2, Q_\tau] = 1$, (5.3) implies that $L_2 Q \neq QL_2$. Since, using Lemma 3.5(i) (d), $1 \neq Q_{\rho\tau} \leq C_Q(L_2)$, by Lemma 3.4 we have $Q_\rho, Q_\tau \leq N_Q(L_2)$, and $N_Q(N_Q(L_2))^* \leq N_Q(L_2)$.

Since $1 \neq N_W(L_2) \leq W_\sigma$, at least one of $[N_W(L_2), \rho]$ and $[N_W(L_2), \tau]$ must be non-trivial. Suppose $V = [N_W(L_2), \rho] \neq 1$. Because V normalizes $O_q(QW)$ and $O_q(QW) \cap N_Q(L_2)$ and $Q_\rho \leq N_Q(L_2)$, I(2.14) (i) gives

$$O_q(QW) = (O_q(QW) \cap N_Q(L_2)) C_{O_q(QW)}(V).$$

However, since W permutes with both P and Q, (5.3) (ii) gives $C_Q(V) = 1$ and hence $O_q(QW) \leq N_Q(L_2)$. But this is not possible since $N_Q(N_Q(L_2))^* \leq N_Q(L_2)$.

It only remains to consider the situation when $P^* \not\leq N_P(Q)$. Appealing to Lemma 5.3 (ii) gives $L_2 Q = QL_2$. By Lemma 3.5(ii) (b) $Q^* = Q_\rho \neq Q$ and so, as $W = W_\sigma$,

$$1 \neq [Q, \sigma] \trianglelefteq QW.$$

Since $[Q, \sigma] \leq O_q(QL_2)$, we obtain

$$O_{\pi_2}(QL_2) \leq \mathscr{P}_{L_2}(W) = 1 \tag{5.8}$$

Since $p = 2$ by Lemma 3.5 (ii) (a), $WL_2 \neq L_2 W$, (5.8) and Glauberman's ZJ-theorem yield $O_w(WQ) \neq 1$. If $[O_w(WQ), \rho] \neq 1$, then either $Q_\rho \leq \mathscr{P}_Q(P)$ or $O_p(PW) \leq \mathscr{P}_P(Q)$ by I(5.8) (c). But $Q^* \leq \mathscr{P}_Q(P)$ and Lemma 3.5(ii) (h) show that neither of these can occur. Therefore

$$1 \neq O_w(WQ) \leq W_\rho. \tag{5.9}$$

Also from (5.8), since $(QL_2)_{\sigma\tau} = 1$, we have $L_2 = L_{2_\tau}$ by I(4.5). Hence, using I(2.3) (x),

$$W_\rho \leq [N_W(L_2), \sigma\tau] \leq C_W(L_2),$$

and then $N_G(O_w(WQ)) \geq W, L_2$ by (5.9) contrary to $WL_2 \neq L_2 W$. This completes case 2 and also the proof of the lemma.

Lemma 5.7. *Suppose $PL_i \neq L_i P$ where $i = 2$ or 3. Then*

(i) $L_i^* \not\leq N_{L_i}(P)$ *and* $Z(P) \leq N_P(L_i)$ *; and*

(ii) $L_i B$ *is a soluble Hall subgroup of* G.

(ii) From Lemma 3.4(i)(b), (ii) (e) and (g) Z(P) $\leq N_P(B)$. If $L_{23} \neq 1$, then by Lemma 5.2(vi) *PL,* $\neq L_{23}$ *P* and it is easy to see that $N_P(Q) = N_P(L,)$. Hence, using Lemma 5.7(i) and the definition of K , we have Z(P) $\leq N_P(K)$.

We now analyse the factorization obtained in Lemma 5.8 beginning with

Lemma 5.9. *If U is an cr-invariant Sylow* u*-subgroup of B , then either*

(i) $P^* \leq N_P(U)$ *; or*

(ii) $P_\rho \leq \mathscr{P}_P(U)$ *and* $U_\sigma, U_\tau \leq \mathscr{P}_U(P)$.

Proof. Suppose $P^* \nleq N_P(U)$. From $Z(P) \leq P_{\sigma\tau}$ and I(2.3) (xi) *[Z(P), N,(P)] = 1*, and so $U^* \nleq$ N,(P) by Lemma 3.5(i)(a).

Therefore, by Lemma 3.5, either

(a) $U_{\alpha_i} \leq \mathscr{P}_U(P)$ and $P_{\alpha_j}, P_{\alpha_k} \leq \mathscr{P}_P(U)$; or

(b) $P_{\alpha_i} \leq \mathscr{P}_P(U)$ and $U_{\alpha_j}, U_{\alpha_k} \leq \mathscr{P}_U(P)$

(where {i, j, k} = Λ).

If (b) holds, then Z(P) $\leq P_{\sigma\tau}$ and Lemma 3.5 (ii) (e) imply $\alpha_i = \rho$ and $\{\alpha_j, \alpha_k\} = \{0, \tau\}$. So to complete the proof of the lemma we must show (a) cannot occur.

Assume (a) holds. Then $u = 2$ by Lemma 3.5 (ii) (a) and hence, by our origina1 choice of notation, $P^* \leq N_P(Q)$. Also, by Lemma 3.5 (ii) (c), $\alpha_i \neq \rho$ since $P_{\sigma\tau} \neq 1$. Without loss of generality we may suppose

$$U_\tau \leq \mathscr{P}_U(P) \quad \text{and} \quad P_\rho, P_\sigma \leq \mathscr{P}_P(U).$$

From Lemma 3.5(ii) we have

(5.11) (i) $\mathscr{P}_U(P) = N_U(P)$

(ii) $P^* = P_\tau > \mathscr{P}_P(U)$

(iii) $[N_U(P), \rho], [N_U(P), \sigma] \leq U_\tau$,

and $N_U(R) \leq$ N,(P) for all non-trivial α-invariant subgroups R of U_τ.

Suppose $O_u(QU)_\tau = 1$. Then $[[Q,\tau], O_u(QU)] = 1$ by I(2.11), and hence, using I(2.3) (v), $N_G([Q, \tau]) \geq P_\tau, O_u(QU)$. Since $[Q, \tau] \neq 1$ by Lemma 3.5(i) (e), either $P_\tau \leq \mathscr{P}_P(U)$ or $O_u(QU) \leq \mathscr{P}_U(P)$ must hold. But both alternatives are impossible, and so $O_u(QU)_\tau \neq 1$ must hold. Then, by (5.11) (iii), $Z(O_u(QU)) \leq$ *N,(P).* Because $O_u(QU) \nleq N_U(P)$, (5.11) (iii) implies $Z(O_u(QU)) \leq U_{\rho\sigma}$, whence $[Q, Z(O_u(QU))] = 1$ by I(2.3) (xi). Consequently $Z(O_u(QU))$ normalizes both $N_P(Q)$ and *P.* Employing I(2.14) (ii) yields

$$P = N_P(Q)C_P(O_u(QU)) = N_P(Q)\mathscr{P}_P(U).$$

But $\mathscr{P}_P(U) < P_\tau \leq N_P(Q)$ by (5.11)(ii) implies $P = N_P(Q)$, a contradiction. Therefore (a) cannot hold, and so we have prove the lemma.

Lemma 5.10. ***Let U be a non-trivial α-invariant Sylow u -subgroup of B , then*** $P^* \leq N_P(U)$.

Proof. Suppose the lemma is false. Then $P_\rho \leq \mathscr{P}_P(U) = N_P(U)$ and $U_\sigma, U_\tau \leq$ @u(P) by Lemma 5.9. So p = 2 by Lemma 3.5(ii) (a). By (5.1) O,(H) $\not\leq$ N,,(U). If $O_2(H)_\rho \neq 1$, then Lemma 3.5 (ii) (d) implies $Z(O_2(H)) \leq N_P(U)$, whence $Z(O_2(H))_\rho = 1$. Hence $Z(O_2(H)) \leq P_{\sigma\tau}$ by Lemma 3.5(ii)(f). Using I(2.3) (xi) we conclude that

$$Z(P) \cap Z(O_2(H)) \leq Z(H).$$

But then, by Lemma 5.8 (ii), $(Z(P) \cap Z(O_2(H)))^G$ is a non-trivial proper cr-invariant norma1 subgroup of G . Therefore

(5.12) $$O_2(H)_\rho = 1.$$

Let $\tilde{A}$ denote the α-invariat Hall $2'$-subgroup of *A*. Then (5.12) and I(2.14) (ii) imply

(5.13) $$\tilde{A} = C_{\tilde{A}}(O_2(H))\tilde{A}_\rho$$

In order to make use of (5.13) we must modify the factorization G = *HK* . First we prove

(5.14) $\langle K, \tilde{A}_\rho, Z(P)\rangle$ is a proper cr-invariant subgroup of G.

Let $\tilde{K}$ denote the α-invariant Hall π'_{23}-subgroup of *B* . Let W be an cr-invariant Sylow w-subgroup of $\tilde{A}$. We now show that $W_\rho \leq \mathscr{P}_{\tilde{A}}(\tilde{K})$, and clearly only need to examine the case $W \not\leq \mathscr{P}_{\tilde{A}}(\tilde{K})$. By Lemma 5.2(iii) one of the following holds

(a) $\tilde{K} = L_jB, j \neq 1$ and $WL_j \neq L_jW$

(b) $\tilde{K} = L_jL_kB, j \neq 1 \neq k$ and $L_jW \neq WL_j, L_kW = WL_k$.

(c) $\tilde{K} = L_jL_kB, j \neq 1 \neq k$ and $L_jW \neq WL_j, L_kW \neq WL_k$.

Suppose (a) holds. Then applying I(2.26) with $M = L_j$, $L = BW$ and $H = W$ (note that $G \neq L_j(BW)$) gives that the Sylow w-subgroup of $\mathscr{P}_{WB}(L_j)$ is $\mathscr{P}_W(L_j)$. In particular

$\mathscr{P}_W(L_j)$ permutes with B, and hence $\mathscr{P}_W(L_j) \leq \mathscr{P}_{\tilde{A}}(\tilde{K})$. For case (b), but in I(2.26) taking $L = L_k BW$, we also obtain $\mathscr{P}_W(L_j) \leq \mathscr{P}_{\tilde{A}}(\tilde{K})$. In case (c) the same arguments yield $N_W(L_j) \cap N_W(L_k) = \mathscr{P}_W(L_j L_k) \leq \mathscr{P}_{\tilde{A}}(\tilde{K})$.

Since Q is not star-covered, if $L_j W \neq W L_j$, then I(5.8) (f) shows $L_j^* \nleq N_{L_j}(W)$. Hence, for $j \neq 1$, $W_p \leq \mathscr{P}_W(L_j) = N_W(L_j)$. Therefore, by the above, we have that $W_p \leq \mathscr{P}_{\tilde{A}}(\tilde{K})$, as required.

Because W was an arbitrary α-invariant Sylow subgroup of $\tilde{A}$ it follows that $\tilde{A}_p \leq \mathscr{P}_{\tilde{A}}(\tilde{K})$. By I(4.4) $O_q(\tilde{\mathscr{P}}_{\tilde{A}}(\tilde{K})) \neq 1$ and so, as $[L_{23}, Q] = 1$, $K \leq N_G(O_q(\tilde{K}\mathscr{P}_{\tilde{A}}(\tilde{K})))$. Let F denote the α-invariant Hall $2'$-subgroup of $N_G(O_q(\tilde{K}\mathscr{P}_{\tilde{A}}(\tilde{K})))$. Then $K, \tilde{A}_p \leq F$. As G contains no non-trivial proper α-invariant normal subgroups $O_{\pi(K)'}(F) = 1$. Hence, by [Theorem 1; 1], there is a non-trivial characteristic subgroup C of K such that $C \trianglelefteq F$. Appealing to Lemma 5.8(ii) we have

$$\langle K, \tilde{A}_p, Z(P)\rangle \leq N_G(C) \neq G,$$

which proves (5.14).

If $K = L_{23} B$, then Lemma 5.2(iii), $O_q(BA) \neq 1$ and [Theorem 1; 1] yield that

$$M = \langle K, \tilde{A}, Z(P)\rangle \neq G.$$

Set $D = Z(P) \cap Z(O_2(PL_2)) \cap Z(O_2(PL_3))$. Note that $D \neq 1$. Employing Lemma 5.2(i), (ii) and 5.3(i) gives

$$G = HK = C_G(D)M,$$

and then $D^G \leq M \neq G$, a contradiction. So we may suppose $K \neq L_{23} B$ and so $H = L_1 L_j A (j \neq 1)$ or $L_1 A$. In the former case set $E = Z(P) \cap O_2(PL_j) \cap O_2(A)$ and in the latter $E = Z(P) \cap O_2(A)$. Observe that $E \neq 1$. By Lemmas 5.2(ii) and 5.3(i) and (5.13) $H = C_H(E)\tilde{A}_p$. Therefore

$$G = HK = C_G(E)\langle K, \tilde{A}_p\rangle,$$

whence, using (5.14),

$$E^G \leq \langle K, \tilde{A}_p Z(P)\rangle \neq G,$$

a contradiction which completes the proof of Lemma 5.10.

Lemma 5.11. *Suppose that* $PL_i \neq L_iP$ *(where* $i = 2$ *or* 3 *) and that* $Z(J(P)) \leq N_P(Q)$. *Then* $Z(J(P)) \leq N_P(L_i)$.

Proof. Suppose the lemma is false, and assume $i = 3$. Put $R = Z(J(P))$. By Lemma 5.10, $P^* \leq N_P(Q)$ and by Lemma 5.7(i), $P_\rho, P_\sigma \leq N_P(L_3)$. Of course we also have $L_3Q = QL_3$.

If $R_\sigma \neq R^*_{\sigma_{(\rho\tau)}}$, then $F = O_p(P_\sigma L_{3\sigma}) \cap R \neq 1$ by I(4.5) whence, as $L_{3_\sigma} \trianglelefteq P_\sigma L_{3\sigma}$, $C_G(F) \geq L_{3_\sigma}, R$. Then $R \leq N_P(L_3)$ by Lemma 3.3(i). Therefore $R_\sigma = R^*_{\sigma_{(\rho\tau)}}$ and, similarly, $R_\rho = R^*_{\rho_{(\sigma\tau)}}$. As $P_{\sigma\rho} \leq C_P(L_3)$, $R_{\rho\sigma} = 1$ and consequently $R^* = R_\tau$. So $[Q, [R, \tau]] = 1$ by I(2.8). By I(2.13) and Lemma 3.5(i) (c) $O_q(QL_3) \neq 1$. Thus $[R, \tau] \leq N_P(L_3)$. Since $R \not\leq N_P(L_3)$ by supposition, $[R, \tau] = 1$. From $R \leq P_\tau$ we conclude that $[R, N_{L_3}(P)] = 1$, whence $N_{L_3}(P) = 1$. Thus $\mathscr{P}_{L_3}(P) = 1$ by Lemma 3.4(ii)(a). Consequently as $N_P(Q) \not\leq N_P(L_3)$, $N_{L_3}(Q_1) = 1$ for all non-trivial characteristic subgroups Q_1 of Q. In particular $L_3 \trianglelefteq L_3 Q$ by I(2.6). So $[[Q, \tau], L_3] = 1$. Since $[Q, \tau] \neq 1$, we then obtain, using I(2.3) (viii),

$$R \leq P_\tau \leq N_P(L_3),$$

a contradiction. This completes the proof of the lemma.

Conclusion of the proof of Theorem 5.1

Set $D = Z(P) \cap O_p(H)$. Since P is not star-covered, $D \neq 1$. If $Z(J(P))_\rho = Z(J(P))^*_{\rho_{(\sigma\tau)}}$, $Z(J(P))_\sigma = Z(J(P))^*_{\sigma_{(\rho\tau)}}$ and $Z(J(P))_\tau = Z(J(P))^*_{\tau_{(\rho\sigma)}}$ holds, then $N_H(Z(J(P))) = C_H(Z(J(P)))P$ by I(6.4). Hence $H \leq C_G(D)$ by I(2.6), and then Lemma 5.6(ii) implies that $D^G \neq G$, a contradiction.

Therefore we must have, say $Z(J(P))_\rho \neq Z(J(P))^*_{\rho_{(\sigma\tau)}}$. Let U be a non-trivial α-invariant Sylow subgroup of B. By I(4.5) $Z(J(P)) \cap O_p(P_\rho U_\rho) \neq 1$ and hence, using Lemmas 5.10 and 3.5(i)(a) we obtain $Z(J(P)) \leq N_P(U)$. Thus $Z(J(P)) \leq N_P(B)$. Appealing to Lemmas 5.2(vi) and 5.11 then yields $Z(J(P)) \leq N_P(K)$. By I(2.6), $D^H \leq Z(J(P))$. But then $D^G \leq N_G(K) \neq G$, which is the final contradiction. Thus we have proved Theorem 5.1.

6. FACTORIZATIONS FOR G

We now assemble the result of the two previous sections so as to obtain global information about G.

6.1. Let P be an α-invariant Sylow p-subgroup of G of type Λ. Then either

(i) P permutes with at least two of L_1, L_2 and L_3; or

(ii) with a possible re-ordering of 1, 2, 3, $G = (L_1 L_{12})(L_2\, L_3\, L_{23})$ with $L_1 L_{12}$ and $L_2\, L_3\, L_{23}$ soluble Hall subgroups.

Proof. Suppose $PL, \neq L_2 P$ and $PL, \neq L_3 P$. Then $L_2 L_3 = L_3 L_2$ by Lemma 4.1. From Lemma 4.6 we **have**

(6.1) **(i)** $P_\rho, \dot{P}_\tau \leq N_P(L_2)$ **and** $P_\rho, P_\sigma \leq N_P(L_3)$;
(ii) $Z(P) \leq P_{\sigma\tau}$; and
(iii) P is not star-covered.

From **(6.1)(ii)** and $PL, \neq L_2 P$, $PL, \neq L_3 P$ *we* note that

$$(6.2) \qquad L_1 P = PL, \quad \text{and} \quad L_{12} = L_{13} = 1.$$

Now let W be an α -invariant Sylow w -subgroup of L and suppose $L_1 W \neq WL$, . By Theorem 5.1 $PW = WP$. Combining **(6.1)(iii)**, I(4.4) and **I(5.8)(f)** we deduce that W_σ, $W_\tau \leq N_W(L_1)$. **From** $[L_1, Z(P)] = 1$ **and** $O_p(PW) \neq 1$ **we obtain** $O_w(PW) \leq N_W(L_1)$. By Lemma **3.4(ii)(d)** and **I(4.6)**, $N_W(L,) \leq W_\rho$, and thus $W = W_\rho$. Then Lemma 3.4 shows **that** W must permute with both L_2 and L_3 . A **further consequence** of $W = W_\rho$, using **I(2.3)(ix)** and (6.1) (i), is

$$(6.3) \qquad P = P_\rho O_p(PW) = N_P(L_i) O_p(PW) \quad (i = 2, 3).$$

By **I(2.10)(ii)** and I(6.1) $W \neq W_\sigma W_\tau$ and so, **as** WL, L_3 admits $\sigma\tau$ fixed-point-freely, $O_w(WL, L_3) \neq 1$ by **(I(2.10)** (iii). So at **least one of** $[O_w(WL_2 L_3), \sigma]$ and $[O_w(WL_2 L_3), \tau]$ is non-trivial. Suppose $[O_w(WL, L_3), \sigma] \neq 1$. Then $[O_w(WL,), \sigma] \neq 1$. Since $W_\sigma \leq N_W(L_3)$, an application of I(5.8) (c) gives either $O_p(PW) \leq N_P(L_3)$ or $L_{3_\sigma} \leq \mathscr{P}_{L_3}(P)$. The former possibility together with (6.3) contradicts $PL, \neq L_3 P$ whilst the latter is **untenable** by (6.1) (i) and Lemma 3.3(i). Thus **there** is no α**-invariant** Sylow subgroup W of L for which $WL, \neq L_1 W$ and so, by Theorem 5.1, LL, is a **soluble** subgroup of G. Since we **also have** $G = (LL_1)(L_2 L_3 L,)$, the lemma is proved.

Lemma 6.2. *At least two of L_1, L_2 and L_3 permute.*

Proof. We suppose the lemma is false and deduce a contradiction. As $\mathscr{L}_i$ is nilpotent for **all** $i \in \Lambda$, **we have** $L_{12} = L_{13} = L_{23} = 1$. Theorem 4.3 is available and so we may assume that

$$L_1 = L_{1_\tau}, L_2 = L_{2_\rho} \quad \text{and} \quad L_3 = L_{3_\sigma}.$$

By **I(2.3)** (ix) we **have**

$$(6.4) \qquad \begin{aligned} &\mathscr{M}(\pi_1, \pi_2) = \{L_1 N_{L_2}(L_1), L_2\}, \mathscr{M}(\pi_1, \pi_3) = \{L_1, L_3 N_{L_1}(L_3)\} \text{ and} \\ &\mathscr{M}(\pi_2, \pi_3) = \{L_2 N_{L_3}(L_2), L_3\}. \end{aligned}$$

Let T denote the a-invariant Sylow 2-subgroup of G. By I(2.24) T is not contained in G_ρ, G_σ nor G_τ. Therefore $2 \notin \pi_1 \cup \pi_2 \cup \pi_3$, and so T must be of type Λ. By Lemma 4.1 T must permute with at least two of L_1, L_2 and L_3. Therefore there are, essentially, two cases to examine:

Case 1. T permutes with L_2 and L_3 but does not permute with L_1; and

Case 2. T permutes with L_1, L_2 and L_3.

Case 1. As $L_1 = L_1$ and $TL_1 \neq L_1T$, it follows that $T_\sigma, T_\tau \leq N_T(L_1)$ and, furthermore, that $[T_\sigma, L_1] = 1$ because $[N_T(L_1), \rho\tau] \leq C_T(L_1)$ by 1(2.3)(x) and I(2.11). Since $[T_{\rho\sigma}, L_3] = 1$ and $L_1 L_3 \neq L_3 L_1$, this implies $T_{\rho\sigma} = 1$, and so TL_1 admits $\rho\sigma$ fixed-point-freely. Hence $L_2 = N_{L_2}(T) O_{\pi_2}(TL_2)$ by I(2.10) (i). Now $[T, \sigma] \leq O_2(TL_2)$, $[T, \sigma] \trianglelefteq$ TL, and $[T, \sigma] \neq 1$, so $O_{\pi_2}(TL_2) \leq \mathscr{P}_{L_2}(L_3) = 1$ by (6.4). Thus $L_2 = N_{L_2}(T)$.

Since $C_T(L_1) \neq 1$ and $T \nleq G_\tau$, $[N_T(L_1), \tau] \neq 1$, and because $[T, \tau] \leq O_2(TL_1)$, wemayinferthat $O_{\pi_3}(TL_3) \leq \mathscr{P}_{L_3}(L_1) = 1$, by (6.4). So $L_3 = N_{L_3}(J(T)) C_{L_3}(Z(T))$ by I(2.6). Now a further appeal to (6.4) gives $L_3 \leq N_{L_3}(L_2)$, which disposes of case 1.

Case 2. As $1 \neq [T, \rho] \leq O_2(TL_1)$ and $[T, \rho] \trianglelefteq TL_1$, we conclude using (6.4) that $O_{\pi_1}(TL_1) = 1$. Likewise, for $i \in \Lambda$, we obtain $O_{\pi_i}(TL_i) = 1$ and hence $L_i = N_{L_i}(J(T)) C_{L_i}(Z(T))$ by I(2.6). We claim that for each $i \in \Lambda$ $N_{L_i}(J(T)) \neq 1 \neq C_{L_i}(Z(T))$. For suppose, say, that $C_{L_1}(Z(T)) = 1$. Then $L_1 = N_{L_1}(J(T))$. The shape of $\mathscr{M}(\pi_1, \pi_3)$ gives $N_{L_3}(J(T)) = 1$, whence $L_3 = C_{L_3}(Z(T))$. Now the shape of $\mathscr{M}(\pi_2, \pi_3)$ implies $C_{L_2}(Z(T)) = 1$. Therefore $L_2 = N_{L_2}(J(T))$ and so $L_1 L_2 = L_2 L_1$, a contradiction. Hence $C_{L_i}(Z(T)) \neq 1$, and a similar argument shows $N_{L_i}(J(T)) \neq 1$, as claimed.

From $N_{L_i}(J(T)) \neq 1 \neq C_{L_i}(Z(T))$ for $i = 1, 2$, (6.4) dictates that $L_2 = N_{L_2}(J(T)) C_{L_2}(Z(T)) \leq N_{L_2}(L_1)$. This finishes case 2 and the proof of the lemma.

Theorem 6.3. *With a possible re-ordering of* 1, 2, 3, *either*

(i) *$G = (LL_1 L_3 L_{13}) L_2$ with $LL_2 L_3 L_{23}$ a soluble Hall subgroup; or*

(ii) *$G = (LL_1)(L_2 L_3 L_{23})$ with LL_1 and $L_2 L_3 L_{23}$ both soluble Hall subgroups.*

Proof. Recall that L is a soluble Hall subgroup by Theorem 5.1 and that, if $L_{ij} \neq 1$ ($i, j \in \Lambda$, $i \neq j$), then $L^*_{ij} \neq L_{ij}$.

We break the proof into two parts depending on whether or not all of L_{12}, L_{13} and L_{23} are trivial. First suppose that, say, $L_{23} \neq 1$. Clearly then $\mathscr{L}_2 \mathscr{L}_3 = \mathscr{L}_3 \mathscr{L}_2$. Suppose P

is an cu-invariant Sylow subgroup of L which permutes with L_{23}. Since $\mathscr{L}_2$ and $\mathscr{L}_3$ are nilpotent and $O_{\pi_{23}}(PL_{23}) \neq 1$, it follows that P permutes with $\mathscr{L}_2$ and $\mathscr{L}_3$. On the other hand, if Q is an cr-invariant Sylow subgroup of L which does not permute with L_{23}, then Z(Q) $\leq Q_{\sigma\tau}$ by Lemma 3.2 and hence $Q\mathscr{L}_1 = \mathscr{L}_1$ Q. Let L^+ (respectively L^-) denote the group generated by those cr-invariant Sylow subgroups of L which permute (respectively do not permute) with L_{23}. Then G = $(\mathscr{L}_2\mathscr{L}_3L^+)(L^-\mathscr{L}_1)$ with $\mathscr{L}_2\mathscr{L}_3L^+$ and $L^-\mathscr{L}_1$ soluble Hall subgroups of G. Since G contains no non-trivial proper α-invariant normal subgroups, $L_{12} = L_{13} = 1$ whence $G = (L_2L_3L_{23}L^+)(L^-L_1)$. Iftheconclusionof the theorem were false there would exist an cr-invariant Sylow subgroup P of L^+ such that $PL, \neq L_1 P$ and an α-invariant Sylow subgroup Q of L^- such that QZ, $\neq L_{23}$ Q . However $PL_{23} = L_{23}P$ and $QL, = L,$ Q , a configuration which is impossible by Lemma 4.5.

Now we consider the case $L_{12} = L_{13} = L_{23} = 1$. By Lemma 6.2 we may assume that $L_2L_3 = L_3L_2$. In view of Lemma 6.1, we may suppose for each α-invariant Sylow subgroup P of L that P permutes with at least two of L_1, L_2 and L_3. Therefore G = (L, L_3 L') (L-L,) where L^+ (respectively L^-) are the subgroups of-L generated by those α-invariant Sylow subgroups of L which permute with L_2 and L_3, (respectively L_1). Again, if the theorem does not hold then it is possible to select α-invariant Sylow subgroups P and Q of (respectively) L^+ and L^- such that $PL, \neq L_1P$ and, say, $QL_2 \neq L_2Q$. Since $PL_2 = L_2P$ and $QL, = L_1$ Q , Theorem 4.4 denies the credibility of this situation. Therefore, in this case also, either $G = (L_2L_3L)L_1$ orG= $(L_2L_3)(LL_1)$.

7. MORE ON FACTORIZATIONS

In this, the final section, we examine the possible factorizations of G as predicted by Theorem 6.3. We begin with a hypothesis.

Hypothesis 7.1. *(i)* G = $K\mathscr{L}_i$ *where* $i \in \Lambda$

(i) K is an α-invariant soluble subgroup of G with $\pi(K) \cap \hat{\pi}_i = \phi$.

Theorem *7.2. Hypothesis 7.1 does not hold.*

Proof. We show that Hypothesis 7.1 leads to a contradiction. Without loss of generality we take $i = 1$. Clearly we must have $\mathscr{L}_1 \neq 1$. Put $\tilde{K} = N_K(\mathscr{L}_1)$. Because G contains no non-trivial proper α - invariant normal subgroups and G = $K\mathscr{L}_1$, $\mathscr{L}_1$ cannot normalize any non-trivial α-invariant subgroups stet of K. Thus, if H is a proper α-invariant subgroup of G containing $\mathscr{L}_1$ then $H \leq N_G(\mathscr{L}_1)$ by I(2.13). So we have shown that

(7.1) (i) $N_G(\mathscr{L}_1)$ is the unique maximal α-invariant subgroup of G containing $\mathscr{L}_1$;

(ii) $O_{\hat{\pi}_1'}(N_G(\mathscr{L}_1)) = 1$;and

(iii) $N_G(\mathscr{L}_1) = \tilde{K}\mathscr{L}_1$ with $\tilde{K} \leq K_\rho$.

Since $[K_{\sigma\tau}, \mathscr{L}_1] = 1$ by I(3.13)(iii), (7.1)(ii) implies

(7.2) $\sigma\tau$ acts fixed-point-freely upon K.

(7.3) Let $p \in \pi(K)$ and let P be the α-invariant Sylow p-subgroupof K. Then $P \not\leq \tilde{K}$.

For suppose $P \leq \tilde{K}$. Then we must have $O_p(K) = 1$. So $P = P_\sigma P_\tau$ by (7.2) and I(2.10) (iii). Since $P = P_\rho$ by (7.1)(iii) and $P \in Syl_p G$, I(6.1) and I(6.4) combine to yield a contradiction. Thus $P \not\leq \tilde{K}$, as asserted.

We now come to the heart of the proof of the theorem, namely that of showing

(7.4) $$K_\sigma, K_\tau \leq \tilde{K}.$$

First we note some easy reductions. Since $[K_\sigma, L_{,,}] = [K_\tau, L_{12}] = 1$, if we have $L_{12} \neq 1 \neq L_{13}$, then (7.1)(i) yields (7.4). So, without loss of generality, we may assume $L_{12} = 1$. If $L_1 = 1$, then $\mathscr{L}_1 = L_{13}$ and so $[\mathscr{L}_1, K,] = 1$, which implies $K_\sigma = 1$ by (7.1) (ii). Then K is nilpotent by I(2.2) (i), whence, by I(2.5), G is soluble, a contradiction. Therefore, in proving (7.4), we may suppose $\mathscr{L}_1 = L_1 L_{13}$ with $L_1 \neq 1$.

Before proceeding further it is convenient to rule out a particular situation.

(7.5) $$L_{1_\sigma} \neq L_{1_\tau}$$

Suppose $L_{1_\sigma} = L_{1_\tau}$ were to hold. Then by I(6.4).

(7.6) Every proper α -invariant subgroup of G has a normal p-complement for each p $\in \pi_1$.

Hence $\pi_1 = \{2\}$ by Thompson's normal p-complement theorem. From I(2.1)(v) we see that L_1 normalizes $\tilde{K}$. Hence

$$(C_{\tilde{K}}(L_{13}))^{\mathscr{L}_1 \tilde{K}} = (C_{\tilde{K}}(L_{13}))^{L_1 \tilde{K}} \leq \tilde{K},$$

and hence $C_{\tilde{K}}(L_{13}) = 1$ by (7.1)(ii). Now $L_{13} \neq 1$ would yield $1 \neq K_\sigma \leq C_{\tilde{K}}(L_{13})$ and so we deduce that $L_{13} = 1$. Thus $\mathscr{L}_1 = L_1$ and clearly, $\tilde{K} = 1$.

Foreach $p \in \pi(K)$, by (7.3), $PL, \neq L_1 P$ where P is the a-invariant Sylow p-subgroup of K. It then follows easily that if at least one of P_σ and P_τ is non-trivial, then $L_1^* \leq N_{L_1}(P)$. Hence $L_1^* \leq N_{L_1}(LL_2 L_3)$. Because $LL, L_3 \neq 1$ and $LL_2 L_3 \trianglelefteq K$ by I(2.8) and (7.2), (7.6) then yields that $L_1^* \leq N_{L_1}(K)$. From (7.2) $G_{\sigma\tau} = L_{1_{\sigma\tau}}$ and thus we have verified all the hypotheses of I(6.2) with γ = or. As a consequence G has a normal 2-complement, which is impossible. Therefore $L_{1_\sigma} \neq L_{1_\tau}$ holds.

(7.7) If $L \neq 1$, then (7.4) holds.

We begin by establishing

(7.8) (i) L_1 does not permute with any (non-trivial) a-invariant Sylow p-subgroups **of** K **of type** Λ **; and**

(ii) $L_1 L_i \neq L_i L_1$ for $i = 1, 2$ (provided $L_1 \neq 1$).

If $L_{13} = 1$, then (7.8) follows immediately from (7.3). So while **proving** (7.8) we may suppose $L_{13} \neq 1$. Let P be a (non-trivial) α**-invariant** Sylow p-subgroup of K of type Λ . Suppose ***PL,,*** $= L_{13} P$ weretohold. **By I(2.8) and I(6.1),** $L_{13} \not\leq G_\sigma$ andso $O_{\pi_{13}}(PL_{13}) \neq 1$ by I(4.5). Consequently $P \leq \tilde{K}$ by (7.1) (i), **contrary** to (7.3). Hence ***PL,,*** $\neq L_{13}$ ***P.*** From **Lemma 3.2** $\mathscr{M}(p, \pi_{13}) = \{L_{13} N_P(L_{13}), P\}$ **with** $C_P(L_{13}) \neq 1$. **Clearly** $N_P(L_{13}) \leq \tilde{K} \leq K_\rho$. ***Thus P =*** P_ρ by I(2.3) (v). Now, if it were the case that ***PL, = L, P,*** **then I(2.3)(ix)** would yield $L_1 \trianglelefteq L_1 P$ whence $P \leq \tilde{K}$, against (7.3). Hence ***PL,*** $\neq L_1 P$ holds and we **have** proved (7.8) (i).

We now prove **(ii).** Since $L_{13} \neq 1$, (7.3) **forces** $L_3 = 1$. Thus we only **need** show $L_1 L_2 \neq L_2 L_1$. **Supppose** $L_1 L_2 = L_2 L_1$ were to hold. Then (7.3) implies $O_{\pi_1}(L_1$ ***L,)*** $= 1$ and so $L_1 = L_{1_\sigma}$ by 1(2.13)(i). Since $L \neq 1$ by hypothesis we may **choose** ***P to bea*** **(non-trivial)** α **-invariant** Sylow p-subgroup of ***K*** of type Λ . By part (i), L_1 ***P# PL,*** . Consulting Lemma 3.3 and using I(2.3) (x) yields first $C_P(L_1) \neq 1$, and then $Z(P) \leq P_\sigma$. However $[L_{13}, P_\sigma] = 1$ and consequently ***PL,,*** $= L_{13}$ ***P.*** So $P\mathscr{L}_1 = \mathscr{L}_1$ ***P,*** **contrary** to (7.3). From **this** contradiction we deduce that $L_1 L_2 \neq L_2 L_1$. The proof of (7.8) is complete.

(7.9)

$$Z(L_1) \neq Z(L_1)_{\sigma\tau}$$

Suppose $Z($ ***L,)*** $= Z(L_1)_{\sigma\tau}$ were to hold and let ***P*** be **an** α**-invariant** Sylow p-subgroup of $L, p \in \pi(L)$. By I(2.3) (xi) $[N_P(L_1), Z($***L,)]*** $= 1$. From (7.8) (i) ***PL,*** $\neq L_1 P$ and so, by Lemma 3.3, either $L_1^* \leq N_{L_1}($ ***P)*** or $P_\sigma, P_\tau \leq N_P($ ***L,)*** . Consequently $Z(L_1) \leq N_{L_1}$ ***(P)*** **;** this is **clear** in the first case and in the latter case follows from [$N_P($***L,)***, $Z($***L,)]*** $=$ $1 \neq N_P(L_1)$. Therefore $Z(L_1) \leq N_G(L)$. A similar argument shows that $Z(L_1) \leq N_G(L_2 L_3)$ and so

$$Z(L_1) \leq N_G(LL_2 L_3).$$

Recalling that $1 \neq$ ***LL,*** $L_3 \trianglelefteq$ ***K*** we **see** that ***(K,*** $Z($ ***L,))*** is a **proper** α**-invariant** subgroup ofG.Now

$$1 \neq Z(L_1)^G = Z(L_1)^K \leq \langle K, Z(L_1) \rangle,$$

a contradiction. Thus we must **have** $Z($ ***L,)*** $\neq Z(L_1)_{\sigma\tau}$.

We are moving **closer** to verifying (7.7).

(7.10) Let $P \in \pi(L)$ and let P be the α-invariant Sylow p-subgroup of L. Then $P_\sigma, P_\tau \leq N_P(L_1) \leq \tilde{K}$.

We only need show that $P_\sigma, P_\tau \leq N_P(L_1)$, since $N_P(L_1) \leq \tilde{K}$. If $P_\sigma, P_\tau \not\leq N_P(L_1)$, then by Lemma 3.4(i) (c) and (d) either $Z(L,) = Z(L_1)_{\sigma\tau}$ or $L_{1_\sigma} = L_{1_\tau}$. By (7.5) and (7.9) neither of these possibilities can occur. Thus P_σ, $P_\tau \leq N_P(L,)$.

(7 .11) $$L_{2_\tau} \leq K.$$

Suppose (7.11) is false. Then $L_2 \neq 1$ and so $L_1 L_2 \neq L_2 L_1$. Since $N_K(L_1) \leq \tilde{K}, L_{2_\tau} \not\leq N_{L_2}(L_1)$ and so $L_{1_\tau} \leq N_{L_1}(L_2)$. Let P be some fixed α-invariant Sylow p-subgroup of $L, p \in \pi(L)$. By (7.8)(i) and (7.10) $PL, \neq L_1 P$ with P_σ, $P_\tau \leq N_P(L,)$. Hence

(7.12) $$\mathscr{P}_{L_1}(P) = N_{L_1}(P) \leq L_{1_{\sigma\tau}}.$$

by Lemma 3.4(ii)(c) and (f).

It is claimed that

(7.13) (i) $C_{L_1}(L_2) = 1$; and
(ii) $L_1^* = L_{1_\sigma}$.

Clearly (i) implies (ii) by I(2.1 1), so we only need prove (i).

Suppose C_{L_1} $(L,) \neq 1$ and argue for a contradiction. Hence $Z(L,) \leq N_{L_1}$ $(L,)$ and then $Z(L_1) \leq L_{1_\sigma}$. Using I(2.3)(x) we then obtain

$$[[N_P(L_1), \sigma], Z(L_1)] = 1$$

Because $P_\sigma, P_\tau \leq N_P(L_1) \leq \tilde{K} \leq K_\rho$, we have $[N_P(L_1), \sigma] \neq 1$. Therefore, either

$$Z(L_1) \leq \mathscr{P}_{L_1}(P) \leq L_{1_{\sigma\tau}} \quad \text{or}$$
$$C_P([N_P(L_1), \sigma]) \leq N_P(L_1) \leq P_\rho.$$

By (7.9) only the latter can hold. Then $P = P_\rho$ by 1(2.3)(v), whence $\mathscr{P}_{L_1}$ $(P) = 1$.

If $O_{\pi_2}(PL_2) \neq 1$, then since $N_G(O_{\pi_2}(PL_2)) \geq P, C_{L_1}(L_2)$, we obtain

$$1 \neq C_{L_1}(L_2) \leq \mathscr{P}_{L_1}(P) = 1.$$

Hence 0, *(PL,)* = 1. Since $P = P_\rho$, I(2.3)(ix) forces $L_2 = L_{2_\rho}$. But then $N_{L_1}(L_2) \trianglelefteq L_2 N_{L_1}$ *(L,)* and $N_{L_1}(L_2) \neq 1$ imply $L_1 L_2 = L_2 L$, . This contradicts (7.8)(ii) and so verifies (7.13).

(7.14) $$L_1 \neq L_{1_\sigma}$$

Suppose $L_1 = L_{1_\sigma}$ were to hold. Then $P_\sigma, P_\tau \le N_P($*L,)*$) \le P_\rho$ yields, using 1(2.3)(x), Z(P) $\le P_\sigma$. But $[P_\sigma, L_{13}] = 1$ then implies that $PL_{13} = L_{13}P$. From $P = P_\rho$ *we* see that $L_{13} \trianglelefteq L_{13}P$. Since $P \not\le \tilde{K}$ by (7.3), we conclude that $L_{13} = 1$. Hence $\mathscr{L}_1 = L_1 \le G_\sigma$. Now I(2.3) (ix) gives $[G, \sigma] \le K \neq$ G. But G does not have any non-trivial proper α-invariant normal subgroups and therefore $L_1 \neq L_{1_\sigma}$.

Since $P_\sigma \le N_P($*L,)*$)$,
$$L_1 = L_{1_\sigma} C_{L_1}(P_\sigma)$$
by (7.13) and I(2.14)(ii). Because $\mathscr{P}_{L_1}(P) \le L_{1_{\sigma\tau}}$ and $L_1 \neq L_{1_\sigma}$ by (7.14), we see that $C_{L_1}(P_\sigma) \not\le \mathscr{P}_{L_1}(P)$. Consequently

(7.15) $$Z(P)_\sigma = 1 \text{ and } Z(P) \le N_P(L_1) \le \tilde{K}.$$

Because $K_{\sigma\tau} = 1$ and $L^*_{2\langle\sigma\tau\rangle} = L_{2_\tau}$, $[P_\tau, L_2] =$ 1,andso $P_\tau \le C_G(L_2)$. Hence,since $P_\tau \le N_P(L_1)$,
$$[N_{L_1}(L_2), P_\tau] \le C_G(L_2) \cap L_1 = C_{L_1}(L_2).$$
Thus [N_{L_1} *(L,)* , P_τ] = 1 by (7.13)(i). If Z(P), $\neq$ 1, then
$$L_{1_\tau} \le N_{L_1}(L_2) \le \mathscr{P}_{L_1}(P).$$

However, we already have $P_\sigma, P_\tau \le N_P($*L,)*$)$ and so $L_{1_\tau} \not\le \mathscr{P}_{L_1}(P)$ by Lemma 3.3(i). Therefore Z(P), = 1 and hence, appealing to (7.15), $Z(P)^*_{\langle\sigma\tau\rangle} = 1$. By I(2.8) and I(2.9), since $K_{\sigma\tau} = 1$, we see that
$$Z(P) \le O_p(K) \text{ and } [Z(P), N_K(P)] = 1$$

Hence, since K has Fitting length at most two, Z(P) $\le$ Z(K) . Consequently, by (7.15), we have
$$1 \neq Z(P)^G = Z(P)^{\mathscr{L}_1} \le N_G(\mathscr{L}_1) \neq G,$$

which is not possible. With this contradiction we have established (7.11).

By a similar argument (and noting that $L_3 \neq 1$ implies $L_{13} = 1$) to the one used to prove (7.11) we also obtain

(7.16) $$L_{3_\sigma} \leq \tilde{K}.$$

Combining (7.10), (7.11) and (7.16) yields (7.7).

(7.17) If $L = 1$, then (7.4) holds.

By I(2.5) K is not nilpotent, and so $L_2 \neq 1 \neq L_3$. Because $L_{13} \neq 1$ implies $L_3 = 1$ by (7.3), we also have $\mathscr{L}_1 = L_1$. In order to show (7.4) holds, because of the symmetry of the arguments, we must show that the two possibilities

$$L_{1_\sigma} \leq N_{L_1}(L_3), \quad L_{1_\tau} \leq N_{L_1}(L_2) \quad \text{and}$$
$$L_{1_\sigma} \leq N_{L_1}(L_3), \quad L_{2_\tau} \leq N_{L_2}(L_1)$$

cannot occur.

Case 1. $L_{1_\sigma} \leq N_{L_1}(L_3), L_{1_\tau} \leq N_{L_1}(L_2)$.

By (7.5) $L_{1_\sigma} \neq L_{1_\tau}$ and so we may assume that, say, $L_{1_\tau} \nleq L_{1_\sigma}$. Then $C_{L_1}(L_2) \neq 1$ by I(2.13). Hence $Z(L_1) \leq N_{L_1}(L_2)$ and so $Z(L_1) = Z(L_1)_\sigma$. But then $Z(L_1) \leq L_{1_\sigma} \leq N_{L_1}(L_3)$. Therefore $Z(L_1) \leq N_G(L_2 L_3)$. Since $1 \neq L_2 L_3 \trianglelefteq K$, $\langle Z(L_1), K \rangle \neq G$, and so $Z(L_1)^G$ is a non-trivial proper α-invariant normal subgroup of G. Consequently $L_{1_\sigma} \leq N_{L_1}(L_3), L_{1_\tau} \leq N_{L_1}(L_2)$ cannot hold.

Case 2. $L_{1_\sigma} \leq N_{L_1}(L_3), L_{2_\tau} \leq N_{L_2}(L_1)$.

Suppose for the moment that $L_{1_\sigma} \leq L_{1_\tau}$. So $L_1^* = L_{1_\tau}$ and by I(2.3)(ix), $L_1 \neq L_{1_\tau}$. By I(2.14) (ii)

$$L_1 = L_{1_\tau} C_{L_1}(L_{2_\tau}).$$

Clearly $C_{L_1}(L_{2_\tau}) \nleq L_{1_\sigma}$. So if $C_{L_1}(L_{2_\tau}) \leq N_{L_1}(L_2)$, I(5.6) forces the contradiction.

$$1 \neq L_{2_\tau} \leq N_{L_2}(L_1) = 1$$

Thus $C_{L_1}(L_{2_\tau}) \not\leq N_{L_2}(L_2)$ and consequently $Z(L_2)_\tau = 1$ and $Z(L_2) \leq N_{L_2}(L_1)$. Since $K_{\sigma\tau} = 1$ and $Z(L_2)^*_{\langle\sigma\tau\rangle} = Z(L_2)_\tau = 1$, I(2.8) and I(2.9) yield $Z(L_2) \trianglelefteq K$. Hence

$$Z(L_2)^G = Z(L_2)^{L_1} \leq N_G(L_1),$$

an untenable situation from which we conclude $L_{1_\sigma} \not\leq L_{1_\tau}$.

From $L_{1_\sigma} \not\leq L_{1_\tau}$, we deduce $C_{L_1}(L_3) \neq 1$ whence $Z(L_1) \leq N_{L_1}(L_3)$ with $Z(L_1) \leq L_{1_\tau}$. We now aim to show that $L_3 = L_{3_\sigma}$. Suppose $L_3 \neq L_{3_\sigma}$. Then $1 \neq [L_3, \sigma] \leq 0, (L_2 L_3)$. Because $L_{1_\sigma} \leq N_{L_1}(L_3)$, I(2.3) (viii) gives

$$O_{\pi_2}(L_2 L_3), L_{1_\sigma} \leq N_G([L_3, \sigma]).$$

Hence either $O_{\pi_2}(L_2 L_3) \leq N_{L_2}(L_1)$ or $L_{1_\sigma} \leq N_{L_1}(L_2)$. The former possibility implies, using 1(2.13)(i), that

$$L_2 = O_{\pi_2}(L_2 L_3) L_{2_\tau} \leq N_{L_2}(L_1),$$

contradicting $L_1 L_2 \neq L_2$ L, . Thus we have $L_{1_\sigma} \leq N_{L_1}($L,$)$, and so $L_{1_\sigma} \leq N_{L_1}(L_2 L_3)$. Because K, $L_{1_\sigma} \leq N_G(L_2 L_3) \neq$ G we conclude that $Z($L,$)$, = 1. But then σ acts fixed-point-freely upon $Z(L_1) N_{L_2}(L_1)$, whence $[Z(L_1), N_{L_2}(L_1)] = 1$. Because $N_{L_2}(L_1) \neq 1$, thisyields $Z(L_1) \leq N_{L_1}(L_2)$. Hence $Z(L_1) \leq N_{L_1}(L_2 L_3)$, and then G contains a non-trivial proper α-invariant normal subgroup. Therefore we must have $L_3 = L_{3_\sigma}$. Recalling that $Z($L,$) = Z($L,$)$, $\leq N_{L_1}(L_3)$, this gives $Z($L,$) = Z(L_1)_{\sigma\tau}$, which is not possible by *(7.9).*

This completes the analysis of Case 2 and the proof of (7.17).

Combining (7.7) and (7.17) establishes (7.4).

Using (7.4) we readily complete the proof of Theorem 7.2. Let P be an arbitrary α-invariant Sylow p-subgroup of K. Since K has Fitting length at most two, $K = N_K($P$)$ 0, (K) by a Frattini argument. Set $M = O_{p'}(K)$. From I(2.14) (ii), I(3.8) and (7.4)

$$[P, M] = [P^*_{\langle\sigma\tau\rangle}, M] \leq \tilde{K} \leq N_G(\mathscr{L}_1).$$

Now [P, $M] \trianglelefteq K$ and thus, as G contains no non-trivial proper α-invariant normal subgroups, $[P, M] = 1$. Hence $P \trianglelefteq K$ and so we deduce that K is nilpotent. By I(2.5) this is not possible and Theorem 7.2 is established.

We now investigate another kind of factorization.

Hypothesis 7.3. G $= (LL_1)(L_1 L_3 L_{23})$ *with* LL_1 *and* $L_2 L_3 L_{23}$ *soluble Hall subgroups of* G.

Let L^+ (respectively L^-) be the subgroup of L generated by the α-invariant Sylow subgroups of L which permute with both L_2 and L_3 (respectively do not permute with both L_2 and L_3). Clearly $L = L^+ L^-$ and $L^+ \cap L^- = 1$. Before considering Theorem 7.6, the last major result of this paper, we prove two preliminary lemmas.

Lemma *7.4. Assume Hypothesis 7.3 holds, and let* P *be a (non-trivial)* α*-invariant Sylow* p*-subgroup of* L^-. *Then* P *permutes with one of* L_2 *and* L_3.

Proof. Suppose $PL_2 \neq L_2 P$ and $PL_3 \neq L_3 P$, and argue for a contradiction. So, by Lemma 4.6, $P_\rho, P_\tau \leq N_P(L_2)$ and $P_\rho, P_\sigma \leq N_P(L_3)$, and, appealing to I(4.5),

$$1 \neq R = O_p(LL_1) \cap Z(P) \leq N_P(L_2 L_3).$$

If $N_G(Z(P)) = PC_G(Z(J(P)))$, then, as $Z(P) \leq Z(J(P))$, $R \leq Z(LL_1)$. Now $L_2 L_3 \neq 1$ by Theorem 7.2 and so $\langle R, L_2 L_3 L_{23}\rangle \leq N_G(L_2 L_3) \neq G$. Then R^G is a non-trivial proper α-invariant normal subgroup of G. Consequently, from Lemma 4.6(iv), we have J(P) contained in at least one of $N_P(L_2)$ and $N_P(L_3)$.

Set $S = R^{LL_1}$. Using I(2.6) we see that $S \leq Z(J(P)))$. If $J(P) \leq N_P(L_2) \cap N_P(L_3)$, then clearly $S \leq N_G(L_2 L_3) \neq G$. So we have $G = N_G(S) N_G(L_2 L_3)$ which implies that S^G is a non-trivial proper α-invariant normal subgroup of G. So to complete the proof of the lemma we have, without loss of generality, to dispose of the case when

(7.18) $$J(P) \leq N_P(L_2) \text{ and } J(P) \nleq N_P(L_3).$$

Suppose (7.18) holds. If $L_{23} \neq 1$, then it is straightforward to show that $PL_{23} \neq L_{23} P$ and (hence) $N_P(L_2) = N_P(L_{23}) = N_P(L_3)$, which contradicts (7.18). Therefore $L_{23} = 1$.

Since $J(P) \leq N_P(L_2)$,

$$J(P) = C_{J(P)}(L_2) J(P)_\sigma$$

by (I.(2.13) (i). Because $P_\sigma \leq N_P(L_3)$ and $J(P) \nleq N_P(L_3)$, $C_P(L_2) \nleq N_P(L_3)$. Thus $O_{\pi_2}(L_2 L_3) = 1$. Then I(2.13) gives

(7.19) $$L_2 = L_{2_\tau} \quad \text{and} \quad L_3 \trianglelefteq L_2 L_3.$$

Since $C_P(J(P)) \leq J(P) \leq N_P(L_2)$ and P is not star-covred by Lemma 4.6(iii), I(2.3)(v) implies $[N_P(L_2), \tau] \neq 1$. So, using (7.19), we have

(7.20) $$1 \neq [N_P(L_2), \tau] \leq C_P(L_2).$$

Clearly, from (7.19), $N_P(L_3) \leq N_P(L_1)$. Hence, by I(2.11), and (7.20),

$$[N_P(L_3), \tau] \leq C_P(L_3) \cap C_P(L_2) \leq C_P(L_2 L_3).$$

Because G = $(LL_1)(L_1 L_3)$ and G contains no non-trivial **proper α-invariant** normal subgroups, we deduce that $[N_P(L_3), \tau] = 1$. Consequently

(7.21) $$P^* = P_\tau \leq N_P(L_2).$$

From Lemma 4.6 $Z(P) \leq Z(PL_1) \cap N_G(L_2 L_3)$ and so we observe that $L \neq P$. Let Q be an α-invariant Sylow q-subgroup of L where $q \in \pi(L) \setminus \{p\}$; *the* existence of Q provides some useful leverage. First we prove

(7.22) (i) $QL_2 \neq L_2 Q$ with $Q_\rho, Q_\tau \leq N_Q(L_2)$;
(ii) $Q = Q_\sigma$; and
(iii) $QL_3 = L_3 Q$.

Suppose $QL_2 = L_2 Q$. Then applying I(5.8) (f) with $L = Q$, $M = P$ and $N = L_2$ yields, since $P^* \leq N_P(L_2)$, that $O_q(QL_2) = 1$. However $L_2 = L_{2_\tau}$ and $L_{2_\sigma} = 1$, then force $Q = Q_{\sigma\tau}$, which is not possible. Therefore $QL_2 \neq L_2 Q$, and because $L_2 = L_{2_\tau}$ we must have $Q_\rho, Q_\tau \leq N_Q(L_2)$. This proves (i).

From (7.21) and I(4.5), $[P, \tau] \leq O_p(LL_1)$. Hence, (7.20) forces $O_p(LL_1) \leq N_Q(L_1)$. Consequently $Q = Q_\sigma$ by I(4.6) and Lemma 3.4(ii)(d), and we have (ii). Part (iii) follows from (ii), using Lemmas 3.3 and 3.4(i)(f).

From $Q = Q_\sigma$, we have $Q_\rho \leq [Q, \tau]$. Hence $L_2 = L_{2_\tau}$ and (7.22) (i) imply that $Q_\rho \leq C_Q(L_2)$. Since $[Q, \tau] \leq O_q(QL_3)$, we have

$$Q_\rho \leq O_q(QL_3) \cap C_Q(L_2).$$

Hence

$$N_G(Q_\rho) \geq O_{\pi_3}(QL_3), L_2.$$

Note that $N_{L_3}(Q_\rho) = L_3$ would imply that Q_ρ^G was a non-trivial **proper α-invariant normal** subgroup of G. So

(7.23) $$N_{L_3}(Q_\rho) \neq L_3.$$

By (7.22)(ii) $Q = Q_\sigma$ and so

$$[L_3, \sigma] \leq O_{\pi_3}(QL_3) \leq N_{L_3}(Q_\rho).$$

Hence $L_3 = N_{L_3}(Q_p)L_{3_\sigma}$. *Now* L_2 normalizes L_3 by (7.19) and clearly normalizes N_{L_3} (Q,) and since $L_{2_\sigma} = 1$ using I(2.3) (x) we deduce that

$$L_3 = N_{L_3}(Q_p)C_{L_3}(L_2).$$

In particular, $C_{L_3}(L_2) \neq 1$ by (7.23).

Now $N_G(L_2) \geq C_{L_3}(L_2), N_P(L_2)$ and so, because of (7.18) $N_P(L_2) \nleq N_P(L_3)$, we have

$$C_{L_3}(L_2) \leq \mathscr{P}_{L_3}(P) = N_{L_3}(P)$$

with $C_{L_3}(L_2)$ normalizing $N_P(L_2)$. Since $P^* \leq N_P(L_2)$ by (7.21), I(2.14)(ii) gives

$$P = N_P(L_2)C_P(C_{L_3}(L_2)).$$

But $C_{L_3}(L_2) \neq 1$ impliesthat $C_P(C_{L_3}(L_2)) \leq N_P(L_2)$, contradicting $PL_2 \neq L_2P$. Thus we have shown that (7.18) is untenable and so the proof of the lemma is complete.

Lemma ***7.5. Assume Hypothesis 7.3 holds. Then one of L-L, and L-L, is a soluble Hall subgroup of* G .**

Proof. If the lemma were false, then there would exist α-invariant Sylow p - and q-subgroups P and Q of L^- (with $p \neq q$) such that

$$PL_2 \neq L_2P \quad \text{and} \quad QL_3 \neq L_3Q.$$

Then, by Lemma 7.4, **PL,** $= L_3P$ and $QL_2 = L_2Q$. Such aconfiguration, since ***PQ = QP*** by Hypothesis 7.3, is not possible by Theorem 4.4. This proves the lemma.

Theorem ***7.6. Hypothesis 7.3 does not hold.***

Proof. We suppose Hypothesis 7.3 pertains and seek a contradiction. For Theorem 7.2 we deduce

(7.24) $$L_2 \neq 1 \neq L_3 \quad \text{and} \quad L \neq 1.$$

Lemma 7.5 yields, without loss of generality, that ***L-L,*** is a soluble Hall subgroup of G and therefore LL_2 is a soluble Hall subgroup of G. Consequently, appealing to Theorem 7.2 again, we have

(7.25) $$L_1L_2 \neq L_2L_1.$$

From the definition of L^- we also note that

(7.26) $PL_3 \neq L_3 P$ for all (non-trivial) cu-invariant Sylow p-subgroups of L^-.

(7.27) If P is an cr-invariant Sylow p-subgroup of L, then P is star-covered.

Suppose P is not star-covered. Then $O_p(LL_1) \neq 1$ and, of course, $D = Z(P) \cap O_p(LL_1) \neq 1$, with, by I(2.6), $D^{LL_1} \leq Z(J(P))$. First we consider the case when $P \leq L^+$. If P permutes with L_{23}, then $O_p(LL_1)^G$ would be a non-trivial proper cu-invariant normal subgroup of G. So $PL_{23} \neq L_{23}P$ and, by Lemma 3.2, $\mathscr{P}_P(L_{23}) = N_P(L_{23})$ with $[L_{23}, P_\rho] = 1$. If $Z(J(P))_\rho \neq 1$, then $D^{LL_1} \leq Z(J(P)) \leq N_G(L_{23})$. Since $G = (LL_1)(L_2 L_3 L_{23}) = N_G(D^{LL_1}) N_G(L_{23})$, this is not possible. Whereas $Z(J(P))_\rho = 1$ yields, using I(2.6), $LL_1 = C_{LL_1}(D) L_1$. Then, since $Z(P), G_\rho \leq N_G(L_{23})$, we obtain $G = N_G(L_{23}) C_G(D)$ with $D \leq N_G(L_{23})$, again an impossible situation. Thus we conclude that $P \leq L^-$. Since P permutes with L_1 and L_2 but not L_3 and, by (7.25), $L_1 L_2 \neq L_2 L_1$, Lemma 4.7 implies that $N_P(L_3) \neq 1$. Hence $P_\rho, P_\sigma \leq N_P(L_3)$ by Lemma 3.4 (i)(a) and then 0, $(L_1 L_2) = 1$ by I(5.8)(f). So $L_3 \trianglelefteq L_2 L_3 L_{23}$. Then, because $G = (LL_1) N_G(L_3)$, $Z(J(P)) \not\leq N_G(L_3)$. Therefore $P_{\rho\sigma} \neq 1$ by Lemma 4.7. Recalling that $[P_{\rho\sigma}, L_3 L_1] = 1$, $O_{\pi_2}(L_1 L_3) = 1$ implies that $N_G(L_3)$ contains a non-trivial cr-invariant normal $\pi(L_2 L_3 L_1)'$-subgroup. Such a configuration cannot occur and so we have shown that P must be star-covered.

(7.28) (i) If $L_{23} \neq 1$, then $PL_{23} \neq L_{23} P$ for each non-trivial α-invariant Sylow subgroup P of L^-.

(ii) For each α-invariant Sylow subgroup P-of L^+, $PL_{23} = L_{23} P$.

(i) Let P be as in (i), and suppose $PL_{23} = L_{23} P$. By I(2.8) and I(6.1) $L_{23}^* \neq L_{23}$ and hence $O_{\pi_{23}}(PL_{23}) \neq 1$ by I(4.5). But then $PL_3 = L_3 P$, contradicting (7.26). Therefore $PL_{23} \neq L_{23} P$.

(ii) Suppose $PL_{23} \neq L_{23} P$. By (7.27) P is star-covered, and so $N_P(L_{23}) \leq P_\sigma$ or P_τ by Lemma 3.2 and I(2.3)(viii). Hence $P = P_\sigma$ or P_τ by I(2.3) (v). But then one of $L_3 \trianglelefteq PL_3$ and $L_2 \trianglelefteq PL_2$ must hold, which forces $PL_{23} = L_{23} P$, a contradiction. This proves (ii).

(7.29) $$L^- \neq 1$$

For $L^- = 1$ implies that $L = L^+$ whence, using (7.28)(ii), $LL_2 L_3 L_{23}$ is a soluble Hall subgroup and $G = L_1 (LL_2 L_3 L_{23})$. Theorem 7.2 rules out this situation, and so $L^- \neq 1$.

We now explore the consequences of (7.26).

(7.30) Let P be a (non-trivial) cu-invariant Sylow p-subgroup of L^-. Then $P_\rho, P_\sigma \leq N_P(L_3)$.

Suppose (7.30) were false and argue for a contradiction. Then $L_3^* \leq N_{L_3}(P)$ by (7.26) and Lemma 3.3. If $Z(P) \leq P_{\sigma\tau}$ were to hold, then I(2.3) (xi) yields $Z(P) \leq \mathscr{P}_P(L_3) = 1$. So $Z(P) \not\leq P_{\sigma\tau}$ and hence Lemma 3.2 implies $PL_{,,} = L_{23} P$. Therefore $L_{23} = 1$ by (7.28)(i).

Now let Q be an arbitrary non-trivial α-invariant Sylow subgroup of L^- (so $Q L_3 \neq L_3 Q$) and suppose Q,, $Q_\sigma \leq N_Q(L_3)$. Since Q is star-covered by (7.27), Lemma 3.4 (ii)(e) implies Q = Q,. Thus $\mathscr{M}(g, \pi_3) = \{Q, N_Q(L_3) L_3\}$. From Lemma 3.4(i)(c) and (d) either $Z(L_3) = Z(L_3)_{\rho\sigma}$ or $L_{3_\rho} = L_{3_\sigma}$. Then $[Z(L_3), N_Q(L_3)] = 1$ by I(6.4) which contradicts the shape of $\mathscr{M}(q, \pi_3))$. Thus $Q_\rho, Q_\sigma \leq N_Q(L_3)$ cannot hold and so $L_3^* \leq N_{L_3}(Q)$.

Because $L_3^* \leq N_{L_3}(P)$, $P_{\rho\sigma} = 1$ by Lemma 3.4(i)(b) and so $[P_\rho, L_2] = 1$ by I(3.6)(ii). The shape of $\mathscr{M}$ (p, π_3) then dictates that $O_{\pi_2}(L_2 L_3) = 1$. Hence

(7.31) $$L_2 = L_{2_\tau}.$$

Since 0, $(L, L_3) = 1$, clearly $L_{3_\rho} \neq L_{3_\sigma}$ by I(6.4) and therefore, using Lemma 3.4(i)(c) and (d) we obtain $Z(L_3) = Z(L_3)_{\rho\sigma} \leq N_{L_3}(Q)$ for each cu-invariant Sylow subgroup Q of L^-. Hence

(7.32) $$Z(L_3) = Z(L_3)_{\rho\sigma} \leq N_{L_3}(L^-).$$

We now demonstrate that $L_3 \trianglelefteq L_2 L_3 L^+$. By I(2.13) this will follows if we could show that $J = O_{\pi_3'}(L_2 L_3 L^+) = 1$. Because $O_{\pi_2}(L_2 L_3) = 1$ we have $J \leq L^+$, and hence $J^G = J^{(L_1 L^-)} \leq L; L$. Thus $J = 1$.

If $Z(L_3) \leq N_{L_3}(L_,)$, then, together with (7.32), we would have $Z(L_,) \leq N_G(L_1 L)$. Since

$$G = (L_1 L)(L_2 L_3) = (L_1 L^-)(L_2 L_3 L^+) = (L_1 L^-) N_G(Z(L_3))$$

this-yields that $Z(L_3)^G$ is a non-trivial proper α-invariant normal subgroup of G. Therefore $Z(L_3) \not\leq N_{L_3}(L_1)$.

Now we show that $Z(L_3) \not\leq N_{L_3}(L_,)$ leads to a contradiction. Suppose $L_1 L_3 \neq L_3 L_1$. By (7.32) $L_{3_\sigma} \not\leq N_{L_3}(L_1)$, and so $L_{1_\sigma} \leq N_{L_1}(L_3)$. But $Z(L_3) = Z(L_3)_{\rho\sigma}, N_{L_1}(L_3) \neq 1$ and I(2.3) (xi) force $Z(L_3) \leq N_{L_3}(L_,)$. Consequently $L_1 L_3 = L_3 L_1$. Since $[P_\sigma, L_1] = 1$ (because $P_{\rho\sigma} = 1$) and $\mathscr{M}(p, \pi_3) = \{L_3, N_{L_3}(P) P\}$, $O_{\pi_1}(L_1 L_3) = 1$ whence $L_1 = L_{1_\tau}$. However $L_2 = L_{2_\tau}$ by (7.31) and so $L_1 L_2 = L_2 L_1$, against (7.25). This is the desired contradiction that establishes (7.30).

Combining (7.27), (7.30) and Lemma 3.4(ii)(e) gives

(7.33) (i) $L^- \leq G_\tau$.
(ii) $\mathscr{M}(p, \pi_3) = \{P, N_{L_3}(P) L_3\}$ for each cr-invariant Sylow p-subgroup P of L^-.

In deducing the fina1 contradiction we shall need the following observation

(7.34) $$L_{3_\rho} \neq L_3 \neq L_{3_\sigma}.$$

Let P be a non-trivial α-invariant Sylow p-subgroup of L^-. From (7.30) and (7.33) $P = P_\tau$, P_ρ, $P_\sigma \leq N_P(L_3)$ and $P_{\rho\sigma} = 1$. So $(PL_1)_{\rho\sigma} = 1 = (PL,)$, and therefore $[P_\rho, L_2] = 1 = [P_\sigma, L_1]$.

Suppose $L_3 = L_{3_\sigma}$ holds. Then $[P_\rho, L_3] = 1$ by 1(2.3)(x). Recalling that $[P_\rho, L_{23}] = 1$ by Lemma 3.2, we then have that P_ρ centralizes $L_2 L_3 L_{23}$, which is not possible. Now we consider the possibility $L_3 = L_{3_\rho}$. Then $[P_\sigma, L_3] = 1$. This implies $PL, = L_{23} P$. For $PL, \neq L_{23} P$ implies Z(P) $\leq P_{\sigma\tau}$, which contradicts $PL_3 \neq L_3 P$. Hence $L_{23} = 1$ by (7.28) (i) and so P_σ centralizes $L_2 L_3 L_{23} = L_2 L_3$, which is not possible. This proves (7.34).

(7.35) A contradiction.

Let P be a fixed (non-trivial) α-invariant Sylow p-subgroup of L^-. Since $P = P_\tau$ by (7.33)(i), I(2.3) (ix) and I(2.13) imply

$$N_G([L_2, \tau]) \geq P, O_{\pi_3}(L_2 L_3).$$

If $[L_2, \tau] \neq 1$, then (7.33)(ii) forces $O_{\pi_3}(L_2 L_3) = 1$. But then $L_3 = L_{3_\sigma}$, against (7.34). Therefore

(7.36) $$L_2 = L_{2_\tau}.$$

Clearly $(PL_2)_{\rho\sigma} = 1$ and so, since $P_\rho, P_\sigma \leq N_P(L_3)$, I(5.8)(f) (with $L = L_2$, $M = P, N = L_3$) gives $O_{\pi_2}(L_2 L_3) = 1$. We may now argue as earlier to obtain $L_3 L_{23} \trianglelefteq \trianglelefteq L_2 L_3 L_{23} L^+$. Hence

(7.37) $$L_3 \trianglelefteq L_2 L_3 L_{23} L^+.$$

If $L_3 L_1 = L_1 L_3$, then, as $L_3 \neq L_{3_\rho}$ by (7.34). $O_{\pi_3}(L_3 L,) \neq 1$ whence $L_1 L_{23} = L_{23} L_1$. Therefore using (7.33) (i) and (7.36)

$$G = (L_1 L_3 L_{23} L^+)(L^- L_2) = (L_1 L_3 L_{23} L^+) G_\tau.$$

This cannot happen since $L_1 L_3 L_{23} L^+$ is a soluble subgroup, and so we infer that $L_1 L_3 \neq L_3 L_1$. Hence either $L_{1_\sigma} \leq N_{L_1}(L_3)$ or $L_{3_\sigma} \leq N_{L_3}(L_1)$.

Suppose $L_{1_\sigma} \leq N_{L_1}(L_3)$ holds. Then, by (7.30),

$$(L^- L_1)_\rho, (L^- L_1)_\sigma \leq N_G(L_3).$$

Now L^-L_1 admits $\rho\sigma$ fixed-point-freely and so, since

$$G = (L_2 L_3 L_{23} L^+)(L^- L_2) = N_G(L_3)(L^- L_2)$$

by (7.37), the argument used at the conclusion of the proof of Theorem 7.2 will prove that $L_1^- L_1$ is nilpotent. Since $[P_\rho, L_2] = 1$ because $(PL_2)_{\rho\sigma} = 1$), we obtain $L_1, L_2 \leq C_G(P_\rho)$, which contradicts (7.25). So $L_{1_\sigma} \not\leq N_{L_1}(L_3)$.

It only remains to consider the case $L_{3_\sigma} \leq N_{L_3}(L_1)$. If $C_{L_3}(L_1) \neq 1$, then (7.33) (ii) forces $O_{\pi_1}(PL_1) = 1$. Therefore, as $(PL_1)_{\rho\sigma} = 1$, $L_1 = L^*_{1\langle\rho\sigma\rangle} = L_{1_\sigma}$. Hence $Z(L_3) \leq L_{3_{\rho\sigma}}$ by I(2.3)(x) and I(5.1)(b). But then $[Z(L_3), N_P(L_3)] = 1$ by I(2.3)(xi) which is contrary to the form of $\mathcal{M}(p, \pi_3)$. Thus $C_{L_3}(L_1) = 1$, and so $N_{L_3}(L_1) \leq L_3$. So, by (7.34), $L_3^* = L_{3_\rho} \neq L_3$. Since $P_\rho \leq N_P(L_3)$, $L_3 = L_{3_\rho} C_{L_3}(P_\rho)$ by I(2.14)(ii). Since, $C_{L_3}(P_\rho) \neq 1$, using (7.33)(ii) we deduce that $Z(P) \leq N_P(L_3)$ and $Z(P)_\rho = 1$. Because $(PL_1)_{\rho\sigma} = 1$, we have $L_1 = N_{L_1}(Z(P)) O_{\pi_1}(PL_1)$ and so, as $Z(P)_\rho = 1$, $[Z(P), L_1] = 1$. Therefore $Z(P) \leq N_P(L_1) \cap N_P(L_3)$ and so $Z(P)$ normalizes $N_{L_3}(L_1)(\geq L_{3_\sigma})$. Since $L_3 \neq N_{L_3}(L_1)$, I(2.14) (i) and $\mathcal{M}(p, \pi_3)$ give $Z(P) \leq P_\sigma$.

Now $K = L^-L_1$ admits $\rho\sigma$ fixed-point-freely and so $K = N_K(P) O_{p'}(K)$. Combining $Z(P) \leq P_\sigma$ and I(2.3) (xi) gives $[Z(P), N_K(P)] = 1$. By (7.30) $P^*_{\langle\rho\sigma\rangle} \leq N_P(L_3) \neq P$, thence $O_p(K) \neq 1$. Therefore

$$1 \neq D = Z(P) \cap O_p(K) \leq N_G(L_3) \cap Z(K)$$

Consequently, using (7.37),

$$G = (LL_1)(L_2 L_3 L_{23}) = K N_G(L_3) = C_G(D) N_G(L_3),$$

which is not possible.

This verifies (7.35) and completes the proof of Theorem 7.6.

Taken together Theorem 6.3, 7.2 and 7.6 show that G cannot exist, so proving the main theorem of this paper.

REFERENCES

[1] Z. ARAD, G. GLAUBERMAN, *A characteristic subgroup of odd order*, Pacific J. Math. 56 (1975). pp. 305-319.

[2] W. FEIT, J.G. THOMPSON, *Solvability of groups of odd order*, Pacific J. Math. 13 (1963), pp. 775-1029.

[3] D. GORENSTEIN, *Finite groups*. Harper and Row, 1968.

[4] P.J. ROWLEY, *Finite groups which admit a fixed-point-free automorphism of order rst*, Ph.D. Thesis, University of Warwick, 1975.

[5] P.J. ROWLEY, *Solubiiity of finite groups admitting a fixed-point-free aufomorphism of order rst I*, Pacific J. Math. 93 (1981), pp. 201-235.

[6] P.J. ROWLEY, *Solubility of finite groups admitting a fixed-point-free aufomorphism of order rst II*, J. Algebra 83 (1983). pp. 293-348.

[7] P.J. ROWLEY, *Solubility of finite groups admitting a fixed-point-free automorphism of order rst III*, Israel Journal of Mathematics 51 (1985), pp. 125-150.

[8] P.J. ROWLEY, *Solubility of finite groups admitting a fixed-point-free automorphism of order rst IV*, Mathematische Zeitschrift 186 (1984). pp. 435-464.

Received November 24.1988.

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